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Complex Uniform Convexity and Riesz Measures

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Abstract. The norm on a Banach space gives rise to a subharmonic function on the complex plane for which the distributional Laplacian gives a Riesz measure. This measure is calculated explicitly here for Lebesgue L^p spaces and the von Neumann-Schatten trace ideals. Banach spaces that are q-uniformly PL-convex in the sense of Davis, Garling and Tomczak-Jaegermann are characterized in terms of the mass distribution of this measure. This gives a new proof that the trace ideals c^p are 2-uniformly PL-convex for $1 \leq p \leq 2$.

1 Riesz Measures for Banach Spaces

Let X be a complex Banach space. A function $\varphi\colon X \to [-\infty, \infty)$ is said to be *plurisubharmonic* if it is upper semicontinuous and if

$$\varphi(x) \leq \frac{1}{2\pi} \int_0^{2\pi} \varphi(x + e^{i\theta} y)\, d\theta \quad (x, y \in X). \tag{1.1}$$

When $\log \varphi$ is also plurisubharmonic, one says that φ is *log-plurisubharmonic*. If φ is plurisubharmonic, and $F\colon D \to X$ is holomorphic where $D = \{\zeta : |\zeta| < 1\}$, then $\varphi\bigl(F(\zeta)\bigr)$ is subharmonic.

For example, the norm itself is log-plurisubharmonic. Equivalently, $u(\zeta) = \log \|x + \zeta y\|$ defines a subharmonic function on $\mathbb{C}$ for each x and y in X. (To see this, one uses the Hahn-Banach theorem to show that

$$u(\zeta) = \sup\{\log\bigl|\langle x + \zeta y \mid x^* \rangle\bigr| : x^* \in \mathrm{Ball}(X^*)\}, \tag{1.2}$$

where $\log\bigl|\langle x + \zeta y \mid x^* \rangle\bigr|$, is a subharmonic function on the complex plane [20].) It follows from Jensen's inequality applied to e^{pt} that, for $0 < p < \infty$,

$$\frac{1}{2\pi} \int_0^{2\pi} \|x + e^{i\theta} y\|^p\, d\theta \geq \|x\|^p \quad (x, y \in X). \tag{1.3}$$

The extent to which this inequality is strict was used by Davis, Garling, and Tomczak-Jaegermann [12] as a measure of the convexity of the norm. They introduced the *modulus of complex convexity*:

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$$(1.4)\qquad H_p^X(t) = \inf\Big\{ \Big(\frac{1}{2\pi} \int_0^{2\pi} \|x + te^{i\theta} y\|^p \, d\theta \Big)^{1/p} - \|x\| : \|x\| = \|y\| = 1 \Big\}.$$

Such an X is *q-uniformly* PL-*convex* if the modulus of uniform PL-convexity is of power type q; that is, if there exists $\delta(p, X) > 0$ such that $H_p^X(t) \geq \delta(p, X)t^q$ for all $0 < t < 1$ where $q < \infty$. It turns out that the smallest such q is independent of the choice of p, and that necessarily $q \geq 2$ (see [12], Theorem 2.4 and Proposition 3.1). One can characterize this condition in terms of analytic martingales, possibly after introducing an equivalent norm on such an X; see [12, 13, 25]. We shall estimate $H_p^X(t)$ for various X using Riesz measures, which we shall calculate explicitly.

Let Ω be a plane domain and let u be a subharmonic function on Ω with $u \not\equiv -\infty$. Then the distributional Laplacian Δu of u is a positive Radon measure. When multiplied by $1/2\pi$ it is also called the *Riesz measure* of u. In complex notation, which will be more useful for us, $\Delta u = 4\bar{\partial}\partial u$. The Laplacian measures the strictness of the sub-mean inequality for u. This is made precise by the Poisson-Jensen formula (see *e.g.* [20], Theorem 4.5.1), of which the following is the special case that we shall need. Let u be a subharmonic function defined on an open neighbourhood of the closed disc $\bar{D}(w, r)$. Then

$$(1.5)\qquad \frac{1}{2\pi} \int_0^{2\pi} u(w + re^{i\theta}) \, d\theta - u(w) = \frac{2}{\pi} \int_{D(w,r)} \log \frac{r}{|\zeta - w|} \bar{\partial}\partial u(d\zeta).$$

Proposition 1.1 *Let X be a complex Banach space and let $x, y \in X$. Then*

$$(1.6)\qquad \frac{1}{2\pi} \int_0^{2\pi} \|x + te^{i\theta} y\|^p \, d\theta - \|x\|^p = \frac{2}{\pi} \int_{D(0,t)} \log \frac{t}{|\zeta|} \bar{\partial}\partial \|x + \zeta y\|^p(d\zeta).$$

This follows when one applies the Poisson-Jensen formula to $u(\zeta) = \|x + \zeta y\|^p$. This formula allows us to estimate the modulus of uniform PL-convexity directly from a knowledge of the Riesz measure $\bar{\partial}\partial\|x + \zeta y\|^p$. Indeed, writing $V_p(t; x, y)$ for the right-hand side of (1.6), we have for $\|x\| = \|y\| = 1$, as $t \to 0$,

$$(1.7)\qquad \Big(\frac{1}{2\pi} \int_0^{2\pi} \|x + te^{i\theta} y\|^p \, d\theta \Big)^{1/p} - \|x\| = \big(1 + V_p(t; x, y)\big)^{1/p} - 1$$

$$(1.8)\qquad = \frac{1}{p} V_p(t; x, y) + O\big(V_p(t; x, y)^2\big).$$

We have thereby proved the following result.

Corollary 1.2 *For any complex Banach space X, the following holds:*

$$(1.9)\qquad \lim_{t \to 0+} \frac{H_p^X(t)}{t^q} = \frac{1}{p} \lim_{t \to 0+} \Big(\frac{1}{t^q} \inf\{V_p(t; x, y) : \|x\| = \|y\| = 1\} \Big).$$

In particular, X is q-uniformly PL-*convex if and only if the right-hand side of* (1.9) *is strictly positive.*

The importance of this result is that, for certain interesting cases, the Laplacian $\bar{\partial}\partial\|x+\zeta y\|^p$ can be calculated, or at least estimated, quite explicitly. In section two we shall do this for the Lebesgue spaces L^p, and in section three we shall do the same for matrices in the von Neumann-Schatten ideals c^p.

In section four we shall use the basic estimates on the Riesz measures associated with the von Neumann-Schatten norms to provide a new proof that c^p has analytic Lusin cotype 2 for $1 \le p \le 2$. In [7] the authors also consider the Riesz measures associated with the norms of certain Banach spaces in the context of analytic Lusin cotype. In Section 5 we present further applications of our estimates on Laplacians of matrix functions.

Note that in general it suffices to obtain lower bounds for $\bar{\partial}\partial\|x+\zeta y\|^p$ for x and y in a dense subset of the unit sphere, since if $x_n \to x$ and $y_n \to y$ then $\bar{\partial}\partial\|x_n+\zeta y_n\|^p \to \bar{\partial}\partial\|x+\zeta y\|^p$ as measures. Certain other aspects of the theory of complex convexity become more transparent when one considers Riesz measures. For instance, the equivalence of the moduli of convexity H_p^X for $0 < p < \infty$ is now seen to be an immediate consequence of the fact that the complex Laplacians $\bar{\partial}\partial\|x+\zeta y\|^p$ $(0 < p < \infty)$ are bounded by constant multiples of each other (since they are just $\bar{\partial}\partial e^{pv(\zeta)}$, where $v(\zeta) = \log\|x+\zeta y\|$).

Not only do we always have $q \ge 2$ in the condition defining q-uniform PL-convexity, we also have the following stronger result.

