TWO PROBLEMS ON FINITE GROUPS WITH k CONJUGATE CLASSES

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1. Introduction

Let G be a finite group of order g having exactly k conjugate classes. Let $\pi(G)$ denote the set of prime divisors of g. K. A. Hirsch [4] has shown that

$$g \equiv k \text{ modulo } 2 \text{ G.C.D.}\{(p^2-1) \mid p \in \pi(G)\} \text{ (provided } 2 \nmid g).$$

By the same methods we prove $g \equiv k$ modulo G.C.D. $\{(p-1)^2 \mid p \in \pi(G)\}$ and that if G is a p-group, $g \equiv k$ modulo $(p-1)(p^2-1)$. It follows that k has the form $(n+r(p-1))(p^2-1)+p^e$ where r and n are integers $\geqq 0$, p is a prime, e is 0 or 1, and $g = p^{2n+e}$. This has been established using representation theory by Philip Hall [3] (see also [5]). If

$$\delta = \text{G.C.D.}\{(p-1)(p^2-1) \mid p \in \pi(G)\}$$

then simple examples show (for $6 \nmid g$ obviously) that $g \equiv k$ modulo δ or even $\delta/2$ is not generally true.

If G is a p-group, W. Burnside [2] and N. Blackburn [1] have shown that the statements G has a conjugate class of maximum order and G has maximum nilpotent class are equivalent. It seems reasonable that if G has minimum (conjugate) class number it would have classes of maximum order; indeed, we show that if $g = p^m$ $(m = 2n+e)$ and $k = n(p^2-1)+p^e$ then G has maximum nilpotent class, and we calculate exactly how many classes G has of each order. Such strong conditions hold for these groups that we can show that they only exist for $m < p+3$. This extends some results we obtained in [5] for 2-groups.

2. Background

Let G denote a finite group of order g, where g has prime decomposition $g = \prod_{i=1}^{n}(p_i^{m_i})$, and let $\pi(G) = \{p_i \mid i = 1, \cdots, n\}$ be the set of primes dividing g. The number of conjugate classes of G will be denoted by $k(G)$;

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often we will simply say that k is the number of classes of G. The classes of G are denoted K_i $(i = 1, \cdots, k)$, as usual ordered, with $K_1 = \{1\}$; $K(x)$ means the class containing x. We denote the lower central series of G by $G \geqq \gamma_2 \geqq \gamma_3 \geqq \cdots$ (γ_1 is left undefined) and the upper central series by $\{1\} \leqq Z_1 \leqq Z_2 \leqq \cdots$. The group generated by $x, y, \cdots$ is denoted $\langle x, y, \cdots \rangle$.

Most of this paper will be concerned with p-groups; that is, $\pi(G) = \{p\}$, $g = p^m$. The phrase "G of order p^m" will mean that G is a group, p a prime, and m a positive integer; we will write $m = 2n+e$ to denote that m and n are integers $\geqq 0$ and e is 0 or 1. In this context we define the function f by $f(p^m) = n(p^2-1)+p^e$, an important expression. The ordered set $(a_0, a_1, \cdots, a_\lambda)$ is called the p-class vector of the p-group G and is used to indicate that G has exactly a_i classes of order p^i $(0 \leqq i \leqq \lambda)$ and no classes of order greater than p^λ.

If G has order p^m, it is well-known (Blackburn [1], p. 52) that G has nilpotent class at most $m-1$. If G has maximum nilpotent class $(m-1)$ then we return to Blackburn (pp. 54 and 57) for the following concepts. Define $\gamma_1 = \gamma_1(G)$ by $\gamma_1/\gamma_4 = C_{G/\gamma_4}(\gamma_2/\gamma_4)$; then G has the characteristic series $G > \gamma_1 > \gamma_2 = Z_{m-2} > \gamma_3 = Z_{m-3} > \cdots > \gamma_{m-1} = Z_1 > 1$ in which successive distinct terms have factor groups of order p. G is said to have maximum degree of commutativity $c(G) = c$ if $[\gamma_i, \gamma_j] \leqq \gamma_{i+j+c}$ for all $i, j = 1, 2, 3, \cdots$ and c is the maximum such integer; obviously $c \geqq 0$.

