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k–GON PARTITIONS

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The object of this note is to present a brief derivation of the main result of Andrews, Paule and Riese (2001).

We obtain a multivariable generating function associated with $t_k(n)$, the number of k–gon partitions of n, that is, those partitions of n,

$$n = a_1 + a_2 + \cdots + a_k$$

in which

$$1 \leqslant a_1 \leqslant a_2 \cdots \leqslant a_k \text{ and } a_1 + a_2 + \cdots + a_{k-1} > a_k.$$

(The case $k = 3$ gives the number of triangles with integer sides and perimeter n.) Andrews, Paule and Riese did this with MacMahon's Partition Analysis (Ω–Calculus), but we do without.

Thus, let

$$a_1 = 1 + \delta_1, \quad a_2 = 1 + \delta_1 + \delta_2, \ \ldots\ , \quad a_k = 1 + \delta_1 + \delta_2 + \cdots + \delta_k,$$

with $\delta_i \geqslant 0$. Then

$$(k-1) + (k-1)\delta_1 + (k-2)\delta_2 + \cdots + 1\delta_{k-1} > 1 + \delta_1 + \delta_2 + \cdots + \delta_k,$$

or,

$$\delta_k < (k-2) + (k-2)\delta_1 + (k-3)\delta_2 + \cdots + 1\delta_{k-2}.$$

It follows that if S denotes the set of all k–gon partitions,

$$G = \sum_S x_1^{a_1} x_2^{a_2} \cdots x_k^{a_k} = \sum x_1^{1+\delta_1} x_2^{1+\delta_1+\delta_2} \cdots x_k^{1+\delta_1+\delta_2+\cdots+\delta_k}$$

where the sum is taken over all $\{\delta_1, \delta_2, \ldots, \delta_k\}$ with $\delta_i \geqslant 0$ and $\delta_k < (k-2) + (k-2)\delta_1 + (k-3)\delta_2 + \cdots + 1\delta_{k-2}$.

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Thus, if we set

$$X_i = x_i x_{i+1} \dots x_k, \quad i = 1, \dots, k,$$

we find that

$$\begin{aligned}
G &= \sum X_1^{1+\delta_1} X_2^{\delta_2} X_3^{\delta_3} \cdots X_k^{\delta_k} \\
&= \sum X_1^{1+\delta_1} X_2^{\delta_2} \cdots X_{k-1}^{\delta_{k-1}} \left(\frac{1 - X_k^{(k-2)+(k-2)\delta_1+(k-3)\delta_2+\cdots+1\delta_{k-2}}}{1 - X_k} \right) \\
&= \frac{X_1}{1 - X_k} \sum X_1^{\delta_1} X_2^{\delta_2} \cdots X_{k-1}^{\delta_{k-1}} \\
&\qquad - \frac{X_1 X_k^{k-2}}{1 - X_k} \sum \left(X_1 X_k^{k-2}\right)^{\delta_1} \left(X_2 X_k^{k-3}\right)^{\delta_2} \cdots \left(X_{k-2} X_k\right)^{\delta_{k-2}} X_{k-1}^{\delta_{k-1}} \\
&= \frac{X_1}{(1 - X_1)(1 - X_2) \cdots (1 - X_k)} \\
&\qquad - \frac{X_1 X_k^{k-2}}{1 - X_k} \frac{1}{(1 - X_{k-1})(1 - X_{k-2} X_k)(1 - X_{k-3} X_k^2) \cdots \left(1 - X_1 X_k^{k-2}\right)},
\end{aligned}$$

which is [**1**, Theorem 1].

Of course, if we put $x_i = q$ for all i, we obtain [1, Corollary 1],

$$\sum_{n \geqslant 0} t_k(n) q^n = \frac{q^k}{(1 - q)(1 - q^2) \cdots (1 - q^k)} - \frac{q^{2k-2}}{1 - q} \frac{1}{(1 - q^2)(1 - q^4) \cdots (1 - q^{2k-2})}.$$

REFERENCES

[**1**] G.E. Andrews, P. Paule and A. Riese, MacMahon's partition analysis IX: k-gon partitions, *Bull. Austral. Math. Soc.* **64** (2001), 321–329..

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