Proposition 1.3 *Let x and y be vectors in a complex Banach space. Then there exists $x' \in \operatorname{span}\{x, y\}$ such that*

$$\frac{1}{2\pi}\int_0^{2\pi}\|x'+te^{i\theta}y\|\,d\theta = \|x'\| + O(t^2) \quad \text{as } t \to 0. \tag{1.10}$$

Proof Define $u(\zeta) = \|x+\zeta y\|$, and set

$$A = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})\,d\theta - u(0). \tag{1.11}$$

Let $B > A$ and set $v(\zeta) = u(\zeta) - B|\zeta|^2$. Then v does not satisfy the sub-mean inequality (1.1) on the unit circle, so it is not a subharmonic function. By Blaschke's theorem [20, Theorem 3.7], there exists w in the unit disc such that

$$\limsup_{r\to 0+}\left(\frac{\frac{1}{2\pi}\int_0^{2\pi} v(w+re^{i\theta})\,d\theta - v(w)}{r^2}\right) < 0. \tag{1.12}$$

This means that, for all sufficiently small $r > 0$,

$$\frac{1}{2\pi}\int_0^{2\pi} u(w+re^{i\theta})\,d\theta - u(w) < Br^2.$$

The result follows upon taking $x' = x + wy$.

Remark 1.4 It can happen that $\bar{\partial}\partial\|x+\zeta y\|^p$ contains a singular part. For example, if $X = \ell_2^\infty$ with $x = (1,0)$ and $y = (0,1)$, then $u(\zeta) = \|x+\zeta y\|_\infty^2 = \max\{1, |\zeta|^2\}$,

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and a calculation then shows that $\bar{\partial}\partial u = \mu + 2\nu$, where μ is Lebesgue area measure on $\{\zeta : |\zeta| > 1\}$ and ν is Lebesgue linear measure on $\{\zeta : |\zeta| = 1\}$.

Proposition 1.5 *Suppose that for some x and y in a complex Banach space X, the Riesz measure $\bar{\partial}\partial\|x + \zeta y\|$ contains a non-zero singular part ν with respect to Lebesgue area measure m. Then for ν-almost all ζ, the following holds:*

$$\lim_{r\to 0+}\left(\frac{\frac{1}{2\pi}\int_0^{2\pi}\|x + (\zeta + re^{i\theta})y\|\,d\theta - \|x + \zeta y\|}{r^2}\right) = \infty. \tag{1.13}$$

Proof For ν-almost all ζ, it holds that

$$\frac{\nu\big(D(\zeta, r)\big)}{m\big(D(\zeta, r)\big)} \to \infty \quad \text{as } r \to 0+\,. \tag{1.14}$$

We select such a ζ and use simple estimates on (1.6) to obtain the lower bound

$$\frac{1}{2\pi}\int_0^{2\pi}\|x + (\zeta + re^{i\theta})y\|\,d\theta - \|x + \zeta y\| \geq \frac{2\log 2}{\pi}\nu\big(D(\zeta, r/2)\big)\,. \tag{1.15}$$

On combining (1.14) and (1.15), we obtain the required result (1.13).

However, from the point of view of complex convexity, the singular part is unimportant, since in Corollary 1.2 we take the *infimum* over x, y. So it is the absolutely continuous part that matters. This explains the pertinence of Blaschke's Theorem which essentially says that, when $u(\zeta) = \|x + \zeta y\|$,

$$\limsup_{r\to 0+}\left(\frac{\frac{1}{2\pi}\int_0^{2\pi}u(w + re^{i\theta})\,d\theta - u(w)}{r^2}\right) \geq A \quad \text{for all } w \tag{1.16}$$

if and only if

$$\bar{\partial}\partial u \geq Am \quad \text{as measures on } D.$$

We remark that the same theorem of Blaschke was used by Haagerup to prove that if X is 2-uniformly PL-convex, then for all $x, y \in X$,

$$\frac{1}{2\pi}\int_0^{2\pi}\|x + e^{i\theta}y\|^2\,d\theta \geq \|x\|^2 + I_{2,2}\|y\|^2, \tag{1.17}$$

where $I_{2,2} = 2\lim_{t\to 0+} H_X^2(t)/t^2$; (see [12], Proposition 2.5).

2 Riesz Measures for Lebesgue Spaces

Let dx be a positive Radon measure on a metric space S, and let L^p be the usual scale of Lebesgue spaces for $1 \leq p < \infty$. A classical result of Mazur [18] asserts that the norm on L^p for $1 < p < \infty$ is Fréchet differentiable at all points other than the origin. Here we shall compute the Laplacian of the norms.

Theorem 2.1

(i) *Let f and g belong to L^p where $1 \le p \le 2$. Then the density of the Riesz measure with respect to Lebesgue area measure satisfies*

$$\|f + \zeta g\|_{L^p}^{2-p}\, \bar{\partial}\partial \|f + \zeta g\|_{L^p}^{p} \ge (p^2/4)\|g\|_{L^p}^2. \tag{2.1}$$

(ii) *The Lebesgue space L^p is 2-uniformly* PL-*convex for $1 \le p \le 2$, and p-uniformly* PL-*convex for $2 \le p < \infty$.*

Proof First suppose that f and g are step functions. Then the Riesz measure of $\|f + \zeta g\|_{L^p}^p$ has density

$$\frac{2}{\pi}\bar{\partial}\partial\|f + \zeta g\|_{L^p}^p = \frac{p^2}{2\pi}\int_S |f(x) + \zeta g(x)|^{p-2}|g(x)|^2\, dx \tag{2.2}$$

with respect to area measure on D. This follows by direct calculation of partial derivatives.

Now take $1 \le p \le 2$, and let $f, g \in L^p$ have $\|f\|_{L^p} = \|g\|_{L^p} = 1$. One can check that $\psi(\zeta) = \int_S |f(x) + \zeta g(x)|^{p-2}|g(x)|^2\, dx$ is finite for almost all ζ and defines a locally integrable function. Indeed, for φ a smooth function which is supported inside $D(0, R)$, we have

$$\iint_{D(0,R)} \varphi(\zeta)\psi(\zeta) m(d\zeta) = \int_S |g(x)|^p \int_{D(0,R)} \varphi(\zeta)\left|\frac{g(x)}{f(x) + \zeta g(x)}\right|^{2-p} m(d\zeta)\, dx$$

where we have a uniform bound on the inner integral, since

$$\iint_{D(0,R)} \left|\frac{g(x)}{f(x) + \zeta g(x)}\right|^{2-p} m(d\zeta) \le C(p, R) < \infty. \tag{2.3}$$

Let (f_n) and (g_n) be sequences of step functions such that $f_n(x) \to f(x)$ and $g_n(x) \to g(x)$ almost everywhere and in L^p as $n \to \infty$. Then by Fatou's lemma we have the inequality

$$\liminf_{n\to\infty} \int_S |f_n(x) + \zeta g_n(x)|^{p-2}|g_n(x)|^2\, dx \ge \int_S |f(x) + \zeta g(x)|^{p-2}|g(x)|^2\, dx$$

on the set $\{\zeta : \psi(\zeta) < \infty\}$. Now by Hölder's inequality we have

$$\Big(\int_S |f(x) + \zeta g(x)|^{p-2}|g(x)|^2\, dx\Big)^{p/2}\Big(\int_S |f(x) + \zeta g(x)|^p\, dx\Big)^{(2-p)/2} \ge \int_S |g(x)|^p\, dx, \tag{2.4}$$

and likewise for the f_n and g_n. Now by the triangle inequality, for f and g on the unit sphere of $L^p(S)$ and $\zeta \in D$, we have

$$\Big(\int_S |f(x) + \zeta g(x)|^p\, dx\Big)^{1/p} \le 2. \tag{2.5}$$

and similarly for the f_n and g_n. Hence we have the distributional inequality of measures

$$\frac{2}{\pi}\bar\partial\partial\|f+\zeta g\|^p_{L^p}(d\zeta)\geq\frac{p^2}{2\pi}2^{p-2}m(d\zeta), \tag{2.6}$$

since a similar inequality holds for $f_n+\zeta g_n$, and consequently

$$\begin{aligned} V(t)&=\frac{2}{\pi}\int_{D(0,t)}\log\frac{t}{|w|}\bar\partial\partial\|f+wg\|^p_{L^p}(dw)\\ &\geq\frac{c_p}{2\pi}\int_0^t\int_0^{2\pi}r\log\frac{t}{r}\,dr\,d\theta=\frac{c_p}{4}t^2. \end{aligned} \tag{2.7}$$

Thus the criterion of Corollary 1.2 is satisfied.

