GRAPH-COLOURING AND COMBINATORIAL NUMBERS

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1. Introduction. We shall use the term *n-configuration* for the complete graph having n vertices, which we visualize as a regular convex polygon. The sides and diagonals will be referred to as *edges*. A complete sub-configuration of p vertices (subset of p vertices with all interconnecting edges) will be called a p-tuple. A 3-tuple will be called a triple or triangle, a 4-tuple a quadruple, etc.

Suppose that all of the edges of an n-configuration are coloured, in an arbitrary way, either red or blue **(1)**. Then the Ramsey combinatorial number $N(p, q; 2)$ may be described as the smallest integer such that if $n \geqslant N(p, q; 2)$, the n-configuration will certainly contain either a red p-tuple (a p-tuple all of whose edges are coloured red) or a blue q-tuple. The theorem of Ramsey asserts that the numbers $N(p, q; 2)$, as well as more general but analogous numbers $N(p_1, \ldots, p_t; r)$, always exist; see **(6**, pp. 38–43**)**. As indicated in **(6)**, up to the year 1963, essentially all of the known exact information concerning the numbers $N(p, q; 2) = N(q, p; 2)$ was derived in **(1)**, and this information is the following: $N(4, 4; 2) = 18$, $N(3, 3; 2) = 6$, $N(3, 4; 2) = 9$, $N(3, 5; 2) = 14$, $N(3, 6; 2) = 17$, 18, or 19. This paper includes a proof that $N(3, 6; 2) = 18$. The referee points out that approximately simultaneously with the author, the value $N(3, 6; 2) = 18$ was established independently by Gerzson Kéry **(5)**, by J. G. Kalbfleisch (1966 Ph.D. dissertation, University of Waterloo, Waterloo, Canada), and by J. E. Graver (not yet published). Graver has proved also that $N(3, 7; 2) = 23$, and found bounds for $N(3, 8; 2)$ and $N(3, 9; 2)$. The author is indebted to the referee for calling his attention to the work of Kéry, Kalbfleisch, and Graver. (See **(2)** and **(4)**.)

The following "verticial" analogous number to $N(p, q; 2)$ suggests itself: $S(p, q; 2)$ is the smallest integer such that if $n \geqslant S(p, q; 2)$, the n-configuration must contain, *for each vertex*, either a red p-tuple containing the vertex or a blue q-tuple not containing the vertex. It is shown in §3, for $p \geqslant 2$, $q \geqslant 2$, that necessarily $S(p, q; 2) = \infty$, i.e., that the number $S(p, q; 2)$ does not exist.

In this paper, several new kinds of combinatorial numbers, which, like the Ramsey numbers, do exist, are introduced, and their values, or bounds for their values, are determined. For example, a number $K(p, q; 2)$ is obtained by slight modification of the above definition of $S(p, q; 2)$: $K(p, q; 2)$ is the smallest integer such that if $n \geqslant K(p, q; 2)$, then the n-configuration is sure to contain, for each vertex, either a like (red or blue) p-tuple containing the

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vertex, or a like q-tuple not containing the vertex. For three colours of edges, the number $M(p, q - x, x; 2)$ is defined to be the smallest integer such that if $n \geqslant M(p, q - x, x; 2)$, then the n-configuration surely contains either a like p-tuple, or a j, k, q-tuple (i.e., a q-tuple in which the edges are of at most two different colours). In §5, a three-colour verticial number $V(p, q - x, x; 2)$, with, just as in the case of $M(p, q - x, x; 2)$, two rather than three alternatives, is introduced, and determinations of its values are made.

In §4, as a side-product, we obtain the following recurrence inequality for $N(3, s, u; 2)$, $3 \leqslant s$, $3 \leqslant u$:

$$N(3, s, u; 2) \leqslant (3q - 1),$$

where $q = N(s, u; 2)$. Except for $N(3, 3, 3; 2) = 17$, as is shown in **(1**, pp. 4–5**)**, this seems to be new. (See the sentence in **(6)** which ends at the top of p. 43.) As far as I know, the lower bound 11 for $N(4, 4; 3)$, established in §2, also is new. (See note added in §2.)

2. Notation; bounds for $N(4, 4; 3)$**.** We use the notation of **(6)** for the Ramsey numbers: $N(p_1, \ldots, p_t; r)$ is regarded as referring to an n-configuration (or $(n - 1)$-dimensional simplex), in which each r-tuple (or $(r - 1)$-dimensional simplicial face) is coloured one of t colours $c_1, c_2, \ldots$, or c_t. A p-tuple, $p \geqslant r$, is of *colour* c_i if all of its r-subtuples are of colour c_i. Always $n \geqslant r$, and $p_i \geqslant r$ for $i = 1, \ldots, t$. In accordance with the theorem of Ramsey **(6)**, if $n \geqslant N(p_1, \ldots, p_t; r)$, then no matter how the r-tuples of the n-configuration are coloured with the t colours, necessarily at least one of the following alternatives holds: the configuration contains a p_1-tuple of colour c_1, or a p_2-tuple of colour $c_2, \ldots$, or a p_t-tuple of colour c_t. For $n < N(p_1, \ldots, p_t; r)$, we say that an n-configuration, with a particular t-colouration of its r-tuples, is *non-degenerate* with respect to $N(p_1, \ldots, p_t; r)$ if none of the alternatives is true.

THEOREM 2.1. *We have* $11 \leqslant N(4, 4; 3) \leqslant 19$.

Proof. Assume that there exists a 19-configuration which is non-degenerate with respect to $N(4, 4; 3)$. Consider the 18-configuration of the other ends of the edges from any fixed vertex v of the 19-configuration. In the 18-configuration, colour edge ij red if the 3-tuple vij of the 19-configuration is coloured red; colour edge ij blue if the 3-tuple vij is coloured blue. Since $N(4, 4; 2) = 18$, the 18-configuration must contain either a red 4-tuple or a blue 4-tuple. The 3-tuples of the 4-tuple whose edges are all of one colour must be coloured all of the opposite colour, in order to avoid a 4-tuple including v having 3-tuples all of one colour. Therefore the 19-configuration is degenerate, contrary to our assumption. This establishes that $N(4, 4; 3) \leqslant 19$.

