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A NOTE ON THE *p*-SUPERSOLUBILITY OF FINITE GROUPS

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Abstract

In this paper, we investigate the influence of certain subgroups of fixed prime power order on the p-supersolubility of finite groups.

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1. Introduction

Throughout this paper, all groups are assumed to be finite. The terminology and notation employed agree with standard usage, as in Doerk and Hawkes [4]. Throughout, G always denotes a finite group, p denotes a prime and $Z_{\mathfrak{U}}(G)$ is the $\mathfrak{U}$-hypercentre of G, that is, the product of all normal subgroups H of G such that all G-chief factors of H are cyclic.

Recall that a subgroup H of G is said to be S-permutable [5] in G if H permutes with all the Sylow subgroups of G. This notion is useful in establishing results concerning the group structure. The study of the generalisations of S-permutability has become a fruitful research area. For example, Ballester-Bolinches and Pedraza-Aguilera [3] defined H to be S-permutably embedded in G if every Sylow subgroup of H is also a Sylow subgroup of some S-permutable subgroup of G, and Skiba [11] called H weakly S-permutable in G if there is a subnormal subgroup T of G such that $G = HT$ and $H \cap T \leq H_{sG}$, where H_{sG} is the subgroup of H generated by all those subgroups of H which are S-permutable in G. In order to unify the above-mentioned subgroups, Li *et al.* [8] introduced the following concept.

Definition 1.1. Let H be a subgroup of G and let H_{eG} denote the subgroup of H generated by all those subgroups of H which are S-permutably embedded in G. If there is a subnormal subgroup T of G such that $G = HT$ and $H \cap T \leq H_{eG}$, then H is called E–S-supplemented in G.

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We say that G is p-nilpotent if every p-chief factor of G is central and G is p-supersoluble if every p-chief factor of G is cyclic. In [8], the authors obtained the following result: let P be a Sylow p-subgroup of a group G, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. Suppose that there exists a subgroup D of P with $1 < |D| < |P|$ such that every subgroup H of P with order $|D|$ and every cyclic subgroup of P with order 4 (if P is a nonabelian 2-group and $|D| = 2$) is E–S-supplemented in G. Then G is p-nilpotent. From [6, Lemma 2.6], we know that p-supersolubility of G implies p-nilpotency of G whenever $(|G|, p-1) = 1$. In this paper, we remove the hypothesis $(|G|, p-1) = 1$ in the Theorem from [8] to arrive at the following main result. Our main result also extends the work of Li *et al.* [7].

THEOREM 1.2. *Let E and X be p-soluble normal subgroups of G with $F_p(E) \leq X \leq E$, where p is a prime divisor of $|E|$. Suppose that G/E is p-supersoluble, and a Sylow p-subgroup P of X has a subgroup D with $1 < |D| < |P|$ such that every subgroup H of P with order $|D|$ and every cyclic subgroup of P with order 4 (if P is a nonabelian 2-group and $|D| = 2$) is E–S-supplemented in G. Then G is p-supersoluble.*

REMARK 1.3. Generally, Theorem 1.2 does not hold if we remove the hypothesis that E is p-soluble. For example, let $G = \mathbb{Z}_5 \times A_5$, where A_5 is the alternating group of degree 5. It is not hard to see that each subgroup of G with order 5 is E–S-supplemented in G. However, G is not 5-supersoluble.

2. Preliminaries

LEMMA 2.1 [8, Lemma 2.2]. *Suppose that H is E–S-supplemented in G.*

(1) *If $H \leq L \leq G$, then H is E–S-supplemented in L.*
(2) *If $N \trianglelefteq G$ and $N \leq H \leq G$, then H/N is E–S-supplemented in G/N.*
(3) *If H is a π-subgroup and N is a normal π'-subgroup of G, then HN/N is E–S-supplemented in G/N.*

LEMMA 2.2 [8, Theorem 1.4]. *Let E be a normal subgroup of a group G and $X \leq E$. Suppose that for every noncyclic Sylow subgroup P of X, there exists a subgroup D of P with $1 < |D| < |P|$ such that every subgroup H of P with order $|D|$ and every cyclic subgroup of P with order 4 (if P is a nonabelian 2-group and $|D| = 2$) is E–S-supplemented in G. If $X = E$ or $X = F^*(E)$, then every G-chief factor of E is cyclic.*