For $2<p<\infty$ we argue differently, observing that

$$w\mapsto\int_0^{2\pi}|w+re^{i\theta}g(x)|^{p-2}\,d\theta/(2\pi)$$

is subharmonic and hence satisfies

$$\int_0^{2\pi}|f(x)+re^{i\theta}g(x)|^{p-2}\frac{d\theta}{2\pi}=\int_0^{2\pi}|f(x)e^{i\theta}+rg(x)|^{p-2}\frac{d\theta}{2\pi}\geq r^{p-2}|g(x)|^{p-2}. \tag{2.8}$$

Hence we can satisfy the condition of Corollary 1.2 by using (2.8) and Fubini's Theorem:

$$\begin{aligned} V(t)&=\frac{2}{\pi}\int_{D(0,t)}\log\frac{t}{|w|}\bar\partial\partial\|f+wg\|^p_{L^p}(dw)\\ &\geq p^2\int_S|g(x)|^2\int_0^t\log\frac{t}{r}\int_0^{2\pi}|f(x)+re^{i\theta}g(x)|^{p-2}\frac{d\theta}{2\pi}r\,dr\,dx\\ &\geq p^2\int_S|g(x)|^p\int_0^t r^{p-1}\log\frac{t}{r}\,dr\,dx\\ &\geq t^p\int_S|g(x)|^p\,dx. \end{aligned} \tag{2.9}$$

Remark 2.2 When $0<p<1$, the Lebesgue space L^p is a quasi-Banach space for the quasi-norm $\|f\|_{L^p}=(\int|f(x)|^p\,dx)^{1/p}$. Theorem 2.1 extends without change to this context.

The following result illustrates the possible properties that the density of a Riesz measure can have.

Theorem 2.3

(i) *For $2\leq p<\infty$, the function $L^p\times L^p\to[0,\infty)$ defined by $(f,g)\mapsto\bar\partial\partial\|f+\zeta g\|^p_{L^p}$ is continuous.*

(ii) *For $2 \leq p < \infty$, the function $L^p \times L^p \to [0, \infty)$ defined by $(f, g) \mapsto \bar{\partial}\partial \| f + \zeta g\|^p_{L^p}$ is plurisubharmonic.*

(iii) *For $1 \leq p < 2$, the function $L^p \to [0, \infty)$ defined by $f \mapsto \bar{\partial}\partial \| f + \zeta g\|^p_{L^p}$ is not necessarily subharmonic.*

Proof (i) This follows from (2.2) when one applies Hölder's inequality in the obvious way.

(ii) Let f, f_1, g and g_1 belong to L^p. We need to show that

$$\int_S |f(x) + \eta f_1(x) + \zeta g(x) + \zeta\eta g_1(x)|^{p-2}|g(x) + \eta g_1(x)|^2 \, dx \tag{2.10}$$

is subharmonic in η. The product of subharmonic functions need not be subharmonic, but the product of log-subharmonic functions is verily subharmonic; so let us consider this stronger condition. First, $g(x)+\eta g_1(x)$ and $f(x)+\eta f_1(x)+\zeta g(x)+\zeta\eta g_1(x)$ are holomorphic in η, and, as $p - 2 \geq 0$, we deduce that

$$(p - 2)\log|f(x) + \eta f_1(x) + \zeta g(x) + \zeta\eta g_1(x)| + 2\log|g(x) + \eta g_1(x)| \tag{2.11}$$

is subharmonic. It follows by Jensen's inequality that the integrand of (2.10) is subharmonic; hence the whole integral is subharmonic, moreover log-subharmonic by Radó's criterion [21, Theorem 2.6.5; 15, Corollary 2.6.9].

(iii) When $p - 2 < 0$ and g is a continuous function of compact support, the function

$$\psi(a) = \int_S |a + \zeta g(x)|^{p-2}|g(x)|^2 \, dx \quad (a \in \mathbb{C}) \tag{2.12}$$

is positive and decays to zero as $|a| \to \infty$. By the maximum principle, ψ cannot be subharmonic on $\mathbb{C}$; see [21].

Hilbert Space For $f, g \in L^2$, the norm satisfies $\bar{\partial}\partial\|f + \zeta g\|^2_{L^2} = \|g\|^2_{L^2}$. One can even calculate the Laplacian of $\log\|f + \zeta g\|_{L^2}$, and this has been extensively studied in the theory of several complex variables [16]. The following result shows that the Riesz measure can distinguish between one- and two-dimensional subspaces of Hilbert space.

Proposition 2.4 *Let x and y belong to the unit sphere of Hilbert space. Then*

$$\int_0^{2\pi} \log\|x + te^{i\phi}y\| \frac{d\phi}{2\pi} = 2^{-1}t^2 \sin^2\Theta(x, y) + O(t^3) \quad (t \to 0) \tag{2.13}$$

where $\Theta(x, y)$ is the angle between x and y.

Proof One expands $2^{-1}\log\|x + te^{i\theta}y\|^2$ using Maclaurin's theorem.

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3 Riesz Measures for Trace Ideals

For $1 \leq p < \infty$, let c^p denote the von Neumann-Schatten ideal of compact operators A on separable complex Hilbert space for which the singular numbers, listed according to multiplicity, form a sequence in ℓ^p; or equivalently, the (quasi)-norm $\|A\|_{c^p} = (\operatorname{trace}\{(A^*A)^{p/2}\})^{1/p}$ is finite.

Dixmier showed that for $1 < p < \infty$ the space c^p is uniformly convex. Carlen and Lieb [9] used an optimal uniform convexity inequality to show that the norm on c^p $(1 < p < \infty)$ is continuously differentiable in Fréchet's sense at all points other than the origin. While c^1 is not uniformly convex, it is 2-uniformly PL-convex, as was shown by Haagerup [12]. We present below a new proof of this result which places it within the general framework of the theory of operator convexity. We shall use without further comment the non-commutative Hölder inequality [22, Theorem 2.8] to show that various expressions involving the trace are well defined.

3a p Even

It is much simpler to compute the Riesz measure when p is an even integer, and we begin with this case.

Proposition 3.1 *For $p = 2, 4$ and 6, the operator ideal c^p is p-uniformly* PL*-convex. Moreover*

(3.1)
$$V_p(t; A, B) \geq \frac{t^2}{4} \operatorname{trace}\{(A^*A)^{(p-2)/2}B^*B\} + \frac{t^p}{p^2} \operatorname{trace}\{(B^*B)^{p/2}\} \quad (0 < t < 1).$$

When proving such lower bounds, our calculations will often exploit the following simple lemma, the proof of which is left to the reader.

Lemma 3.2 *Suppose that X, Y and C are matrices with $X, Y \geq 0$.*
Then $\operatorname{trace}(CXC^*Y) \geq 0$.

Proof of Proposition 3.1 Let $p = 2k$, and let A and B belong to c^p. We proceed to differentiate $\|A + \zeta B\|_{c^p}^p$, using Leibniz' rule to obtain

$$\begin{aligned}\bar{\partial} \operatorname{trace}\{[(A+\zeta B)^*(A+\zeta B)]^k\} \\ = \sum_{j=0}^{k-1} \operatorname{trace}\{[(A+\zeta B)^*(A+\zeta B)]^j B^*(A+\zeta B)[(A+\zeta B)^*(A+\zeta B)]^{k-j-1}\},\end{aligned}$$

which simplifies by the cyclical property of the trace to

$$k \operatorname{trace}\{B^*(A+\zeta B)[(A+\zeta B)^*(A+\zeta B)]^{k-1}\}. \tag{3.2}$$

We differentiate once more to obtain

$$\tag{3.3} \bar{\partial}\partial \operatorname{trace}\{[(A+\zeta B)^*(A+\zeta B)]^k\} = k\sum_{j=0}^{k-2} \operatorname{trace}\Big\{B^*(A+\zeta B)[(A+\zeta B)^*(A+\zeta B)]^j \times (A+\zeta B)^* B[(A+\zeta B)^*(A+\zeta B)]^{k-j-2}\Big\} + k \operatorname{trace}\{B^*B[(A+\zeta B)^*(A+\zeta B)]^{k-1}\}.$$