A 9-configuration which is non-degenerate with respect to $N(4, 4; 3)$ may be constructed as follows. Number the vertices 1 to 9. The triangles 123, 456, 789, 174, 258, 369, 159, 267, 348, 186, 924, 357; 125, 379, 468, 134, 578, 269, 167, 594, 238, 189, 247, 356; 128, 374, 569, 179, 358, 246, 136, 489, 257, 145,

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239, 678 are three disjoint Steiner systems of triangles. Colour red all of these triangles, except 136 of the third system; and also colour red 169, 234, 679, and 137, for a total of 39 red triangles. Colour the remaining 45 triangles blue.

Let a 10th vertex 0 be added to the 9-configuration. By digital enumeration and checking, it may be seen that red and blue colourations of the 36 new triangles involving vertex 0 are possible, such that none of the 4-tuples containing vertex 0 fails to have 3-tuples of both colours. Therefore we have $N(4, 4; 3) > 10$, as required.*

Following is an interesting problem on the connection between $N(4, 4; 3)$ and the Euclidean geometry of an $(n - 1)$-simplex. A Euclidean triangle may have at most one obtuse ($\geqslant 90°$) angle; call such a triangle *obtuse*. Call a Euclidean triangle in which all three angles are acute ($<90°$), an *acute* triangle. Call a tetrahedron *obtuse* if all of its triangular faces are obtuse; *acute* if all its faces are acute; otherwise *normal*. What is the largest n for which a Euclidean $(n - 1)$-simplex exists, in which all of the tetrahedral faces are normal? Clearly $n < N(4, 4; 3)$, and because of the geometrical (and symmetrical!) constraint, one would expect that $n < N(4, 4; 3) - 1$.

Hereafter in this paper, except in the first statement of Theorem 4.6, r will always be 2; i.e., 2-tuples or edges will be coloured, with the number of colours t being 2 or 3.

3. Verticial numbers. As an example, we establish the well-known result that $N(3, 3; 2) = 6$. A pentagon with outer edges red and with blue diagonals contains neither a red triangle nor a blue triangle. We say accordingly that this 5-configuration is *non-degenerate with respect to* $N(3, 3; 2)$, and its existence proves that $N(3, 3; 2) > 5$.

The following argument establishes that $N(3, 3; 2) \leqslant 6$. Of the five edges issuing from any vertex of a 6-configuration with edges of two colours, at least three must be of the same colour, say red. Consider the triangle which has for its vertices the three other ends of the three red edges from the vertex. If the sides of this triangle are all blue, the 6-configuration contains a blue triangle; otherwise it contains a red triangle involving the original vertex. Combining this with the preceding paragraph, we have shown that $N(3, 3; 2) = 6$.

The definitions of $S(p, q; 2)$ and $K(p, q; 2)$, which were given in §1, obviously are suggested by the fact that the above sort of argument applies at each vertex of any configuration.

THEOREM 3.1. *For all $p \geqslant 2$, $q \geqslant 2$, we have that $S(p, q; 2) = \infty$; i.e., for arbitrarily large n, there exists an n-configuration with a vertex for which there is neither a red p-tuple containing the vertex nor a blue q-tuple not containing the vertex.*

**Note added in proof.* In his dissertation, Kalbfleisch established that $N(4, 4; 3) > 11$. I now have examples showing that the Euclidean n described below is greater than 6, and that the corresponding metric n is greater than 8.

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Proof. To show that $S(3, 2; 2) = \infty$, at one vertex of an n-configuration, with n arbitrarily large, assign one edge from the vertex to be red, and let all other edges from the vertex be blue. Let all other edges of the n-configuration be red. Then, as required, there is neither a red triangle containing the vertex nor a blue edge not containing the vertex. To show that $S(2, q; 2) = \infty$ for $q \geqslant 2$, let all edges from the vertex be blue and all other edges red. Then, as required, there is neither a red 2-tuple containing the vertex nor a blue q-tuple not containing the vertex. Finally, to show that $S(p, q; 2) = \infty$ for $p \geqslant 3, q \geqslant 3$, let $(p - 1)$ edges from the vertex be red, and let the remaining edges from the vertex be blue. Let one edge joining other ends of the $(p - 1)$ red edges be blue, and let all other edges of the n-configuration be red. Then, as required, the n-configuration contains neither a red p-tuple containing the vertex nor a blue q-tuple not containing the vertex.

THEOREM 3.2. *We have* $K(3, q; 2) \leqslant 2q$.

Proof. For any configuration with $n \geqslant 2q$ vertices, since there are at least $q + (q - 1)$ edges from each vertex, we see that at least q edges from the vertex must be alike (say red). Consider the q-sub-configuration formed by the vertices at the other ends of the q like edges. Either it is a like q-tuple (all edges say blue), or there is a like triangle (all edges say red) containing the vertex, as required.

Let an m-configuration be called *non-degenerate with respect to* $K(p, q; 2)$ if there is some vertex for which there is neither a like p-tuple containing the vertex nor a like q-tuple not containing the vertex. For non-degeneracy, it is necessary that $m < K(p, q; 2)$. Similarly, with respect to $N(p, q; 2)$, call the configuration *non-degenerate* if the configuration contains neither a red p-tuple nor a blue q-tuple. (For $p < q$, a configuration non-degenerate with respect to $N(p, q; 2)$ might contain a blue p-tuple.)

THEOREM 3.3. *With respect to* $K(3, 3; 2)$ *or* $N(3, 3; 2)$, *a* 5-*configuration is non-degenerate if, and only if, it consists of a cycle* (*closed polygon*) *of five red edges and of a* (*complementary*) *cycle of five blue edges.*

Proof. The example, mentioned at the beginning of this section, of the 5-configuration which is non-degenerate with respect to $N(3, 3; 2)$, clearly also is non-degenerate with respect to $K(3, 3\colon 2)$; this shows the sufficiency.