Burnside ([2], section 98) has shown that the conjugate classes of a non-abelian group G of order p^m all have order at most p^{m-2}. In fact the statements that G contains a class of maximum order and that G has maximum nilpotent class are equivalent:

2.1 THEOREM. (Burnside [2], section 98). *If G is a non-abelian group of order p^m containing a conjugate class of order p^{m-2} then G has nilpotent class $m-1$.*

2.2 THEOREM. *Let G be a non-abelian group of order p^m with nilpotent class $m-1$. Then*

(i) *G has p-class vector (p, p^2-1) if $m = 3$, (p, p^2-1, p^2-p) if $m = 4$, and $(p, p-1, p^2-1, p^2-p)$ or $(p, p^3-1, 0, p^2-p)$ if $m = 5$,*

(ii) *(Blackburn [1], 2.11 and 3.8) $c(G) > 0$ if m is odd, $m = 4$, or $m \geqq p+2$, and so*

(iii) *$c(G/Z) > 0$ if $m \geqq 4$,*

(iv) *(Blackburn [1], 2.8) $c(G) > 0$ if and only if $\gamma_1 = C_G(Z_2)$, and*

(v) *(Blackburn [1], 2.14 and the corollaries of 2.15) G has exactly (p^2-p) conjugate classes of order p^{m-2} if $c(G) > 0$, and $(p-1)^2$ otherwise.*

3. The relation $g \equiv k$

K. A. Hirsch [4] has shown $g \equiv k$ modulo $2(\text{G.C.D.}\{(p^2-1) \mid p \in \pi(G)\})$ if g is odd, and modulo 3 if g is even but $3 \nmid g$. Also, for p-groups, Philip Hall [3] proved by representation theory that $k = (n+r(p-1))(p^2-1)+p^e$, where $g = p^{2n+e}$ and $r \geqq 0$. In this section we wish to use Hirsch's extremely elementary group-theoretic approach to establish Hall's theorem and, in some cases, improve Hirsch's results. Throughout, let $\delta = \delta(G) = \text{G.C.D.}$ $\{(p^2-1)(p-1) \mid p \in \pi(G)\}$. We assume $6 \nmid g$, so that $\delta > 1$.

3.1 Lemma. *Let $\{\{1\} = H_1, H_2, \cdots, H_\lambda\}$ be the set of all cyclic primary subgroups of G, $|H_i| = q^s$, $q \in \pi(G)$, for $i > 1$, and let $\rho(1) = 1$, $\rho(H_i) = q^{2(s-1)}$ (q^2-1). Then $gk \equiv \sum_{i=1}^{\lambda} \rho(H_i)$ and $\rho(H_i) \equiv q^2-1$ (for $i > 1$) modulo δ.*

Proof. This is equivalent to a statement of Hirsch [4]; we outline the proof. We note first that $q(q^2-1) \equiv (q^2-1)$ modulo $(q-1)(q^2-1)$ so that the last statement is proved.

The number of solutions $x, y \in G$ of the equation $[x, y] = 1$ is $\sum_{x \in G}(|C_G(x)|) = \sum_{i=1}^{k}(|K_i|)(g/|K_i|) = gk$. The pair $(x, y) \neq (1, 1)$ is a solution of $[x, y] = 1$ if and only if it is a generator of an abelian subgroup H of G, so $gk = \sum_{H \text{ abelian}, d(H) \leqq 2}(\rho(H))$ where $\rho(H)$ is the number of pairs of generators of H. Let $H = \prod_{i=1}^{n} H_i$, H_i a p_i-group. Then $\rho(H) = \prod_{i=1}^{n} \rho(H_i)$ while if H_i is an abelian p_i-group of type (p_i^s), $(p_i^s, p_i^t)_{s=t}$, or $(p_i^s, p_i^t)_{s>t}$ then $\rho(H_i)$ is $p_i^{2s-2}(p_i^2-1)$, $(p_i^{2s}-p_i^{2s-2})$ $[(p_i^{2s}-p_i^{2s-2})-(p_i^s-p_i^{s-1})]$, or $\varphi(p_i^s)p_i^t\varphi(p_i^t)(p_i^s+p_i^{s-1})$. Since $(p_i^2-1)(p_j^2-j) \equiv 0$ modulo δ, we are done.