Each summand here is non-negative, as one can see by introducing

$$C_j = [(A+\zeta B)^*(A+\zeta B)]^{j/2}(A+\zeta B)^* B[(A+\zeta B)^*(A+\zeta B)]^{(k-j-2)/2}$$

and observing, after re-arranging terms, that the j-th summand is simply $\operatorname{trace}\{C_j^* C_j\}$. The final summand is likewise positive, and we shall bound it below using the observation that for $k = 1, 2, 3$, the operator function $X \mapsto X^{k-1}$ is convex on $\{X \in B(H) : X \geq 0\}$. On account of this, we can set $\zeta = re^{i\theta}$ in the final term of (3.3) and integrate to get

$$\tag{3.4} \begin{aligned} &\int_0^{2\pi} \operatorname{trace}\{(B^*B)^{1/2}[(A+re^{i\theta}B)^*(A+re^{i\theta}B)]^{k-1}(B^*B)^{1/2}\}\frac{d\theta}{2\pi} \\ &\geq \operatorname{trace}\Big\{(B^*B)^{1/2}\Big[\int_0^{2\pi}(A^*A + re^{i\theta}B^*A + re^{-i\theta}A^*B + r^2B^*B)\frac{d\theta}{2\pi}\Big]^{k-1}(B^*B)^{1/2}\Big\} \\ &= \operatorname{trace}\{(B^*B)^{1/2}[A^*A + r^2B^*B]^{k-1}(B^*B)^{1/2}\}. \end{aligned}$$

Lemma 3.3 *For $k = 1, 2, 3$, (and 4, 5), the polynomial in r*

$$\tag{3.5} \operatorname{trace}\{(A^*A + r^2B^*B)^{k-1}B^*B\}$$

has non-negative coefficients and hence is

$$\tag{3.6} \geq \operatorname{trace}\{(A^*A)^{k-1}B^*B\} + r^{2k-2}\operatorname{trace}\{(B^*B)^k\}.$$

The proof of the lemma consists of multiplying out the expression and inspecting the terms, which one shows to be non-negative by using the cyclical property of the trace.

Resuming the proof of the Proposition, we deduce from the lemma and (3.4) that

$$\tag{3.7} V_p(t; A, B) \geq \int_0^t \log\frac{t}{r}\Big(\operatorname{trace}\{(A^*A)^{k-1}B^*B\} + r^{2k-2}\operatorname{trace}\{(B^*B)^k\}\Big)\, r\, dr$$

$$\tag{3.8} = \frac{t^2}{4}\operatorname{trace}\{(A^*A)^{k-1}B^*B\} + \frac{t^{2k}}{(2k)^2}\operatorname{trace}\{(B^*B)^k\}.$$

By Corollary 1.2, this inequality implies that c^p is p-uniformly PL-convex.

3b The Riesz Measure for c^p with p Not an Even Integer, $p > 2$

We do not know if there is an exact analogue of the formula (2.2) for the spaces c^p, but in this section we obtain some closely related inequalities which imply complex uniform convexity for certain values of p.

It is convenient to assume that A and B are finite invertible matrices and set $X(\zeta) = (A+\zeta B)^*(A+\zeta B)$.

Theorem 3.4

(i) *Let $p \geq 2$. Then the density of the Riesz measure satisfies*

$$\bar\partial\partial \|A+\zeta B\|_{c^p}^p \geq (p/2)\operatorname{trace}\{BB^*[(A+\zeta B)(A+\zeta B)^*]^{(p/2)-1}\}. \tag{3.9}$$

(ii) *For $4 \leq p \leq 6$, the space c^p is p-uniformly* PL*-convex.*

Let us write $p/2 = n+\beta$, where $n \geq 1$ is an integer and $0 < \beta < 1$. By a formula familiar from the theory of the Gamma function [14, 6.6.57, p. 553] and the spectral theorem, we have a Bochner-Lebesgue integral

$$X^{p/2} = \frac{\sin\beta\pi}{\pi}\int_0^\infty u^{\beta-1}\frac{X^{n+1}}{u+X}\,du, \tag{3.10}$$

for which the integrand is a rational function of X. We shall calculate its partial derivatives using the following lemma.

Lemma 3.5 *Let $F\colon D \to M_n$ be holomorphic and let $X(\zeta) = F(\zeta)^*F(\zeta)$. Then the following identity holds wherever X is invertible:*

$$\bar\partial\partial X = (\bar\partial X)(u+X)^{-1}(\partial X) + u(\bar\partial X)X^{-1}(u+X)^{-1}(\partial X) \quad (u > 0). \tag{3.11}$$

Proof of Theorem 3.4(i) The cyclical property of the trace gives

$$\begin{aligned}\partial\operatorname{trace}\{X^{n+1}(u+X)^{-1}\} &= (n+1)\operatorname{trace}\{(\partial X)X^n(u+X)^{-1}\}\\ &\quad - \operatorname{trace}\{(\partial X)X^{n+1}(u+X)^{-2}\},\end{aligned} \tag{3.12}$$

which reduces to

$$\operatorname{trace}\{n(\partial X)X^n(u+X)^{-1}\} + \operatorname{trace}\{(\partial X)X^n u(u+X)^{-2}\}. \tag{3.13}$$

We differentiate again to get

(3.14)
$$\begin{aligned}\bar\partial\partial\operatorname{trace}\{X^{n+1}(u+X)^{-1}\} = \operatorname{trace}\Big\{&(\bar\partial\partial X)nX^n(u+X)^{-1} + (\bar\partial\partial X)uX^n(u+X)^{-2}\\ &+ (\partial X)(\bar\partial X^n)\big(n(u+X)+u\big)(u+X)^{-2}\\ &- n(\partial X)X^n(u+X)^{-1}(\bar\partial X)(u+X)^{-1}\\ &- (\partial X)X^n u(u+X)^{-1}(\bar\partial X)(u+X)^{-2}\\ &- (\partial X)X^n u(u+X)^{-2}(\bar\partial X)(u+X)^{-1}\Big\}.\end{aligned}$$

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We number the signed terms on the right-hand side by **(1)**–**(6)** in order of appearance, and combine them to show that the sum is positive. We substitute the identity of the lemma in terms **(1)** and **(2)**, and use the terms arising from the middle term in (3.11) to cancel **(4)** and **(6)** by the cyclical property of the trace, and we thereby show that

(3.15)
$$\begin{aligned}\mathbf{(1)}+\mathbf{(2)}+\mathbf{(4)}+\mathbf{(5)}+\mathbf{(6)} &= \operatorname{trace}\Big\{u(\bar\partial X)X^{-1}(u+X)^{-1}(\partial X)nX^n(u+X)^{-1}\\ &\qquad + u(\bar\partial X)X^{-1}(u+X)^{-1}(\partial X)uX^n(u+X)^{-2}\\ &\qquad - (\bar\partial X)(u+X)^{-2}(\partial X)X^n u(u+X)^{-1}\Big\}\\ &= \operatorname{trace}\Big\{u(\bar\partial X)X^{-1}(nu+nX-u)(u+X)^{-2}\\ &\qquad\qquad \times(\partial X)X^n(u+X)^{-1}\\ &\qquad + u^2(\bar\partial X)X^{-1}(u+X)^{-1}(\partial X)X^n(u+X)^{-2}\Big\},\end{aligned}$$

a term which is positive for $n \geq 1$.