To show necessity, call the *red number* of a vertex the number of red edges which issue from the vertex. In consequence of the edge-argument given at the beginning of this section, it is clear that in order that a 5-configuration not contain a like triangle, the red number of each vertex must be exactly 2. This implies that the vertices may be arranged in order so that the red edges are the outer edges of the pentagon, with the diagonals blue. The five-pointed star formed by the diagonals is a blue 5-cycle.

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THEOREM 3.4. *We have the following table of exactly determined values of* $K(p, q; 2)$ *for some* p, q, *and the following equation and inequalities for other* p, q. *For all* $q \geqslant 3$, *we have* $K(q, q; 2) = N(q, q; 2)$; *for* $p > q$, $K(p, q; 2) \leqslant N(q, q; 2) + 1$; *and for* $3 < p < q$, $K(p, q; 2) \geqslant K(p - 1, q; 2) + 1$, *which implies that* $K(p, q; 2) \geqslant 2q + p - 3$.

DETERMINED VALUES OF THE NUMBER $K(p, q; 2)$

$p \diagdown q$	3	4	5 . . .	q
3	6	8	10	$2q$
4	7	18		
5	7			
.				
.				
.				
p	7			

Proof. Since a non-degenerate pentagonal configuration exists, by Theorem 3.2 we have $K(3, 3; 2) = N(3, 3; 2) = 6$. Also by Theorem 3.2, $K(3, 4; 2) \leqslant 8$. In a 6-configuration, with vertices 1, 2, 3, 4, 5, 6, let the eight edges 13, 14, 15, 16, 25, 26, 35, 56 be red and the remaining seven edges blue. The 6-configuration contains no like 4-tuple. Consider the 7-configuration which is formed from the 6-configuration by addition of a vertex 7, with red edges 27, 37, 47 and blue edges 17, 57, 67. Then, for vertex 7, there is neither a like triangle containing the vertex nor a like 4-tuple not containing the vertex. Therefore this 7-configuration is non-degenerate with respect to $K(3, 4; 2)$, and it is established that $K(3, 4; 2) = 8$.

Again in a 6-configuration, let the 5-cycle of edges 23, 34, 45, 56, 62 be blue and the remaining 10 edges red. With respect to $K(4, 3; 2)$, this 6-configuration is non-degenerate; for the configuration of the five vertices other than 1 is the pentagonal configuration which contains no like triangle, and therefore there is neither a like 4-tuple containing 1 nor a like triangle not containing 1. On the other hand, for each vertex of any 7-configuration, since $N(3, 3; 2) = 6$, we have that the configuration of the other six vertices contains a like triangle. Thus $K(4, 3; 2) = 7$. Similarly $K(p, 3; 2) = 7$ for all $p > 4$, and for $p > q > 3$, $K(p, q; 2) \leqslant N(q, q; 2) + 1$.

By Theorem 3.2, we have that $K(3, q; 2) \leqslant 2q$. For $q \geqslant 5$, with respect to $K(3, q; 2)$, the following $(2q - 1)$-configuration is non-degenerate. The configuration contains a $(q - 1)$-tuple, with vertices $1, \ldots, (q - 1)$, which has all sides red; and a $(q - 1)$-tuple, with vertices $q, (q + 1), \ldots, (2q - 2)$, which has all sides blue. The edges from the vertex $(2q - 1)$ to $1, \ldots, (q - 1)$ are all blue; and those from $(2q - 1)$ to $q, \ldots, (2q - 2)$ are all red. Thus there is no like triangle at vertex $(2q - 1)$. For any q-tuple to have all sides red, it is necessary that it include $1, \ldots, (q - 1)$, and for it to have all sides blue, it is necessary that it include $q, \ldots, (2q - 2)$. Therefore in order to obtain the

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desired conclusion that $K(3, q; 2) = 2q$, we assign the following alternating cycle of edges: 1 to q is red, q to 2 is blue, 2 to $(q + 1)$ is red, ..., $(q - 1)$ to $(2q - 2)$ is red, $(2q - 2)$ to 1 is blue. The other edges between $1, \ldots, (q - 1)$ and $q, \ldots, (2q - 2)$ may be assigned arbitrarily to be red or blue. Thus each q-tuple must contain both a red edge and a blue edge, as required for the conclusion. Accordingly we have $K(3, q; 2) = 2q$ for all $q \geqslant 3$.

Since $N(4, 4; 2) = 18$, we have that for $n = 17$, there exists a configuration which contains neither a 4-tuple with all sides red nor a 4-tuple with all sides blue. For this configuration, it is true that for each vertex, there is neither a like 4-tuple containing the vertex nor a like 4-tuple not containing the vertex. Therefore $K(4, 4; 2) > 17$. For $n = 18$, however, any configuration must contain a like 4-tuple. For each vertex of the 4-tuple, there is a like 4-tuple containing the vertex. For every other vertex, the same like 4-tuple is a like 4-tuple not containing the vertex. Therefore $K(4, 4; 2) = 18$. Similarly $K(p, p; 2) = N(p, p; 2)$ for $p > 4$.

Finally for $3 < p < q$, to show that $K(p, q: 2) \geqslant K(p - 1, q; 2) + 1$, consider a configuration with $n = K(p - 1, q; 2) - 1$ which is such that there is an exceptional vertex, for which there is neither a like $(p - 1)$-tuple containing the vertex nor a like q-tuple not containing the vertex. Then for any new configuration including this configuration and one additional vertex, there cannot be a like p-tuple containing the exceptional vertex: such a p-tuple would have to contain a like $(p - 1)$-tuple containing the exceptional vertex. The set of vertices of the original configuration, other than the exceptional vertex, which are vertices of $(q - 1)$-tuples with all sides red, and the similar set of vertices of $(q - 1)$-tuples with all sides blue, have at most one vertex in common. Assign each edge from the new vertex to the vertices of the first set to be blue, and each edge to the vertices of the second set, except the common vertex if there is one, to be red. Then the exceptional vertex is still exceptional in the new configuration, i.e., there is neither a like p-tuple containing it nor a like q-tuple not containing it. Thus the new configuration is non-degenerate with respect to $K(p, q; 2)$, as required.