LEMMA 2.3 [2, Lemma 2.10]. *Let p be a prime and G a group.*

(1) *$Soc(G) \leq F_p^*(G)$.*
(2) *$O_{p'}(G) \leq F_p^*(G)$. In fact, $F^*(G/O_{p'}(G)) = F_p^*(G/O_{p'}(G)) = F_p^*(G)/O_{p'}(G)$.*
(3) *If $F_p^*(G)$ is p-soluble, then $F_p^*(G) = F_p(G)$.*

LEMMA 2.4 [12, Theorem C]. *Let E be a normal subgroup of G. If every G-chief factor of $F^*(E)$ is cyclic, then every G-chief factor of E is also cyclic.*

Lemma 2.5 [7, Theorem 3.1]. *Let p be a fixed prime dividing the order of G and L a p-soluble normal subgroup of G such that G/L is p-supersoluble. If there exists a Sylow p-subgroup P of L such that every maximal subgroup of P is E–S-supplemented in G, then G is p-supersoluble.*

Combining [8, Lemma 2.2(5)] and [11, Lemma 2.11] gives the following lemma.

Lemma 2.6. *Let N be an elementary abelian normal p-subgroup of a group G. If there is a subgroup D of N with $1 < |D| < |N|$ such that every subgroup of N with order $|D|$ is E–S-supplemented in G, then there exists a maximal subgroup M of N such that M is normal in G.*

Lemma 2.7 [9, Lemma 2.3]. *Suppose that H is S-permutable in G and let P be a Sylow p-subgroup of H. If $H_G = 1$, then P is S-permutable in G.*

Lemma 2.8 [10, Lemma A]. *If P is an S-quasinormal p-subgroup of a group G for some prime p, then $N_G(P) \geq O^p(G)$.*

Lemma 2.9 [1, Theorem 2.1.6]. *If G is p-supersoluble and $O_{p'}(G) = 1$, then the Sylow p-subgroup of G is normal in G.*

Lemma 2.10 [8, Theorem 1.5]. *Let P be a Sylow p-subgroup of a group G, where p is a prime divisor of $|G|$ with $(|G|, p - 1) = 1$. Suppose that there exists a subgroup D of P with $1 < |D| < |P|$ such that every subgroup H of P with order $|D|$ and every cyclic subgroup of P with order 4 (if P is a nonabelian 2-group and $|D| = 2$) is E–S-supplemented in G. Then G is p-nilpotent.*

3. Main results

Theorem 3.1. *Let P be a Sylow p-subgroup of a p-soluble group G, where p is an odd prime divisor of $|G|$. If every cyclic subgroup of P with order p is E–S-supplemented in G, then G is p-supersoluble.*

Proof. Suppose that the theorem is false and let G be a counterexample of minimal order. Assume that $O_{p'}(G) \neq 1$. From Lemma 2.1(3), it is easy to see that every cyclic subgroup of $P/O_{p'}(G)$ with order p is E–S-supplemented in $G/O_{p'}(G)$. By the minimal choice of G, $G/O_{p'}(G)$ is p-supersoluble and so G is also p-supersoluble. This contradiction implies that $O_{p'}(G) = 1$. Since G is p-soluble, we have $O_p(G) \neq 1$. In view of Lemma 2.3, $F^*(G) = F^*_p(G) = F_p(G) = O_p(G)$. By hypothesis, every cyclic subgroup of $F^*(G)$ with order p is E–S-supplemented in G. Applying Lemma 2.2 shows that G is p-supersoluble. □

Theorem 3.2. *Let P be a Sylow p-subgroup of a p-soluble group G, where p is an odd prime divisor of $|G|$. If there exists a subgroup D of P with $1 < |D| < |P|$ such that every subgroup H of P with order $|D|$ is E–S-supplemented in G, then G is p-supersoluble.*

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Proof. Suppose that the theorem is false and let G be a counterexample of minimal order.

Step 1: $O_{p'}(G) = 1$. If $O_{p'}(G) \neq 1$, then, from Lemma 2.1(3), $G/O_{p'}(G)$ satisfies the hypotheses of the theorem. Thus, $G/O_{p'}(G)$ is p-supersoluble by the minimal choice of G. But then G is p-supersoluble, which is a contradiction.