This leaves us with **(3)**, which by Leibniz' rule contributes

$$\begin{aligned}\operatorname{trace}\Big\{&(\partial X)(\bar\partial X)X^{n-1}\big(n(u+X)+u\big)(u+X)^{-2}\\ &+(\partial X)X(\bar\partial X^{n-1})\big(n(u+X)+u\big)(u+X)^{-2}\Big\}.\end{aligned}\tag{3.16}$$

By Lemma 3.2 and an induction argument, we see that both of these terms have positive traces, so

$$\mathbf{(3)} \geq \operatorname{trace}\Big\{(\partial X)(\bar\partial X)X^{n-1}\big(n(u+X)+u\big)(u+X)^{-2}\Big\}.\tag{3.17}$$

It follows from the spectral theorem that

$$\int_0^\infty u^{\beta-1}\big(n(u+X)+u\big)(u+X)^{-2}\,du = (n+\beta)\pi\operatorname{cosec}(\beta\pi)X^{\beta-1},\tag{3.18}$$

and hence from (3.17) we have

$$\bar\partial\partial\frac{\sin\beta\pi}{\pi}\int_0^\infty u^{\beta-1}\operatorname{trace}\Big\{\frac{X^{n+1}}{u+X}\Big\}\,du \geq (n+\beta)\operatorname{trace}\{(\partial X)(\bar\partial X)X^{n+\beta-2}\}.\tag{3.19}$$

In the original notation, $\bar\partial\partial\operatorname{trace}\{X^{p/2}\}$ equals

$$\begin{aligned}&\bar\partial\partial\operatorname{trace}\{[(A+\zeta B)^*(A+\zeta B)]^{p/2}\}\\ &\quad\geq (p/2)\operatorname{trace}\{(A+\zeta B)^*BB^*(A+\zeta B)[(A+\zeta B)^*(A+\zeta B)]^{(p/2)-2}\}.\end{aligned}\tag{3.20}$$

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We recall that for polynomials P and Q with $Q(x) = xP(x)$ and C any matrix, it holds that $CP(C^*C)C^* = Q(CC^*)$. The stated result (3.9) follows from the spectral theorem, as we can approximate $x^{(p/2)-2}$ on compact subsets of $(0, \infty)$ by polynomials and thereby re-arrange terms.

Proof of Theorem 3.4(ii) Let $\beta = (p-4)/2$, so that $0 < \beta < 1$. Having established (3.9), our proof depends upon the observation that the operator function $Y \mapsto Y^{\beta+1}$ is convex on the cone $B(H)_+ = \{Y \in B(H) : Y \geq 0\}$. This follows from the criterion of [14, Theorem 6.652(4)] since the operator function

$$Y \mapsto \frac{Y^{\beta+1} - t^{\beta+1}}{Y - t} = \frac{\sin \beta\pi}{\pi} \int_0^\infty u^{\beta-1} \left(1 - \frac{u^2}{(u+Y)(u+t)}\right) du \tag{3.21}$$

is monotone increasing on $B(H)_+$ for each $t > 0$.

Consequently, with $Y = (A + re^{i\theta}B)(A + re^{i\theta}B)^*$ we have

$$\int_0^{2\pi} \operatorname{trace}\{BB^*[(A + re^{i\theta}B)(A + re^{i\theta}B)^*]^{1+\beta}\} \frac{d\theta}{2\pi}$$

$$\geq \operatorname{trace}\{BB^*(AA^* + r^2BB^*)^{1+\beta}\} \tag{3.22}$$

$$\geq r^2 \operatorname{trace}\{(BB^*)^{1/2}(AA^* + r^2BB^*)^{\beta/2}BB^*(AA^* + r^2BB^*)^{\beta/2}(BB^*)^{1/2}\}$$

$$= r^2\|(BB^*)^{1/2}(AA^* + r^2BB^*)^{\beta/2}(BB^*)^{1/2}\|_{c^2}^2. \tag{3.23}$$

Now the operator function $Y \mapsto Y^{\beta/2}$ is monotone increasing on $B(H)_+$, so

$$(BB^*)^{1/2}(AA^* + r^2BB^*)^{\beta/2}(BB^*)^{1/2} \geq r^\beta(BB^*)^{1+(\beta/2)} \geq 0. \tag{3.24}$$

Moreover, for operators with $W \geq V \geq 0$ the trace duality gives

$$\begin{aligned} \|W\|_{c^2} &= \sup\bigl\{\operatorname{trace}\{WS\} : S \geq 0, \|S\|_{c^2} \leq 1\bigr\} \\ &\geq \sup\bigl\{\operatorname{trace}\{VS\} : S \geq 0, \|S\|_{c^2} \leq 1\bigr\} = \|V\|_{c^2} \end{aligned} \tag{3.25}$$

since $S^{1/2}WS^{1/2} \geq S^{1/2}VS^{1/2}$. Hence

$$\bigl(\text{right-hand side of (3.23)}\bigr) \geq r^{2+2\beta} \operatorname{trace}\{(BB^*)^{2+\beta}\} = r^{p-2}\|B\|_{c^p}^p. \tag{3.26}$$

Arguing as in (2.8) of Theorem 2.1, we obtain the stated result from (3.26) and Corollary 1.2.

3c The Riesz Measure for c^p With $1 \leq p < 2$

It is known that for $1 \leq p \leq 2$ the space c^p is 2-uniformly PL-convex. The existing proofs involve factorization of the operator-valued Hardy spaces $H^p c^p$, or interpolation through the scale of such spaces [4, 25, 26]. Here we obtain a more elementary proof using the Riesz measure criterion of Corollary 1.2 and a non-commutative analogue of the inequality (2.1).

Theorem 3.6

(i) *For $1 \leq p \leq 2$ the density of the Riesz measure satisfies*

$$\|A+\zeta B\|_{c^p}^{2-p}\bar{\partial}\partial\|A+\zeta B\|_{c^p}^{p} \geq \frac{p^2}{8}\|B\|_{c^p}^{2}. \tag{3.27}$$

(ii) *The von Neumann-Schatten operator ideal c^p is 2-uniformly* PL*-convex for $1 \leq p \leq 2$.*

It is convenient to assume that A and B are both finite and invertible $n\times n$ matrices. Let Ω be the set $\{\zeta \in D : \det(A+\zeta B) \neq 0\}$ on which $A+\zeta B$ is invertible. The Riesz measure of $\|A+\zeta B\|_{c^p}^{p}$ could conceivably have positive atoms on the singular set $\{\zeta : \det(A+\zeta B)=0\}$, but the lower bound that we shall use to verify the hypotheses of Corollary 1.2 will require only the absolutely continuous part of the Riesz measure with respect to area measure. This we obtain by differentiation.

Lemma 3.7 *With $X=(A+\zeta B)^*(A+\zeta B)$, the density with respect to area measure of the Riesz measure for $\|A+\zeta B\|_{c^p}^{p}$ is given by*

$$\begin{aligned}\bar{\partial}\partial&\|A+\zeta B\|_{c^p}^{p}\\ &=\frac{p\sin(p\pi/2)}{2\pi}\int_0^\infty u^{p/2}\operatorname{trace}\{(u+X)^{-1}(\bar{\partial}X)X^{-1}(u+X)^{-1}(\partial X)\}\,du.\end{aligned} \tag{3.28}$$

Proof A calculation similar to the above gives

$$\begin{aligned}\partial\operatorname{trace}\{X^{p/2}\}&=\frac{\sin(p\pi/2)}{\pi}\int_0^\infty u^{p/2}\operatorname{trace}\{(\partial X)(u+X)^{-2}\}\,du\\ &=\frac{p\sin(p\pi/2)}{2\pi}\int_0^\infty u^{(p/2)-1}\operatorname{trace}\{(\partial X)(u+X)^{-1}\}\,du,\end{aligned}$$

by integration by parts, and hence

$$\begin{aligned}\bar{\partial}\partial\operatorname{trace}\{X^{p/2}\}=\frac{p\sin(p\pi/2)}{2\pi}&\int_0^\infty u^{(p/2)-1}\operatorname{trace}\\ &\{(u+X)^{-1}\bar{\partial}\partial X-(u+X)^{-1}(\bar{\partial}X)(u+X)^{-1}(\partial X)\}\,du\end{aligned} \tag{3.29}$$

where in this case, as in Lemma 3.5,

$$\bar{\partial}\partial X-(\bar{\partial}X)(u+X)^{-1}(\partial X)=u(\bar{\partial}X)X^{-1}(u+X)^{-1}(\partial X) \tag{3.30}$$

and so we have the positive expression

$$\bar{\partial}\partial\operatorname{trace}\{X^{p/2}\}=\frac{p\sin(p\pi/2)}{2\pi}\int_0^\infty u^{p/2}\|(u+X)^{-1/2}(\bar{\partial}X)X^{-1/2}(u+X)^{-1/2}\|_{c^2}^2\,du. \tag{3.31}$$

One can easily check that (3.31) converges on Ω. This concludes the proof of the lemma.