4. Three-colour combinatorial numbers. We turn now to configurations with three colours of edges, with however two rather than three alternative sub-configurations.

Definition 4.1. The number $M(p, q - x, x; 2)$ is the smallest integer such that if $n \geqslant M(p, q - x, x; 2)$, then any n-configuration is certain to contain either a p-tuple with all sides of one colour or a q-tuple with at most two colours of sides (which will be called a j, k q-tuple).

Definition 4.2. The number $V(p, q - x, x; 2)$ is the smallest integer such that if $n \geqslant V(p, q - x, x; 2)$, then for each vertex of any n-configuration, there must be either a like p-tuple containing the vertex, or a j, k q-tuple not containing the vertex.

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Let the three colours be red, green, and violet. Then by grouping say green and violet edges as blue, we see that $M(p, q - x, x; 2) \leqslant N(p, q; 2)$. Also, it is evident that $M(p, q - x, x; 2) \leqslant V(p, q - x, x; 2)$, and that for each q, $V(q, q - x, x; 2)$ is either $M(q, q - x, x; 2)$ or $M(q, q - x, x; 2) + 1$. For at each vertex of a configuration with $n = M(q, q - x, x; 2) + 1$, the sub-configuration obtained by omitting the vertex contains either a j, k q-tuple or a like q-tuple (which qualifies as a j, k q-tuple).

THEOREM 4.3. *For* $p \geqslant 3$, *we have* $M(p, 3 - x, x; 2) = 5$. *Also* $M(3, 4 - x, x; 2) = 8$, *while* $N(3, 4; 2) = 9$.

Proof. A square with horizontal sides red, vertical sides green, and diagonals violet contains neither a like triangle nor a j, k triangle; therefore $M(3, 3 - x, x; 2) > 4$. Four edges issue from each vertex of any 5-configuration, of which two must be alike; the two are edges of either a like triangle or a j, k triangle, as required for degeneracy of the configuration with respect to $M(3, 3 - x, x; 2)$. Therefore $M(3, 3 - x, x; 2) = 5$. Similarly for $p > 3$, we have $M(p, 3 - x, x; 2) = 5$. (Recall that a like triangle, or triangle with only one colour j of sides, qualifies as a j, k triangle.)

To show that $M(3, 4 - x, x; 2) \geqslant 8$, we construct a non-degenerate 7-configuration, as follows. In the Steiner triple system on seven elements (**3**, p. 82, or **6**, p. 100), the 4-tuple 1346 contains one edge of each of six of the seven Steiner triangles. The Steiner triangles which contain edges 16 and 34 have 7 as common vertex; the Steiner triangles which contain 13 and 46 have 2 as common vertex; and the Steiner triangles which contain the diagonals 14 and 36 have 5 as a common vertex. The seventh Steiner triangle is 257. This Steiner system (**3**, p. 82) also may be described as follows: Let equally spaced points on a circle be numbered in order as 1243675. Then the seven Steiner triangles are 123 and the six further triangles obtained by rigid rotations of 123. Assign edge 12 to be red, 23 to be green, and 31 to be violet; and preserve the same edge assignments for the six rigid isomorphs of 123. This assigns all edges of the 7-cycle 12436751 as red, all edges of the 7-cycle 14652371 as green, and all of the 7-cycle 13547261 as violet. Clearly any triangle of the 7-configuration must have its sides contained in either two or three of the 7-cycles; therefore there is no like triangle. Of the thirty-five 4-tuples, twenty-eight contain Steiner triangles which are three-colour triangles. The remaining seven 4-tuples, namely 1247, 1256, 1346, 1357, 2345, 2367, 4567, each contain edges of all three colours. Therefore there is also no j, k, 4-tuple in the 7-configuration; thus the configuration is non-degenerate. Since $M(3, 4 - x, x; 2) \leqslant N(3, 4; 2) = 9$, so far we have established that $M(3, 4 - x, x; 2)$ is either 8 or 9. (Less symmetric non-degenerate 7-configurations also exist.)

To show that necessarily $M(3, 4 - x, x; 2) = 8$, we note first that in any 8-configuration which is non-degenerate with respect to $N(3, 4; 2)$, the red number of each vertex must be 2 or 3. For if the red number of some vertex were $\geqslant 4$, either the configuration of the four other ends of the four red edges

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would be a blue 4-tuple or there would be a red triangle involving the vertex, contrary to non-degeneracy of the 8-configuration with respect to $N(3, 4; 2)$. Similarly, if the red number of some vertex were <2, there would be six blue edges from the vertex, and thus since $N(3, 3; 2) = 6$, the 8-configuration would have to contain either a blue 4-tuple involving the vertex or a red triangle.

LEMMA 4.4. *In any 8-configuration which is non-degenerate with respect to $N(3, 4; 2)$, necessarily the red number is 3 for at least four of the eight vertices of the configuration.*

Proof. If p vertices are 3-red, and the remainder 2-red, then the total number of red edges is $[3p + 2(8 - p)]/2$, whence p must be even. There obviously are blue quadruples if all eight vertices have red number 2 (cases having no red triangle: an 8-cycle, or two 4-cycles, of red edges). We consider all possible red colourations, having no red triangle, with two 3-red and six 2-red vertices. All cases are covered under the headings 1 to 5 below. Let edges of the 8-configuration be classified as follows: outer edges 12, 23, . . . , 78, 81 are *skip*-0 edges; edges 13, 24, . . . are *skip*-1 edges; . . . ; edges joining vertices with k vertices in between are *skip-k* edges.

1. The outer 8-cycle of skip-0 edges are red, and there is a skip-2 or skip-3 red edge, say 14 or 15. In either case, quadruple 2468 is blue.

2. There is a 7-cycle 23456782 of red edges, and edges 21, 14, or 21, 15, are red. In either case, quadruple 1368 is blue.