Step 2: $|D| > p$. This follows from Theorem 3.1.

Step 3: $|P : D| > p$. This follows from Lemma 2.5.

Step 4: For any minimal normal subgroup N of G, we have $p < |N|$. Since G is p-soluble and $O_{p'}(G) = 1$, it follows that $N \leq P$. Assume that $|N| = p$ and consider the factor group G/N. By Lemma 2.1(2) and Step 2, we know that G/N satisfies the hypotheses of the theorem. Hence, G/N is p-supersoluble by the minimal choice of G. But then G is p-supersoluble, which is a contradiction.

Step 5: For any minimal normal subgroup N of G, we have $|N| \leq |D|$. This follows from Lemma 2.6.

Step 6: G has the unique minimal normal subgroup N such that G/N is p-supersoluble and $\Phi(G) = 1$. Let N be a minimal normal subgroup of G. If $|N| < |D|$, then, from Lemma 2.1(2), it is easy to see that G/N satisfies the hypotheses of the theorem. Thus, G/N is p-supersoluble by the minimal choice of G. Therefore, we may assume that $|N| = |D|$ by virtue of Step 5. Let K/N be a subgroup of P/N with order p. According to Step 2, N is noncyclic. Hence, there is a maximal subgroup L of K such that $K = LN$. Of course, $|N| = |D| = |L|$. Since L is E–S-supplemented in G, there is a subnormal subgroup T of G such that $G = LT$ and $L \cap T \leq L_{eG}$. If $N \nsubseteq O^p(G)$, then $N \cong NO^p(G)/O^p(G) \leq G/O^p(G)$. Since $G/O^p(G)$ is a p-group, it follows that $|NO^p(G)/O^p(G)| = |N| = |D| = p$, contrary to Step 2. Hence, $N \leq O^p(G)$. Since $|G : T|$ is a power of p and T is subnormal in G, $O^p(G) \leq T$. Then $G/N = (K/N)(T/N)$ and $K/N \cap T/N = LN/N \cap T/N = (L \cap T)N/N \leq L_{eG}N/N \leq (LN/N)_{eG/N}$. This shows that every cyclic subgroup of P/N with order p is E–S-supplemented in G/N. By Theorem 3.1, G/N is p-supersoluble. Since the class of all p-supersoluble groups is a saturated formation, the uniqueness of N and $\Phi(G)$= 1 are obvious.

Step 7: Final contradiction. By Step 6, there is a maximal subgroup M of G such that $G = NM$. Furthermore, $P = N(P \cap M)$ and $P \cap M \neq 1$. Pick a maximal subgroup P_1 of P containing $P \cap M$. Then $P = NP_1$ and $N \cap P_1 < N$. If $N \cap P_1 = 1$, then N is of prime order, which is a contradiction. Hence, $N \cap P_1 \neq 1$. By Step 3, we can choose a subgroup H of P_1 containing $N \cap P_1$ such that $|H| = |D|$ and H is normal in P. Furthermore, $N \cap H = N \cap P_1 \neq 1$. By hypothesis, H is E–S-supplemented in G. Then there is a subnormal subgroup T of G such that $G = HT$ and $H \cap T \leq H_{eG}$. Since $|G : T|$ is a power of p and $T \lhd\lhd G$, $N \leq O^p(G) \leq T$. It follows that $N \cap H = N \cap H_{eG}$. Let $U_1, U_2, \ldots, U_s$ be all the nontrivial subgroups of H which are S-permutably embedded in G. For every $i \in \{1, 2, \ldots, s\}$, there is an S-permutable subgroup K_i of G such that U_i is a Sylow p-subgroup of K_i. Suppose that for some $i \in \{1, 2, \ldots, s\}$, we have $(K_i)_G \neq 1$. By Step 6, $N \leq (K_i)_G \leq K_i$. Hence, $N \leq U_i \leq H < P_1$. This contradiction shows that for all $i \in \{1, 2, \ldots, s\}$, we have $(K_i)_G = 1$. By Lemma 2.7, the U_i ($i \in \{1, 2, \ldots, s\}$) are S-permutable in G. It follows that H_{eG} is S-permutable in G and so $N \cap H_{eG} = N \cap H$ is S-permutable in G. Since $N_G(H \cap N) \geq O^p(G)$ by Lemma 2.8 and $H \cap N$ is normal in P by the choice of H, $H \cap N$ is normal in G. Therefore, $H \cap N = 1$ by the minimality of N, which is a contradiction. □