Recall we defined $f(p^{2n+e}) = n(p^2-1)+p^e$.

3.2 Lemma. *$p^m \equiv f(p^m)$ modulo $(p^2-1)(p-1)$.*

Proof. This is trivially true if m is 1 or 2. If $m \geqq 3$ $p^m = p^{m-2}+p^{m-2}$ $(p^2-1) \equiv p^{m-2}+(p^2-1)$ modulo $(p^2-1)(p-1)$. Therefore $p^{2n+e} = (p^e)$ $(p^{2n}) \equiv (p^e)(p^0+n(p^2-1)) \equiv p^e+n(p^2-1)$ modulo $(p^2-1)(p-1)$.

3.3 Corollary. *If $g = \prod_{i=1}^{n} p_i^{m_i}$ then $g^2 \equiv 1+\sum_{i=1}^{n} m_i(p^2-1)$ modulo δ.*

Proof. $g^2 = \prod_{i=1}^{n} p_i^{2m_i} \equiv \prod_{i=1}^{n}(1+m_i(p^2-1)) \equiv 1+\sum_{i=1}^{n} m_i(p_i^2-1)$ since $(p_i^2-1)(p_j^2-1) \equiv 0$ modulo δ.

The following lemma is of some interest in itself, and is modelled on one of Hirsch ([4], p. 99).

3.4 Lemma. *If $p^m \,\|\, g$, p odd, and t is the number of non-trivial cyclic p-subgroups of G then G contains exactly μp^m solutions of the equation $x^{p^m} = 1$, $(p-1) \mid (\mu-1)$, and $t \equiv m+(\mu-1)/(p-1)$ modulo $(p-1)$.*

Proof. By Frobenius' Theorem, G has μp^m solutions of $x^{p^m} = 1$. Each non-trivial solution generates a non-trivial cyclic p-group. Let G have λ_j

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cyclic subgroups of order p^j; each has $\varphi(p^j)$ generators. Therefore $\sum_{j>1}(\lambda_j p^{j-1}(p-1)) = \mu p^m - 1 = \mu(p^m-1)+(\mu-1)$. It follows that $(p-1)\,|\,(\mu-1)$ and $\sum_{j>1}(\lambda_j p^{j-1}) = \mu(p^{m-1}+p^{m-2}+\cdots+p+1)+(\mu-1)/(p-1)$. Since μ and p are congruent to 1 modulo $(p-1)$, we have $\sum_{j>1}\lambda_j \equiv m + (\mu-1)/(p-1)$ modulo $(p-1)$.

3.5 Corollary. *If G has a normal p-Sylow subgroup of order p^m ($p \neq 2$) and t is the number of non-trivial cyclic p-subgroups of G, then $t \equiv m$ modulo $p-1$.*

3.6 Corollary. *If G is a nilpotent group, g odd, then $g \equiv k$ modulo δ.*

Proof. By Corollary 3.5, we have, in Lemma 3.1, $gk \equiv 1+\sum_{i=1}^{n} m_i(p_i^2-1)$. By Corollary 3.3, $gk \equiv g^2$ modulo δ, and the corollary follows since $(g, \delta) = 1$.

By Corollary 3.5 and Lemma 3.2 we have shown that $k = (n+r(p-1))(p^2-1)+p^e$ for a group G of order p^m (p odd, $m = 2n+e$) where r is an integer. By Hirsch's theorem, this is also true for $p = 2$. We will have proved Hall's theorem if we can show that $k \geqq f(p^m)$. This is established in (5) but the following useful lemma, which is quite easy to prove, also shows that $r \geqq 0$.

Let $f_r(p^{2n+e}) = (n+r(p-1))(p^2-1)+p^e$.

3.7 Lemma. *Let G have order p^m and let H be a normal subgroup of G of order p. If $k(G) \leqq f_r(p^m)$, then $k(G/H) \leqq f_r(p^{m-1})$; if $k(G/H) \geqq f_r(p^{m-1})$, then $k(G) \geqq f_r(p^m)$.*

Proof. It is straightforward that $f_r(p^m) = f_{r+1-e}(p^{m-1})+(-1)^e(1-p)$. Hence $f_r(p^m) < f_{r+1}(p^{m-1})$, or $f_r(p^{m-1}) > f_{r-1}(p^m)$. Since $k(G/H) < k(G)$, then, if $k(G) \leqq f_r(p^m)$, $k(G/H) < f_r(p^m) < f_{r+1}(p^{m-1})$ and so $k(G/H) \leqq f_r(p^{m-1})$. Similarly if $k(G/H) \geqq f_r(p^{m-1})$ then $k(G) \geqq f_r(p^m)$.