3. There is a 6-cycle 3456783 of red edges, and edges 81, 12, and 25 are red. In this case quadruple 1357 is blue.

4. If there is a 5-cycle of red edges, there must also be a 6-, 7-, or 8-cycle. The latter cases are already covered.

5. There are two 4-cycles, 12341, 56785, of red edges, and edge 25 is red. In this case quadruple 1368 is blue.

Now to return to the proof that $M(3, 4 - x, x; 2) = 8$, in any 8-configuration which is allegedly non-degenerate with respect to $M(3, 4 - x, x; 2)$, by grouping green and violet edges as blue, we see by Lemma 4.4 that the red number of four of the eight vertices must be 3. Similarly, by grouping red and violet edges together, we see that the green number of four of the eight vertices must be 3; and by grouping red and green edges, that the violet number of four of the eight vertices must be 3. But with only eight vertices, since there are $2 + 2 + 3 = 7$ edges from each vertex, it is not possible to have simultaneously four 3-red, four 3-green, and four 3-violet vertices. Therefore any 8-configuration necessarily is degenerate with respect to $M(3, 4 - x, x; 2)$, and it is established that $M(3, 4 - x, x; 2) = 8$.

THEOREM 4.5. *We have $M(3, 5 - x, x; 2) = N(3, 5; 2) = 14$, while $M(3, 6 - x, x; 2) = 17$, $N(3, 6; 2) = 18$.*

Proof. We shall show that the following 13-configuration is non-degenerate with respect to $M(3, 5 - x, x; 2)$. Let the skip-0 and skip-4 edges be red, the

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skip-1 and skip-2 edges green, and the skip-3 and skip-5 edges violet. Note that in a 13-configuration, the kinds of edges of the following pairs are really the same: skip-1, skip-10; skip-2, skip-9; . . . ; skip-5, skip-6. The condition for skip-a, skip-b, and skip-c edges to be sides of a triangle is

$$a + b + c = 10. \tag{1}$$

No combination of three skip-0, skip-11, skip-4, skip-7 edges can satisfy (1); therefore there is no red triangle. Similarly, no combination of three skip-1, skip-10, skip-2, skip-9 edges can satisfy (1); and no combination of three skip-3, skip-8, skip-5, skip-6 edges can satisfy (1); therefore there is no green or violet triangle.

The condition for five edges to be outer edges of a 5-tuple is

$$a + b + c + d + e = 8, \tag{2}$$

where $a, \ldots, e$ are the respective skips of the edges. The possible integral solutions $a, \ldots, e$ of (2) are the permutations of the following: 80000, 71000, 62000, 53000, 44000, 61100, 52100, 43100, 42200, 33200, 51110, 42110, 33110, 22220, 41111, 32111, 22211. All except 80000, 71000, 53000, 44000, 42200, 42110, 22220, 41111, 32111, 22211 have three colours of outer edges. It may be seen that the 5-tuples 80000, 22220, 41111, as well as all permutations of 71000, 53000, 44000, 42200, 42110, 32111, 22211, contain edges of all three colours. Thus the census of 5-tuples is complete, and it is verified that there is no two-coloured 5-tuple. As was observed immediately following Definition 4.2, $M(p, q - x, x; 2) \leqslant N(p, q; 2)$. Since we have just shown that $M(3, 5 - x, x; 2) > 13$, and since $N(3, 5{:}\,2) = 14$, it follows that $M(3, 5 - x, x; 2) = N(3, 5; 2) = 14$.

For each vertex of any 17-configuration, since $16 = 5 + 5 + 6$, there must be six edges of one colour issuing from the vertex. If the 6-tuple formed by the other ends of the six edges is three-coloured, then there is a like triangle involving the vertex. Therefore $M(3, 6 - x, x; 2) \leqslant 17$.

It is shown on pages 4–5 of **(1)** that $N(3, 3, 3; 2) = 17$. Consider the 16-configuration, non-degenerate with respect to $N(3, 3, 3; 2)$, which is described there. This 16-configuration also is non-degenerate with respect to $M(3, 6 - x, x; 2)$. For suppose it were degenerate; then it would have to contain a two-coloured 6-tuple. Since $N(3, 3; 2) = 6$ and since the 16-configuration contains no red triangle, and no like triangle, the 6-tuple must be green-violet. Again since $N(3, 3; 2) = 6$, the 6-tuple must contain a green or a violet triangle, contrary to non-degeneracy of the 16-configuration with respect to $N(3, 3, 3; 2)$. Therefore we have $M(3, 6 - x, x; 2) \geqslant 17$; combining this with the preceding paragraph, it is established that $M(3, 6 - x, x; 2) = 17$.

A 19-configuration, non-degenerate with respect to $N(3, 6; 2)$, cannot have six red edges from a vertex. The number of ends of red edges cannot be odd; therefore it is impossible for the red number to be 5 at all vertices; it must be 4 at one vertex. But then the configuration has 14 blue edges from that vertex,

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and since $N(3, 5; 2) = 14$, either there is a red triangle in the 14-configuration of the other ends, or there is a blue 6-tuple involving the vertex. Thus every 19-configuration is degenerate with respect to $N(3, 6; 2)$, i.e., $N(3, 6; 2) \leqslant 19$.

We defer the proof that $N(3, 6; 2)$ is less than 19 to §6. We show now that $N(3, 6; 2) > 17$, by constructing a 17-configuration which is non-degenerate with respect to $N(3, 6; 2)$. Consider first the following 13-configuration which is non-degenerate with respect to $N(3, 5; 2)$: the skip-0 and skip-4 edges are red, all others blue. Let k-tuples be specified by the skips of their outer edges. The blue 4-tuples in the 13-configuration are of the forms 1323, 2223, 1116. Let the outer edges of a square, with vertices α, β, γ, δ, be red, and the diagonals blue. Red-connect α to vertices 3, 5, 7; β to vertices 2, 9, 11; γ to vertices 4, 6, 8; and δ to vertices 1, 10, 12. Blue-colour all other edges. In the resulting 17-configuration, there are no red triangles, and a blue 6-tuple exists only if one of the blue diagonals $\alpha\gamma$ or $\beta\delta$ is entirely blue-connected to one of the already mentioned 4-tuples 1323, 2223, or 1116. By rotation of trapezoids corresponding to 1323, 2223, around a 13-sided polygon, it is easily verified that such is not the case for those two 4-tuples: each candidate for an entirely blue 6-tuple contains a red edge (e.g., $\alpha 3$, $\beta 9$, etc.). However, the 6-tuple involving the 4-tuple of type 1116 which has vertices 2, 9, 11, 13 is blue; the 6-tuples involving all other 4-tuples 1116 contain a red edge. To remedy the situation with respect to 4-tuple 2, 9, 11, 13, recolour edge 12 blue, and recolour edge 2-13 red. None of the blue 5-tuples so formed in the 13-configuration is entirely blue-connected to any vertex α, β, γ or δ. In this way we accomplish the construction of a 17-configuration which is non-degenerate with respect to $N(3, 6; 2)$, as required. Vertices 2 through 13 are 5-red; vertex 1 is 4-red; and vertices α, β, γ, δ are 5-red.