Theorem 3.3. *Let E be a p-soluble normal subgroup of G and P a Sylow p-subgroup of E, where p is a prime divisor of $|E|$. Suppose that there exists a subgroup D of P with $1 < |D| < |P|$ such that every subgroup H of P with order $|D|$ and every cyclic subgroup of P with order 4 (if P is a nonabelian 2-group and $|D| = 2$) is E–S-supplemented in G. Then $E/O_{p'}(E) \leq Z_{\mathfrak{U}}(G/O_{p'}(E))$.*

Proof. By Lemma 2.1(1), every subgroup H of P with order $|D|$ and every cyclic subgroup of P with order 4 (if P is a nonabelian 2-group and $|D| = 2$) is E–S-supplemented in E. If $p = 2$, then E is 2-nilpotent by Lemma 2.10. In particular, E is 2-supersoluble. If $p > 2$, then E is also p-supersoluble by Theorem 3.2. If $O_{p'}(E) \neq 1$, then, from Lemma 2.1(3), the hypothesis is still true for $(G/O_{p'}(E), E/O_{p'}(E))$. By induction, $E/O_{p'}(E) = (E/O_{p'}(E))/O_{p'}(E/O_{p'}(E)) \leq Z_{\mathfrak{U}}((G/O_{p'}(E))/(O_{p'}(E/O_{p'}(E)))) = Z_{\mathfrak{U}}(G/O_{p'}(E))$. Now assume that $O_{p'}(E) = 1$. By virtue of Lemma 2.9, $P \trianglelefteq E$. Obviously, P is also normal in G. By Lemma 2.2, $P \leq Z_{\mathfrak{U}}(G)$. Since E is p-soluble, it follows from Lemma 2.3 that $F^*(E) = F_p^*(E) = F_p(E) = O_p(E) = P$ and so $F^*(E) \leq Z_{\mathfrak{U}}(G)$. Applying Lemma 2.4, $E \leq Z_{\mathfrak{U}}(G)$. □

Theorem 3.4. *Let E and X be p-soluble normal subgroups of G with $F_p(E) \leq X \leq E$, where p is a prime divisor of $|E|$. Suppose that a Sylow p-subgroup P of X has a subgroup D with $1 < |D| < |P|$ such that every subgroup H of P with order $|D|$ and every cyclic subgroup of P with order 4 (if P is a nonabelian 2-group and $|D| = 2$) is E–S-supplemented in G. Then $E/O_{p'}(E) \leq Z_{\mathfrak{U}}(G/O_{p'}(E))$.*

Proof. By Theorem 3.3, $X/O_{p'}(X) \leq Z_{\mathfrak{U}}(G/O_{p'}(X))$. Since $F_p(E) \leq X \leq E$, it is easy to see that $O_{p'}(X) = O_{p'}(E)$. Hence, $X/O_{p'}(E) \leq Z_{\mathfrak{U}}(G/O_{p'}(E))$. Consequently, $F_p(E)/O_{p'}(E) \leq Z_{\mathfrak{U}}(G/O_{p'}(E))$. Since E is p-soluble, it follows from Lemma 2.3 that $F^*(E/O_{p'}(E)) = F_p^*(E/O_{p'}(E)) = F_p(E)/O_{p'}(E) \leq Z_{\mathfrak{U}}(G/O_{p'}(E))$. Applying Lemma 2.4, $E/O_{p'}(E) \leq Z_{\mathfrak{U}}(G/O_{p'}(E))$. □

Proof of Theorem 1.2. By the conclusion of Theorem 3.4, $E/O_{p'}(E) \leq Z_{\mathfrak{U}}(G/O_{p'}(E))$. Since $(G/O_{p'}(E))/(E/O_{p'}(E)) \cong G/E$ is p-supersoluble, it follows that $G/O_{p'}(E)$ is p-supersoluble and so G is p-supersoluble. □

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