This latter statement, combined with the fact that (obviously) $k(G) \geqq f(g)$ for groups of order p, p^2, and p^3 gives us by induction

3.8 Corollary. *If G has order p^m then $k(G) = (n+r(p-1))(p^2-1)+p^e$, $r \geqq 0$.*

We would like to show that $g \equiv k$ modulo δ for all groups ($6 \nmid g$). By Lemma 3.1 and Corollary 3.3, it seems we would need to extend Corollary 3.5: if $p^m \,\|\, g$, $p \neq 2$, and t is the number of cyclic non-trivial p-subgroups of G then $t \equiv m$ modulo $p-1$. We present some counterexamples to these conjectures.

Let p and q be primes such that $p \mid (q-1)$, and let $1 < \alpha < q$ be such that if $\alpha^\beta \equiv 1$ modulo q then $p \mid \beta$. Let $\mathrm{Fr}(p, q)$ denote the (Frobenius) group $G = \langle x, y \mid x^p = y^q = 1,\ y^x = y^\alpha \rangle$. Then $G = \mathrm{Fr}(p, q)$ contains exactly q p-Sylow subgroups, $g = pq$, and the number of non-trivial cyclic

p-subgroups of G is q. But it is not necessarily true (for example: $p = 7$, $q = 29$) that $q \equiv 1$ modulo $p-1$, or even modulo $(p-1/2)$, so Corollary 3.5 cannot be extended to all groups, except in the form of Lemma 3.4. If $p = 61$, $q = 367$, then $g = 22{,}387$, $k = 67$, $g-k = 22{,}320 = 16 \cdot 9 \cdot 5 \cdot 31$, while $\delta = 32 \cdot 9$ so that $g \equiv k$ modulo $\delta/2$ but not δ. If $p = 7$, $q = 71$, then $g = 497$, $k = 17$, $g-k = 480 = 32 \cdot 3 \cdot 5$, while $\delta = 32 \cdot 9$ so that $g \equiv k$ modulo $\delta/3$ but not δ.

Although we cannot show that $g \equiv k$ modulo δ or even $\delta/2$ in general then, we can still extend Hirsch's result slightly.

3.9 Proposition. *$g \equiv k$ modulo G.C.D.$\{(p-1)^2 \mid p \in \pi(G)\}$ if g is odd.*

Proof. Let

$$\tau = G.C.D.\{(p-1)^2 \mid p \in \pi(G)\} = [G.C.D.\{(p-1) \mid p \in \pi(G)\}]^2 \text{ say.}$$

As every element of G generates a cyclic subgroup of G,

$$g = \textstyle\sum_{H \text{ cyclic}} \varphi(|H|) \equiv \sum_{i=1}^{\lambda} \varphi(|H_i|) \text{ modulo } \tau,$$

where H_i and λ are as in Lemma 3.1, taking $\varphi(1) = 1$. Note that if $|H_i| = q_i^{s_i}$, $q_i \in \pi(G)$, then $\varphi(|H_i|) = q_i^{s_i-1}(q_i-1) \equiv q_i-1$ modulo τ. Therefore $g^2 \equiv [1+\sum_{i=2}^{\lambda}(q_i-1)]^2 \equiv 1+\sum_{i=2}^{\lambda} 2(q_i-1) \equiv 1+\sum_{i=2}^{\lambda}(q_i+1)(q_i-1) \equiv gk$ modulo τ by Lemma 3.1. Since $(g, \tau) = 1$, the proposition follows.

3.10 Corollary. *$g \equiv k$ modulo L.C.M.$[G.C.D.\{(p-1)^2 \mid p \in \pi(G)\}$, $2(G.C.D.\{(p^2-1) \mid p \in \pi(G)\})]$ if g is odd.*

Proposition 3.9 says, for example, if $\pi(G) = \{19,37\}$, then $g \equiv k$ modulo $(18)^2$, whereas Hirsch's theorem states $g \equiv k$ modulo $(16)(18)$.