The proof of the following theorem involves a generalization of the above demonstration that $17 \geqslant M(3, 6 - x, x; 2) \geqslant N(3, 3, 3; 2)$.

THEOREM 4.6. *Any n-configuration, which is non-degenerate with respect to* $N(p, s, u; r)$, $p \leqslant s$, $p \leqslant u$, *also is non-degenerate with respect to* $M(p, q - x, x; r)$, *where* $q = N(s, u; r)$. *In particular, we have*

$$(3q - 1) \geqslant M(3, q - x, x; 2) \geqslant N(3, s, u; 2).$$

Proof. If the configuration were degenerate with respect to $M(p, q - x, x; r)$, then since it does not contain a red p-tuple, it would contain a two-coloured q-tuple. But since $q = N(s, u; r)$ and $s \geqslant p$, $u \geqslant p$, by symmetry in s, u of $N(s, u; r)$ and of $N(p, s, u; r)$, the q-tuple must be green-violet. It then must contain either a green s-tuple, or a violet u-tuple, contrary to non-degeneracy with respect to $N(p, s, u; r)$.

Any $(3q - 1)$-configuration has $(3q - 2) = (q - 1) + (q - 1) + q$ edges from each vertex, of which at least q must be of one colour. Thus there is either a triangle of that colour or a q-tuple of the other two colours. Therefore $(3q - 1) \geqslant M(3, s, u; 2)$.

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THEOREM 4.7. *We have* $10 \leqslant M(4, 4 - x, x; 2) \leqslant 17 < N(4, 4; 2) = 18$.

Proof. At any vertex of a 17-configuration, since $5 + 5 + 6 = 16$, six of the 16 edges issuing from the vertex must be of one colour, say red. Since $N(3, 3; 2) = 6$, the configuration of the six other ends must contain either a red-green triangle, or a violet triangle. In either case, the configuration contains a j, k 4-tuple involving the vertex. Therefore the configuration is degenerate with respect to $M(4, 4 - x, x; 2)$, and it is established that $M(4, 4 - x, x; 2) \leqslant 17$.

We now construct a non-degenerate 9-configuration by use of the Steiner triple system on nine elements (**3**, p. 82). Let all sides of the four Steiner triangles 467, 589, 123, 249 be coloured red. Let all sides of the Steiner triangles 145, 278, 369, 375 be coloured green; and let all sides of the remaining Steiner triangles 168, 179, 256, 348 be coloured violet. Using symmetry, it is easy to verify in turn that no 4-tuple of the 9-configuration fails to contain a red edge, that none fails to contain a green edge, and that none fails to contain a violet edge. Therefore, each 4-tuple contains edges of all three colours, and we have shown that $M(4, 4 - x, x; 2) \geqslant 10$.

5. A three-colour verticial number. The number $V(p, q - x, x; 2)$ was the subject of Definition 4.2, and preliminary comparisons of $V(p, q - x, x; 2)$ with $M(p, q - x, x; 2)$ were made in the paragraph which precedes Theorem 4.3. It may be seen further that for any p, we have $V(p, q - x, x; 2) \leqslant M(q, q - x, x; 2) + 1$. For $p > q$, we have $M(q, q - x, x; 2) \leqslant M(p, q - x, x; 2)$ and $M(q, q - x, x; 2) \leqslant V(p, q - x, x; 2)$. Also, for $p > q$, $V(q, q - x, x; 2) \leqslant V(p, q - x, x; 2)$. Finally, let $W(q, p - x, x; 2)$ be defined as the smallest integer such that if $n \geqslant W(q, p - x, x; 2)$, then for each vertex the configuration is sure to contain either a j, k q-tuple containing the vertex or a like p-tuple not containing the vertex. We notice that $W(q, p - x, x; 2) = V(p, q - x, x; 2)$.

THEOREM 5.1. *For* $p \geqslant 3$, *we have* $V(p, 3 - x, x; 2) = 6$. *For* $3 < q$, *we have* $V(3, q - x, x; 2) \leqslant (3q - 1)$ *and* $V(3, 4 - x, x; 2) = 10$ *or* 11.

Proof. In a square (a 4-tuple), colour the horizontal sides green, the vertical sides violet, and the diagonals red. Abbreviate the respective colours as g, v, r, and refer to such a square as a g, v, r square. All sub-triangles of the square are three-coloured.

A 5-configuration consisting of a g, v, r square, with suitably assigned edges g, g, r, v, from the fifth vertex to the vertices of the square, is non-degenerate with respect to $V(p, 3 - x, x; 2)$ for $p \geqslant 3$; therefore $V(p, 3 - x, x; 2) \geqslant 6$. But for each vertex of any 6-configuration, since $M(3, 3 - x, x; 2) = 5$, the sub-5-configuration obtained by omitting the vertex contains either a j, k triangle or a like triangle (qualifies as a j, k triangle). Therefore any 6-configuration is degenerate with respect to $V(p, 3 - x, x; 2)$ for all $p \geqslant 3$.

In any $(3q - 1)$-configuration, $(q - 1) + (q - 1) + q$ edges issue from each vertex, of which some q must be alike. The sub-q-configuration of the

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other ends of the q like edges is either a j, k q-tuple or there is a like triangle containing the vertex. Therefore for $q > 3$, we have $V(3, q - x, x; 2) \leqslant (3q - 1)$.