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We pause to remark that integrals of this form have been considered in [4, 14, 17]. However, Ando [1] records that the function $(Y, L) \mapsto Y^{-1/2}L^2Y^{-1/2}$ need not be convex on the positive definite matrices, a fact which makes this integral tricky to deal with. In the proof below, we overcome these difficulties by passing to a suitably chosen orthonormal basis; a device related to arguments from [4, 9].

Proof of Theorem 3.6 Let us write $K = X^{-1/2}\partial X$ and observe that this operator function satisfies $K^*K = B^*B$ wherever it is defined. The density of the Riesz measure with respect to area measure is, by the lemma,

$$\bar\partial\partial\|A+\zeta B\|_{c^p}^p = \frac{p\sin(p\pi/2)}{2\pi}\int_0^\infty u^{p/2}\,\mathrm{trace}\{(u+X)^{-1}K^*(u+X)^{-1}K\}\,du. \tag{3.32}$$

We let $(e_j)_{j=1}^n$ be an orthonormal basis consisting of eigenvectors of the positive operator X, let $(x_j)_{j=1}^n$ be the corresponding eigenvalues and set $K_{jk} = \langle Ke_k, e_j\rangle$; all these quantities depend implicitly upon ζ, but this does not have significant consequence in what follows. In terms of these quantities the integral becomes

$$\frac{p\sin(p\pi/2)}{2\pi}\sum_{j,k=1}^n |K_{jk}|^2\int_0^\infty \frac{u^{p/2}}{(u+x_j)(u+x_k)}\,du, \tag{3.33}$$

where for $j = k$ and other instances with $x_j = x_k$ we have

$$\int_0^\infty \frac{u^{p/2}}{(u+x_j)^2}\,du = \frac{p}{2}\int_0^\infty \frac{u^{(p/2)-1}}{u+x_j}\,du = \frac{p\pi}{2}\,\mathrm{cosec}(p\pi/2)x_j^{(p/2)-1}; \tag{3.34}$$

and for all cases wherein $x_j \neq x_k$,

$$\int_0^\infty \frac{u^{p/2}}{(u+x_j)(u+x_k)}\,du = \int_0^\infty \frac{u^{p/2}}{x_j-x_k}\Big(\frac{1}{u+x_k}-\frac{1}{u+x_j}\Big)\,du \tag{3.35}$$

has a left-hand-side which defines a holomorphic function of p for $-2 < \Re p < 2$, so by analytic continuation is equal to

$$\pi\,\mathrm{cosec}(p\pi/2)\frac{x_j^{p/2}-x_k^{p/2}}{x_j-x_k}. \tag{3.36}$$

For $1 \le p \le 2$ we can use the mean value theorem to obtain x_{jk} between x_j and x_k such that

$$0 < \frac{x_j-x_k}{x_j^{p/2}-x_k^{p/2}} = \frac{2}{p}x_{jk}^{1-(p/2)} \le \frac{2}{p}(x_j^{1-(p/2)}+x_k^{1-(p/2)}). \tag{3.37}$$

We introduce $E \in c^q$ where $1/p+1/q = 1$, with $\|E\|_{c^q} = 1$, such that $\mathrm{trace}\{KE\} = \|K\|_{c^p}$ in the standard duality of von Neumann-Schatten ideals [22, p. 45]. Writing

$E_{jk} = \langle Ee_k, e_j \rangle$, we obtain from the Cauchy-Schwarz inequality a decoupling expression

$$\begin{aligned}| \operatorname{trace}\{KE\}|^2 &= \Big| \sum_{j,k=1}^{n} K_{jk}E_{kj} \Big|^2 \\ &\leq \Big(\sum_{j,k=1}^{n} |K_{jk}|^2 \frac{x_j^{p/2} - x_k^{p/2}}{x_j - x_k} \Big) \times \Big(\sum_{j,k=1}^{n} |E_{jk}|^2 \frac{x_j - x_k}{x_j^{p/2} - x_k^{p/2}} \Big)\end{aligned} \tag{3.38}$$

in which we interpret the difference quotient as the corresponding derivative for $j = k$ and $x_j = x_k$. By (3.37), the latest sum is

$$\begin{aligned}&\leq \frac{2}{p} \sum_{j,k=1}^{n} |E_{jk}|^2 (x_j^{1-(p/2)} + x_k^{1-(p/2)}) \\ &= \frac{2}{p} \sum_{j=1}^{n} \|E^* e_j\|_{\ell_n^2}^2 x_j^{1-(p/2)} + \frac{2}{p} \sum_{k=1}^{n} \|Ee_k\|_{\ell_n^2}^2 x_k^{1-(p/2)}.\end{aligned} \tag{3.39}$$

To these sums we apply Hölder's inequality, thus obtaining

$$\leq \frac{2}{p} \Big(\sum_{j=1}^{n} \|E^* e_j\|_{\ell_n^2}^q \Big)^{2/q} \Big(\sum_{j=1}^{n} x_j^{p/2} \Big)^{(2-p)/p} + \frac{2}{p} \Big(\sum_{k=1}^{n} \|Ee_k\|_{\ell_n^2}^q \Big)^{2/q} \Big(\sum_{k=1}^{n} x_j^{p/2} \Big)^{(2-p)/p}. \tag{3.40}$$

Now we can recognise from the latest sum and the identity

$$\Big(\sum_{k=1}^{n} x_k^{p/2} \Big)^{1/p} = (\operatorname{trace}\{X^{p/2}\})^{1/p} = \|A + \zeta B\|_{c^p}, \tag{3.41}$$

the first factor from (3.27). Moreover, the map $c^\nu \to \ell_n^\nu(\ell_n^2)$: $E \mapsto (Ee_k)_{k=1}^n$ is bounded with norm at most one for $\nu = 2$ and $\nu = \infty$, hence also for all $\nu = q$ with $2 \leq q \leq \infty$ by the Riesz-Thorin interpolation theorem extended to the scale of trace ideals [22, Theorem 2.10(b)]. The other terms in (3.40) may be estimated similarly. Since $\|K\|_{c^p} = \|B\|_{c^p}$, it follows from the preceding estimates that

$$\frac{4}{p} \|A + \zeta B\|_{c^p}^{2-p} \, \bar{\partial}\partial \|A + \zeta B\|_{c^p}^p \geq \frac{p}{2} \|B\|_{c^p}^2. \tag{3.42}$$

On integrating this inequality we obtain the lower bound

$$\frac{1}{p} \lim_{t \to 0+} \Big(\frac{1}{t^2} \inf\{V_p(t; A, B) : \|A\|_{c^p} = \|B\|_{c^p} = 1\} \Big) \geq \frac{p}{32} \tag{3.43}$$

as in Theorem 2.1. By Corollary 1.2, it follows from (3.43) that the space c^p is 2-uniformly PL-convex.

4 Analytic Lusin Cotype for Trace Ideals

Uniform PL-convexity is an isometric property of quasi-Banach spaces, and in the development of the theory it has been helpful to introduce related properties which are invariant under linear homeomorphism. In this section we give a new proof that c^p $(1 \leq p \leq 2)$ has the analytic Lusin cotype 2 property introduced by Xu [25, 26]. Previously known proofs have used factorization of c^1-valued holomorphic functions as products of holomorphic c^2-valued functions, in the style of Helson-Lowdenslager and Sarason; see [6].