4. $k(G) = f(g)$

In this section, G will always denote a group of order p^m, p prime. We have shown that if $f_r(p^m) = (n+r(p-1))(p^2-1)+p^e$ (where $m = 2n+e$), then $k(G) = f_r(p^m)$ for some integer $r \geqq 0$. Denote $f_0(p^m)$ by $f(p^m)$; what is the structure of G if $k(G) = f(g)$?

4.1 Lemma. *Let N be a normal subgroup of G of order p. Let $k(G/N) = f_r(p^{m-1})$, $k(G) = f_r(p^m)$, and let G/N have p-class vector $(a_0, a_1, \cdots, a_\lambda)$. Then G has p-class vector $(p^{2-e}, (a_0-1)+(e-1)(p-1), a_1, a_2, \cdots, a_\lambda)$ or $(p, (a_0-1), a_1, \cdots, a_{i-1}, a_i+(1-e)(p^2-p), a_{i+1}+(e-1)(p-1), a_{i+2}, \cdots, a_\lambda)$ for some $0 \leqq i < \lambda$.*

Proof. Let ξ be the canonical map of $G \to G/N$ and let $\bar{K}$ be any conjugate class of G/N. If $1 \neq \xi(x) \in \bar{K}$ then $\xi^{-1}(\bar{K}) = K(x) \cdot N$ is a union of classes of G, and since $|N| = p$, $\xi^{-1}(\bar{K})$ then must be a single class of G

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or a union of p classes of G (obviously $\xi^{-1}(1) = N$ is a union of p classes of order 1). $\xi^{-1}(\bar{K}) \neq N$ is a union of p classes of G if and only if $\bar{K}'$ is, where $\bar{x} \in \bar{K}$ and $\bar{x}^a \in \bar{K}'$ for some $1 < a < p$, so this happens in sets of $p-1$ classes of the same order, over G/N. If we let β denote the number of such sets, then $k(G) = k(G/N)+(p-1)+\beta[p(p-1)-(p-1)]$. Straightforward substitution shows that if $m = 2n+e$, $\beta = 1-e$, and we are done.

4.2 Theorem. *If G has order p^m $(m > 1)$ and $k(G) = f(p^m)$ then G has nilpotent class $m-1$.*

Proof. The theorem is obviously true for $m = 2$ and 3, so suppose $m > 3$, $k(G) = f(p^m)$, and that the theorem is proved for all groups of order p^{m-i} $(1 \leqq i \leqq m-2)$. Take $N \leqq Z_1(G)$, $|N| = p$. By Lemma 3.7 and Corollary 3.8, $k(G/N) = f(p^{m-1})$. By the induction hypothesis, G/N has maximum nilpotent class, so by part (v) of Theorem 2.2, G has p^2-p classes of maximum order, or $(p-1)^2$ perhaps if $p > 2$. By Lemma 4.1 then G has $(p^2-p)-(p-1)$, or $(p-1)^2-(p-1)$ if $p > 2$, classes of order p^{m-2}, at least; that is, G has at least one class of maximum order. The theorem follows by Theorem 2.1.

The p-Sylow subgroup of $\mathrm{Sym}(p^2)$ shows that the converse of Theorem 4.2 is not true. In fact, we must place rather strong conditions upon G in order that $k(G) = f(g)$.

4.3 Theorem. *If $k(G) = f(p^m)$ for a group G of order p^m $(m \geqq 3)$ then either*

(i) *$c(G) = 0$ and G has p-class vector*

$$(p, \underbrace{p-1, \cdots, p-1}_{n-1}, p^2-1, \underbrace{p^2-p, \cdots, p^2-p}_{n-4}, 2(p^2-p), (p-1)^2) \text{ if } n \geqq 4,$$

or $(p, p-1, p-1, 2p^2-p-1, (p-1)^2)$ if $n = 3$; or

(ii) *$c(G) = 1$; for $1 \leqq i \leqq m-2$, if $x \in \gamma_i - \gamma_{i+1}$ then $C_G(x) = \langle x, \gamma_{m-i-1}\rangle$ and $x^p \in \gamma_{m-i-1}$; and G has p-class vector*

$$(p, \underbrace{p-1, \cdots, p-1}_{n-2+e}, p^2-1, \underbrace{p^2-p, \cdots, p^2-p}_{n-1}).$$

Proof. Since each γ_i is a normal subgroup of G and so a union of conjugate classes of G, then $G-\gamma_1$ and $\gamma_i-\gamma_{i+1}$ $(i > 0)$ are unions of classes of G. Note that $|\gamma_i| = p^{m-i}$.