Consider a 9-configuration, consisting of the eight vertices of a cube, and a ninth vertex. Let three vertices of the top face of the cube be coloured r, representing three edges coloured r from the ninth vertex. Similarly let three vertices of the bottom face be coloured g and the two remaining vertices v. Some edges and diagonals of the cube are then coloured in such a way as to avoid like triangles at the ninth vertex. Then passing to an octagonal configuration, it may be seen that it is possible to complete the colouring of the edges in such a way that all 4-tuples of the 8-configuration are r, g, v 4-tuples. This establishes that $V(3, 4 - x, x; 2) \geqslant 10$. (An alternative construction of a non-degenerate 9-configuration uses two g, v, r squares with a common vertex, and two additional vertices.)

As to comparison of $V(p, q - x, x; 2)$ with the Ramsey number $N(p, q; 2)$, the first would be expected to be larger than the second for the reason that it involves a requirement at each vertex, but also it would be expected to be smaller for the reason that the requirement of a like p-tuple or a j, k q-tuple is less stringent than the requirement of a definitely red p-tuple or a definitely blue q-tuple. The first reason predominates for $p = 3$, $q = 4$: $V(3, 4 - x, x; 2) > N(3, 4; 2)$. However $V(3, 5 - x, x; 2) \leqslant 14 = N(3, 5; 2)$, $V(3, 6 - x, x; 2) \leqslant 17 < N(3, 6; 2) = 18$, $V(3, 7 - x, x; 2) \leqslant 20 < N(3, 7; 2) = 23$. Apparently the second reason predominates also for the case $p = q = 4$.

THEOREM 5.2. *We have that* $11 \leqslant V(4, 4 - x, x; 2) \leqslant 18 = N(4, 4; 2)$.

Proof. Given any 18-configuration, by Theorem 4.7 each 17-subconfiguration contains a j, k 4-tuple. Thus the subconfigurations obtained by omitting each vertex in turn all contain j, k 4-tuples, which implies that the 18-configuration is degenerate with respect to $V(4, 4 - x, x; 2)$. Therefore $V(4, 4 - x, x; 2) \leqslant 18$.

To the 9-configuration, in the proof of Theorem 4.7, which is non-degenerate with respect to $M(4, 4 - x, x; 2)$, add a 10th vertex v. There is then no j, k 4-tuple not containing v. Colour edges $v1$, $v7$, $v8$ red; edges $v2$, $v4$, $v6$ green; and the remaining edges, $v3$, $v5$, $v9$, violet. With this colouration, there is no like 4-tuple containing the vertex v. (There is not even a like triangle containing v.) Therefore the 10-configuration is non-degenerate with respect to $V(4, 4 - x, x; 2)$, and it is established that $11 \leqslant V(4, 4 - x, x; 2)$.

6. Proof that $N(3, 6; 2) < 19$**.** Besides Lemma 4.4, we require two similar lemmas concerning classification of 13- and 12-configurations which are non-degenerate with respect to $N(3, 5; 2)$.

LEMMA 6.1. *A* 13-*configuration, non-degenerate with respect to* $N(3, 5; 2)$, *must be exactly* 4-*red at each vertex.*

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Proof. If there are nine blue edges issuing from a vertex, then since $N(3, 4; 2) = 9$, there is either a red triangle in the configuration of the nine other ends, or a blue 4-tuple, and thus in the 13-configuration, a blue 5-tuple which includes the vertex. Therefore the number of red edges from the vertex must be greater than three. If there are five red edges from the vertex, then there is either a red triangle containing the vertex, or the configuration of the five other ends is a blue 5-tuple.

THEOREM 6.2. *An 18-configuration, which is allegedly non-degenerate with respect to $N(3, 6; 2)$, must be exactly 5-red at each vertex.*

Proof. Since $N(3, 5; 2) = 14$, there cannot be 14 blue edges from a vertex; therefore the red number at the vertex is at least 4. There cannot be six red edges, since if so either there would be a red triangle, or the configuration of the six other ends would be a blue 6-tuple. Therefore the red number at each vertex must be either 4 or 5.

Suppose that there is a 4-red vertex. Then the remaining red edges from the four other ends issue to the configuration formed by the remaining 13 vertices of the 18-configuration; the original vertex is blue-connected to each of these 13 vertices. If the 13-configuration contains a blue 5-tuple, then the 18-configuration contains a blue 6-tuple involving the original vertex. Since the red number of each vertex is 4 or 5, there must be at least 12 remaining red edges from the four other ends, and by Lemma 6.1, there cannot be more than 13 such edges. In the former case, by the lemma there must be twelve 5-red vertices in the 13-configuration; and in the latter case, all 13 vertices of the 13-configuration, and one of the four other ends, must be 5-red. Thus, for non-degeneracy, any 4-red vertex must be blue-connected to at least twelve 5-red vertices.

Consider one of the (three or four) 4-red vertices among the four other ends of the red edges from the original 4-red vertex. It is blue-connected to the remaining three of the four other ends; therefore there are only 10 blue edges from it to the 13-configuration, and it is joined by blue edges to at most eleven 5-red vertices. But, as just shown, 12 are necessary for non-degeneracy. Therefore, as required for the theorem, all vertices of the allegedly non-degenerate 18-configuration necessarily are 5-red.

LEMMA 6.3. *A 12-configuration, non-degenerate with respect to $N(3, 5; 2)$, must be 3-red or 4-red at each vertex. Any non-degenerate 12-configuration in which four vertices are 4-red, and the remaining eight 3-red, necessarily is of the following form: the red sub-configuration of the 3-red vertices $1, \ldots, 8$ is an 8-cycle; the red sub-configuration of the 4-red vertices α, β, γ, δ is a 4-cycle; α is red-connected to $1, 4$; β to $3, 6$; γ to $5, 8$; and δ to $2, 7$.* (This is case (1) below.)