Proposition 4.1 *Let $F\colon D \to c^p$ be a holomorphic function and $1 \leq p \leq 2$. Then there exists $C_p < \infty$ such that*

$$\int_0^{2\pi} \Big(\int_0^1 \log \frac{1}{r} \|F'(re^{i\theta})\|_{c^p}^2 \, r\, dr \Big)^{p/2} \frac{d\theta}{2\pi} \leq C_p \sup_{0 \leq r < 1} \int_0^{2\pi} \|F(re^{i\theta})\|_{c^p}^p \frac{d\theta}{2\pi}. \tag{4.1}$$

Proof For $1 \leq p \leq 2$ the Schatten trace ideal c^p is a separable dual space, and hence has the Radon-Nikodým property. It follows that if the right-hand side of (4.1) is finite, then $F(re^{i\theta})$ has limits as $r \to 1-$ for almost all θ, and $F(\zeta)$ may be interpreted for $\zeta \in D$ as the Poisson integral of the function in the Bochner-Lebesgue space $L^p([0, 2\pi]; c^p)$ that is defined by the resulting boundary values. Trigonometric polynomials with coefficients in c^p are dense in $L^p([0, 2\pi]; c^p)$. Consequently, we can assume without loss that $F(\zeta)$ is a polynomial with coefficients from c^p. A further approximation allows us to take these coefficients to be finite matrices so that $F(\zeta)$ is an invertible matrix on D, except on the finite set $\{a_j : j = 1, \dots, m\} = \{\zeta : \det F(\zeta) = 0\}$. Let $a_0 = 0$ and introduce $\Omega_\eta = D - \bigcup_{j=0}^m D(a_j, \eta)$, where $\eta > 0$ is chosen so small that the discs $D(a_j, 2\eta)$ do not overlap and are all contained in D. One can show as in Theorem 3.6 that $\|F(\zeta)\|_{c^p}^p$ is twice continuously differentiable on Ω_η with

$$\|F(\zeta)\|_{c^p}^{2-p} \, \bar\partial\partial \|F(\zeta)\|_{c^p}^p \geq (p^2/8) \|F'(\zeta)\|_{c^p}^2; \tag{4.2}$$

indeed, (3.11) holds with $X = F(\zeta)^* F(\zeta)$. Moreover, we have the differential inequality $|\nabla \|F(\zeta)\|_{c^p}^p| \leq p \|F(\zeta)\|_{c^p}^{p-1} \|F'(\zeta)\|_{c^p}$ for $\zeta \in \Omega_\eta$. Consequently we can apply Green's theorem on Ω_η, and then let $\eta \to 0+$ to deduce

$$\int_0^{2\pi} \|F(e^{i\theta})\|_{c^p}^p \frac{d\theta}{2\pi} - \|F(0)\|_{c^p}^p = \frac{2}{\pi} \iint_D \log \frac{1}{r} \bar\partial\partial \|F(re^{i\theta})\|_{c^p}^p \, r\, dr\, d\theta,$$

since $\{a_j : j \geq 0\}$ has zero area measure.

As (4.2) holds for all but finitely many radial rays, it follows that

$$\begin{aligned} &\int_0^{2\pi} \Big(\int_0^1 \log \frac{1}{r} \|F'(re^{i\theta})\|_{c^p}^2 \, r\, dr \Big)^{p/2} \frac{d\theta}{2\pi} \\ &\quad \leq \frac{8^{p/2}}{p^p} \int_0^{2\pi} \Big(\int_0^1 \log \frac{1}{r} \bar\partial\partial \|F(re^{i\theta})\|_{c^p}^p \, r\, dr \Big)^{p/2} \sup_{0 \leq r < 1} \|F(re^{i\theta})\|_{c^p}^{(2-p)p/2} \frac{d\theta}{2\pi} \end{aligned} \tag{4.3}$$

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which by Hölder's inequality is

$$\leq \frac{8^{p/2}}{p^p}\Big(\int_0^{2\pi}\int_0^1 \log\frac{1}{r}\bar\partial\partial\|F(re^{i\theta})\|_{c^p}^p\, r\,dr\frac{d\theta}{2\pi}\Big)^{p/2} \times\Big(\int_0^{2\pi}\sup_{0\leq r<1}\|F(re^{i\theta})\|_{c^p}^p\frac{d\theta}{2\pi}\Big)^{1-(p/2)}. \tag{4.4}$$

We have already calculated the first of these integrals. By the Hardy-Littlewood maximal theorem in the cases $1 < p \leq 2$, or by the maximal inequality of Bourgain[8] in the case $p = 1$, we have a bound on the second integral

$$\int_0^{2\pi}\sup_{0\leq r<1}\|F(re^{i\theta})\|_{c^p}^p\frac{d\theta}{2\pi} \leq C_p\int_0^{2\pi}\|F(e^{i\theta})\|_{c^p}^p\frac{d\theta}{2\pi}. \tag{4.5}$$

The required result follows from these bounds.

The following result was mentioned in [26]; a different proof was published in [2].

Proposition 4.2 *Let $2 \leq q < \infty$. Then there exists $C_q < \infty$ such that*

$$\int_0^{2\pi}\int_0^1 (1-r)^{q-1}\|F'(re^{i\theta})\|_{c^q}^q\, r\,dr\,d\theta \leq C_q\sup_{0<r<1}\int_0^{2\pi}\|F(re^{i\theta})\|_{c^q}^q\frac{d\theta}{2\pi} \tag{4.6}$$

for all holomorphic functions $F\colon D \to c^q$.

Proof Let $F\colon D \to B(H)$ be a bounded holomorphic function. Then by the Cauchy integral formula, one has the inequality

$$(1-r)\|F'(re^{i\theta})\|_{B(H)} \leq \sup_{\zeta\in D}\|F(\zeta)\|_{B(H)} \quad (re^{i\theta}\in D), \tag{4.7}$$

which one interprets as the limiting case $q = \infty$ of (4.6). Moreover, (4.6) holds for the Hilbert-Schmidt ideal c^2 with $C_2 = 2$ by Proposition 4.1. Pisier [19] has shown that one can interpolate by the real method in the family of c^q-valued Hardy spaces and identify, up to equivalent norms,

$$\big(H^1c^1, H^\infty B(H)\big)_{\psi,q} = H^qc^q \tag{4.8}$$

where ψ satisfies $1/q = (1-\psi)/1 + \psi/\infty$ and $0 < \psi < 1$. By the re-iteration theorem [5, Theorem 3.5.3], we deduce that $\big(H^2c^2, H^\infty B(H)\big)_{\phi,q} = H^qc^q$ when $1/q = (1-\phi)/2 + \phi/\infty$. The inequality (4.6) thus holds for all $q \in [2,\infty]$ by interpolation between the endpoint cases.

For a holomorphic function $F\colon D \to c^q$ with $2 < q < \infty$ the inequality

$$\bar\partial\partial\|F(\zeta)\|_{c^q}^q \geq (q/2)\,\mathrm{trace}\{F'(\zeta)F'(\zeta)^*[F(\zeta)F(\zeta)^*]^{(q/2)-1}\} \tag{4.9}$$

may be established as in Theorem 3.6 since (3.11) holds with $X = F(\zeta)^*F(\zeta)$. It would of interest to deduce (4.6) from this fact.

5 Subharmonicity for Matrix Functions

Let $F\colon D \to M_n$ be a holomorphic matrix function, and set $X(\zeta) = F(\zeta)^*F(\zeta)$. We let $\lambda_1(\zeta) \geq \lambda_2(\zeta) \geq \cdots \geq \lambda_n(\zeta)$ be the eigenvalues of $X(\zeta)$, listed according to multiplicity, and let $s_j(\zeta) = \lambda_j(\zeta)^{1/2}$ be the singular values of $F(\zeta)$. It is well known that $\lambda_1(\zeta) = \|F(\zeta)\|^2_{M_n}$ is a subharmonic function, whereas $\lambda_2(\zeta)$ need not be subharmonic, as examples of Aupetit [3] show. In his thesis [23], White proved that, for any convex and increasing function $h\colon [-\infty, \infty) \to [0, \infty)$ with $h(-\infty) = 0$, the sums $\sum_{j=1}^{k} h\big(\log s_j(\zeta)\big)$ are subharmonic for $k = 1, 2, \ldots, n$. For instance, trace $h\big(\log X(\zeta)\big)$ is subharmonic. In this section, we shall obtain lower bounds on the Laplacians of such expressions for a special class of h. Our main result is a generalization of Theorem 3.4.