First, suppose $c(G) > 0$. By part (v) of Theorem 2.2, $G-\gamma_1$ splits into p^2-p classes of p^{m-2} elements each. Since $[\gamma_i, \gamma_j] \leqq \gamma_{i+j+1}$ then $[\gamma_i, \gamma_{m-i-1}] = 1$ so if $x \in \gamma_i - \gamma_{i+1}$ then $C_G(x) \geqq \langle x, \gamma_{m-i-1}\rangle$. Now $x \in \gamma_{m-i-1}$ if and only if $\gamma_i \leqq \gamma_{m-i-1}$ or $i \geqq (m-1)/2$. Therefore if $1 \leqq i < n$, then $|C_G(x)| \geqq p \cdot p^{i+1}$ so $|K(x)| \leqq p^{m-i-2}$. It follows that $\gamma_i-\gamma_{i+1}$ splits into at least $(p^{m-i}-p^{m-i-1})/p^{m-i-2} = p^2-p$ classes of G if $1 \leqq i \leqq n$. In the same way if $n \leqq i \leqq m$, $\gamma_i - \gamma_{i+1}$ splits into at least $p-1$ classes. Finally $\gamma_m = \{1\}$ is a class of G. Therefore $k(G) \geqq n(p^2-p)+(m-n)(p-1)+1 = f(p^m)$, with equality only if $x \in \gamma_i - \gamma_{i+1}$ implies that $C(x) = \langle x, \gamma_{m-i-1} \rangle$ and $x^p \in \gamma_{m-i-1}$ for $i = 1, \cdots, m-2$. In particular $[\gamma_2, \gamma_{m-3}] = 1$ but $[\gamma_1, \gamma_{m-3}] \neq 1$ so $c(G) = 1$. We note that we have $\gamma_{n-1} - \gamma_n$ splitting into p^2-p classes of order p^{m-n-2} and $\gamma_n - \gamma_{n+1}$ splitting into $p-1$ classes of the same order. To summarize, G must have p-class vector

$$(p, \underbrace{p-1, \cdots, p-1}_{n-2+e}, p^2-1, \underbrace{p^2-p, \cdots, p^2-p}_{n-1}).$$

Suppose now $c(G) = 0$. By part (ii) of Theorem 2.2 $m = 2n$ and $6 \leqq m \leqq p+2$, while by part (iii), $c(G/Z) > 0$. By Lemma 3.7 and Corollary 3.8, $k(G/Z) = f(p^{m-1})$. Hence we can apply the above results and G/Z must have p-class vector

$$(p, \underbrace{p-1, \cdots, p-1}_{n-2}, p^2-1, \underbrace{p^2-p, \cdots, p^2-p}_{n-2}).$$

Now by part (v) of Theorem 2.2, G has exactly $(p-1)^2 = (p^2-p)-(p-1)$ classes of order p^{m-2}. The p-class vector of G now follows by Lemma 4.1, and we are done.

4.4 THEOREM. *If G is a group of order p^m and $m \geqq p+3$ then $k(G) \geqq f_1(p^m)$.*

PROOF. Suppose $g = p^m$, $m \geqq p+3$, and $k(G) = f(p^m)$. Define s and s_1 as generators of G modulo γ_1 and γ_1 modulo γ_2 respectively; define $s_i = [s_{i-1}, s]$ for $i > 1$. Blackburn ([1], 2.9 and 3.8) has shown that s_i and γ_{i+1} generate γ_i then because of Theorem 4.2. By Theorem 4.3, $s_1^p \in \gamma_{m-2} \leqq \gamma_{p+1}$ since $m \geqq p+3$. Therefore $s_1^p s_p \notin \gamma_{p+1}$, contradicting Lemma 3.3 of Blackburn. The theorem follows by Corollary 3.8.

The case of $p = 2$ and $k(G) = f_1(g)$ has been examined in [5].

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