Proof. If the red number at a vertex is less than 3, then there are nine blue edges from the vertex; since $N(3, 4; 2) = 9$, the 12-configuration contains either a red triangle or a blue 5-tuple. If there are five red edges from a vertex, the same is true. Therefore the red number of each vertex must be 3 or 4.

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Next consider the case that a 3-red vertex is red-connected to at least two 4-red vertices. Then in the 8-configuration of the other ends of the blue edges from the 3-red vertex, there are one or two 4-red vertices. If there is one, then there are nine red edges into the 8-configuration, issuing from the other ends of the red edges from the original 3-red vertex. By Lemma 4.4, two of the nine must go to the 4-red vertex, and one to each of the seven 3-red vertices. But then within the 8-configuration, all edges are 2-red, so by Lemma 4.4 that configuration contains a blue 4-tuple. This 4-tuple, and the original 3-red vertex, then form a blue 5-tuple in the 12-configuration. Similarly, if there are two 4-red vertices in the 8-configuration, then within that configuration, only two of the vertices are 3-red, and again by Lemma 4.4, there is a blue 5-tuple in the 12-configuration.

The remaining case is that each 3-red vertex is red-connected to at most one 4-red vertex. Assuming that there is no red triangle, there must be at least eight red edges issuing from the 4-sub-configuration of the four 4-red vertices. If one 3-red vertex were red-connected to no 4-red vertex, then one of the remaining seven would be red-connected to two 4-red vertices, contrary to hypothesis. Thus each of the eight 3-red vertices is red-connected to exactly one 4-red vertex, and the four 4-red vertices are vertices of a red 4-cycle. Within the 8-sub-configuration of the 3-red vertices, each vertex is 2-red. The only two possibilities in the 8-sub-configuration are: (1) there is an 8-cycle of red edges; (2) there are two disjoint 4-cycles of red edges.

Denote the 3-red vertices by the numerals $1, \ldots, 8$, and the 4-red vertices by the letters $\alpha, \ldots, \delta$. In subcase (1), the configuration is non-degenerate if and only if it is as described in the statement of the lemma. Under subcase (2), if the two red 4-cycles are 13571 and 24682, it may be verified that regardless of how the remaining eight red edges connect $1, \ldots, 8$ to $\alpha, \ldots, \delta$, if there is no red triangle, then we have that there is a blue 5-tuple, of a form like $\alpha\gamma 145$, $\beta\delta 145$, $\alpha\gamma 348$, in the 12-configuration. All cases are covered, and thus the second statement of Lemma 6.3 is established.

THEOREM 6.4. *The Ramsey number* $N(3, 6; 2)$ *is* 18.

Proof. In §4 it was proved that necessarily $N(3, 6; 2) \geqslant 18$. By Theorem 6.2, for a non-degenerate 18-configuration, it is necessary that the red number of each vertex be 5. Consider the 12-sub-configuration of the 12 vertices which are blue-connected to a vertex 0. There are 20 red edges to the 12-sub-configuration, from the five other ends $1', \ldots, 5'$ of the five red edges which issue from 0. By Lemma 6.3, we see that eight vertices, say $1, \ldots, 8$, within the 12-configuration must be 3-red (sending two red edges each to the set $1', \ldots, 5'$), and that the remaining four vertices, say $\alpha, \beta, \gamma, \delta$, must be 4-red (sending one red edge each to the set $1', \ldots, 5'$). Also by Lemma 6.3, unless the 12-configuration is the exceptional case (1), if there is no red triangle, then the 12-configuration must contain a blue 5-tuple. Thus the 6-tuple, which is formed by the original vertex and the 5-tuple of the 12-configuration, is blue. Therefore

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unless the blue-connected 12-configuration for *each* vertex of the 18-configuration is the exceptional case (1), necessarily the 18-configuration is degenerate.

The form of case (1) of the 12-configuration, with vertices labelled as above, dictates, to within symmetries, that the red-connected edges are the 8-cycle $1, \ldots, 8$, the 4-cycle $\alpha, \ldots, \delta$, and $\alpha 1$, $\alpha 6$, $\beta 3$, $\beta 8$, $\gamma 2$, $\gamma 5$, $\delta 4$, $\delta 7$. Also $\alpha 1'$, $\beta 2'$, $\gamma 3'$, $\delta 4'$; $15'$, $35'$, $55'$, $75'$, are red. Now on interchange of $5'$ and 0, by the remarks above, $5'$ must play the same role as 0; 1, 3, 5, 7, 0 the same roles as $1'$, $2'$, $3'$, $4'$, $5'$, respectively; and $1'$, $2'$, $3'$, $4'$, 2, 4, 6, 8, α, β, γ, δ must be the vertices of a 12-configuration of the type under case (1). Enumeration, taking advantage of symmetries, is not difficult, and shows that there is no red 8-cycle among the 12 vertices last named. But for case (1), a red 8-cycle is required. Therefore by Lemma 6.3, the 18-configuration must be degenerate, and Theorem 6.4 is proved.

The following lemma was superfluous for the present purpose of establishing that $N(3, 6; 2) < 19$, but seems to be of interest, and therefore it is included here. It should be helpful for the problem of determining and classifying all possible configurations which are non-degenerate with respect to $N(3, 6; 2)$.

LEMMA 6.5. *For $n \geqslant 16$, any n-configuration in which there is a vertex which is red-connected to three 3-red vertices necessarily is degenerate with respect to $N(3, 6; 2)$.*

Proof. By hypothesis, if there is no red triangle, then the triangle formed by the three 3-red vertices is blue. The three 3-red vertices are red-connected to at most six vertices besides the given vertex (which is red-connected to the three 3-red vertices). Since $n \geqslant 16$, there are six further vertices which are entirely blue-connected to the blue triangle formed by the three 3-red vertices. Since $N(3, 3; 2) = 6$ and since there is no red triangle, the 6-configuration formed by the six further vertices contains a blue triangle. The two entirely blue-connected triangles constitute a blue 6-tuple, and the lemma is proved. Of course, if the three 3-red vertices are red-connected to k vertices, with $k < 6$, then the lower bound 16 for n may be reduced to $10 + k$.

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