Theorem 5.1 *Let $F\colon D \to M_n$ be a holomorphic and invertible matrix function with $\|F(\zeta)\|_{M_n} < 1$ for all $\zeta \in D$, and set $X(\zeta) = F(\zeta)^*F(\zeta)$. Suppose that g is continuous on $(-\infty, 0]$ and has derivatives of all orders on $(-\infty, 0)$ which satisfy*

$$g^{(k)}(x) \geq 0 \quad (k = 0, 1, 2, \ldots ; x < 0). \tag{5.1}$$

Then $\Phi(X) = Xg(\log X)$ has $\varphi(\zeta) = \operatorname{trace}\{\Phi(X)\}$ subharmonic on D with

$$\bar\partial\partial\varphi(\zeta) \geq \operatorname{trace}\{(X^{-1}\partial X)^*\Phi(X)(X^{-1}\partial X)\}. \tag{5.2}$$

Proof By Bernstein's Theorem [24], this g is the Laplace transform of some bounded and positive Radon measure on $[0, \infty)$, so that

$$g(u) = \int_0^\infty e^{tu}\mu(dt) \quad (u \leq 0). \tag{5.3}$$

We transform this by setting $x = e^u$ so that

$$xg(\log x) = \int_0^\infty x^{t+1}\mu(dt) \quad (0 < x < 1). \tag{5.4}$$

Since $0 < X < I$, we obtain thereby a formula which expresses $g(\log X)$ as a multiple of a convex combination of X^t for $t \geq 0$.

To verify that $\varphi(\zeta)$ is subharmonic, it suffices to show that $\operatorname{trace}\{X(\zeta)^{t+1}\}$ is subharmonic for each $t \geq 0$. This we do by differentiating various integral formulae such as

$$X^{t+1} = \frac{\sin\beta\pi}{\pi}\int_0^\infty u^{\beta-1}\frac{X^{m+1}}{u + X}\,du \tag{5.5}$$

where $m = \max\{n \in \mathbb{Z} : n < t + 1\} \geq 1$ and $\beta = t + 1 - m$ has $0 < \beta < 1$; the case in which t is itself an integer is simpler.

The calculations of Theorem 3.4 lead to the bound

$$\partial\bar\partial \operatorname{trace}\{X^{t+1}\} \geq (t + 1)\operatorname{trace}\{(\partial X)(\bar\partial X)X^{t-1}\}. \tag{5.6}$$

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On integrating this against $\mu(dt)$, we deduce from the identity $\Phi(X) = \int_0^\infty X^{t+1}\mu(dt)$ that

$$\bar\partial\partial\,\mathrm{trace}\{\Phi(X)\} \geq \int_0^\infty (t+1)\,\mathrm{trace}\{X^{-1}(\partial X)(\bar\partial X)X^{-1}X^{t+1}\}\,\mu(dt) \geq \mathrm{trace}\{X^{-1}(\partial X)(\bar\partial X)X^{-1}\Phi(X)\}, \tag{5.7}$$

as required.

We conclude this paper by considering another family of plurisubharmonic functions on Hilbert space which arises in interpolation theory. A family of norms $\|\,.\,\|_{(\zeta)}$ on $\mathbb{C}^n$, parametrized by $\zeta \in D$, is said to be *subharmonic* if $\zeta \mapsto \|f(\zeta)\|_{(\zeta)}$ is a subharmonic function for each holomorphic function $f\colon D \to \mathbb{C}^n$. The norms are Hilbertian when there exists a positive definite matrix $\Omega(\zeta)$ such that $\langle\Omega(\zeta)\xi,\xi\rangle_{\mathbb{C}^n} = \|\xi\|^2_{(\zeta)}$ for each $\xi \in \mathbb{C}^n$ and $\zeta \in D$. Coifman and Semmes [10] show that, when Ω is twice continuously differentiable, then $\langle\Omega(\zeta)\xi,\xi\rangle^{1/2}_{\mathbb{C}^n}$ is a subharmonic family of norms if and only if

$$\bar\partial\partial\Omega - (\bar\partial\Omega)\Omega^{-1}(\partial\Omega) \geq 0 \quad (\zeta \in D) \tag{5.8}$$

holds as an operator inequality. The curvature of the family is

$$K(w,\xi) = \frac{-1}{\langle\Omega\xi,\xi\rangle}\Big\langle \big(\bar\partial\partial\Omega - (\bar\partial\Omega)\Omega^{-1}(\partial\Omega)\big)\,\xi,\xi\Big\rangle \tag{5.9}$$

where Ω and its partial derivatives are evaluated at $w \in D$. One can check that the curvature is characterized by the variational formula

$$K(w,\xi) = \sup\{-\bar\partial\partial\log\|f(\zeta)\|^2_{(\zeta)}|_{\zeta=w} : f \text{ holomorphic}, f(w) = \xi\}. \tag{5.10}$$

Hence a subharmonic norm family has $K(w,\xi) \leq 0$ and $\bar\partial\partial\log\|f(\zeta)\|^2_{(\zeta)}|_{\zeta=w} \geq -K(w,\xi)\|\xi\|^2_{\mathbb{C}^n}$ for any holomorphic function $f\colon D \to \mathbb{C}^n$ with $f(w) = \xi$. Kobayashi has extended these ideas to the context of norm families on Banach spaces. The following result gives a new method for constructing subharmonic Hilbertian families of norms.

Let us recall that a Borel real function g is *strictly operator monotone* on (a,b) if

$$X < Y \Rightarrow g(X) < g(Y) \tag{5.11}$$

for all self-adjoint and bounded linear operators X and Y with spectra contained in (a,b). Such a g is a function of positive type; see [14, 6.6.40]. In particular, g has derivatives of all orders on (a,b) which satisfy $g^{(k)}(x) > 0$ for all $x \in (a,b)$, as in Theorem 5.1.

Theorem 5.2 *Let g be strictly operator monotone on $(-\infty,1)$, and suppose that $F\colon D \to M_n$ is a bounded holomorphic function with $\sup_{\zeta\in D}\|F(\zeta)\|_{M_n} < 1$ such that $F(\zeta)^{-1}$ exists for each $\zeta \in D$. Then*

$$\Omega(\zeta) = g\big(F^*(\zeta)F(\zeta)\big) - g(0) \quad (\zeta \in D) \tag{5.12}$$

gives a subharmonic family of Hilbertian norms $\langle\Omega(\zeta)\xi,\xi\rangle^{1/2}$.

Proof By Loewner's Theorem [11, p. 86], such a g may be represented as

$$g(x) = \alpha x + \beta + \int_1^\infty \Big(\frac{1}{u-x} - \frac{u}{u^2+1} \Big) \mu(du), \tag{5.13}$$

where $\alpha \geq 0$ and $\beta \in \mathbb{R}$ are constants, and μ is a positive Radon measure on $[1, \infty)$ for which $\int_1^\infty (1+u^2)^{-1} \mu(du) < \infty$. Consequently we have an integral representation

$$\Omega = \alpha F^* F + \int_1^\infty \big((u - F^* F)^{-1} - u^{-1} \big) \mu(du). \tag{5.14}$$

The limit of an increasing sequence of subharmonic functions is again subharmonic; so it is easy to approximate this integral by finite Riemann sums, and hence it suffices to check the subharmonicity conditions for such finite sums.

The function $S_\infty = F^* F$ satisfies (5.8) with equality, while one can check that $S_u = (u - F^*F)^{-1} - u^{-1}$ satisfies

$$\bar\partial \partial S_u - (\bar\partial S_u) S_u^{-1} (\partial S_u) = (u - F^*F)^{-1} F^* \partial F (u - F^*F)^{-1} (\partial F)^* F (u - F^* F)^{-1} \geq 0 \tag{5.15}$$

since $u - F^*F > 0$ for $u \geq 1$. Hence the norms associated with the positive definite matrix functions S_∞ and S_u $(u \geq 1)$ form subharmonic families, and we shall check that the norm family associated with Ω is likewise subharmonic.

Radó [21] showed that a function $w \colon D \to (0, \infty)$ has $\log w(\zeta)$ subharmonic, if and only if $|f(\zeta)| w(\zeta)$ is subharmonic for all scalar-valued holomorphic functions f. Using this characterization, one checks that $\|\xi\|_{(\zeta)} = (\|\xi\|^2_{1,(\zeta)} + \|\xi\|^2_{2,(\zeta)})^{1/2}$ is a subharmonic family of Hilbertian norms whenever $\|\xi\|_{j,(\zeta)}$ $(\zeta \in D)$ are subharmonic families of Hilbertian norms for $j = 1, 2$.

We deduce that $\Omega = \alpha S_\infty + \int_1^\infty S_u \mu(du)$ gives a subharmonic family of Hilbertian norms.

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