Examples of a Method of Developing Logarithms and the Trigonometrical Functions without the Calculus by means of their Addition Formulae and Indeterminate Coefficients.

By JOHN JACK, M.A.

[ABSTRACT.]

The convergence of the series is *assumed.*

The method consists in assuming that the function is equal to a certain power series with undetermined coefficients, substituting these series in the addition formula. This gives an identity.

Ex. gr. $\sin^{-1}x+\sin^{-1}y=\sin^{-1}(x\sqrt{1-y^2}+y\sqrt{1-x^2})$

$$\left.\begin{matrix} a_1x+a_2\psi^2+a_3\psi^3+\cdots \\ +a_1y+a_2y^2+a_3y^3+\cdots \end{matrix}\right\}$$
$$=\left\{\begin{matrix} a_1(x\sqrt{1-y^2}+y\sqrt{1-x^2})+a_2(x\sqrt{1-y^2}+y\sqrt{1-x^2})^2 \\ +a_3(a\sqrt{1-y^2}+y\sqrt{1-x^2})^3+ \end{matrix}\right.$$

Picking out coefficient of y, we get

$$a_1=a_1+\text{function of } x.$$

Now this function of x must $=0$, and therefore the coefficients of the powers of x must each $=0$. From this it can be inferred that the function contains only odd powers of x, and the coefficients can easily be determined. The inverse function can be developed in the same way, and in the case of $\sin x$ or $\cos x$ with greater ease and completeness. I have found the development of $\text{sinam}x$, $\text{cosam}x$, and $\Delta\text{am}x$ and $\text{sinam}^{-1}x$ in the same way. The n^{th} term in the expansion of $\tan x$ is not given by this plan, that of $\sin^{-1}x$ can be inferred by *induction.* No. 2 has been

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done of course in practically the same way, but is given on account of its intimate connection with No. 1. I give

(1) $\log x$	(3) $\sin^{-1}x$	(5) $\tan^{-1}x$
(2) $\log^{-1}x$ or e^x	(4) $\sin x$, $\cos x$	(6) $\tan x$.

The method seems symmetrical and quite elementary. The analogy between sin, cos, tan, and sinh, cosh, tanh, can be readily seen without at all using the imaginary i, by developing by this plan.

1. To develop $\log \overline{1+x}$ in a series of powers of x

$$\log(1+x)(1+y) = \log(1+x) + \log(1+y)$$

Let
$$\log(1+x) = \phi(x)$$

$$\therefore \quad \phi(x) + \phi(y) = \phi(x+y+xy)$$
$$= \phi(x + y\overline{1+x})$$

Let
$$\phi(x) = a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots$$

$$\therefore \quad \left.\begin{matrix} a_1x + a_2x^2 + a_3x^3 + \cdots \\ + a_1y + a_2y^2 + a_3y^3 + \cdots \end{matrix}\right\} \equiv \left\{\begin{matrix} a_1(x + y\overline{1+x}) + a_2(x + y\overline{1+x})^2 \\ + a_3(x+y\overline{1+x})^3 + \cdots \end{matrix}\right.$$

Pick out the coefficient of y.

$$\therefore \quad a_1 = (1+x)(a_1 + 2a_2x + 3a_3x^2 + \cdots)$$
$$= a_1 + (a_1 + 2a_2)x + (2a_2 + 3a_3)x^2 + \cdots$$

and the coefficients of x must vanish

$$\therefore \quad a_1 + 2a_2 = 0$$
$$2a_2 + 3a_3 = 0$$
$$3a_3 + 4a_4 = 0$$
$$4a_4 + 5a_5 = 0$$

and so on.

$$\therefore \quad a_1 = -2a_2 = 3a_3 = -4a_4 = 5a_5 = -, \cdots$$

$$\therefore \quad a^2 = -\frac{a_1}{2},\ a_3 = \frac{a_1}{3},\ a_4 = -\frac{a_1}{4},\ a_5 = \frac{a_1}{5}, \text{ and so on}$$

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$$\therefore \qquad \phi(x) = a_1\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}\cdots\right)$$

$$\therefore \qquad \log(1+x) = a_1\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}\cdots\right)$$

and a_1 must be determined otherwise.

The other expansions are given in abstract.

2. To find the number corresponding to a logarithm, or to develop $log^{-1}x$.

Taking as before $\qquad \phi(x) = \log(1+x)$

$$\phi(x) + \phi(y) = \phi(x + y + xy)$$

Let $\qquad \phi(x) = u \qquad \therefore \quad x = \phi^{-1}(u)$

$\qquad \phi(y) = v \qquad \therefore \quad y = \phi^{-1}(v)$

$$\therefore \qquad u + v = \phi\{\phi^{-1}(u) + \phi^{-1}(v) + \phi^{-1}(u)\phi^{-1}(v)\}$$

$$\therefore \qquad \phi^{-1}(u+v) = \phi^{-1}(u) + \phi^{-1}(v) + \phi^{-1}(u)\cdot\phi^{-1}(v)$$

Let $\qquad \phi^{-1}(\ ) = a_1(\ ) + a_2(\ )^2 + a_3(\ )^3 + \cdots$

Insert these expansions in the equation just given, and pick out the coefficients of v.

From the identity so obtained in powers of u, we get, by equating coefficients of like powers, the required relations between the constants $a_1 a_2 a_3 \cdots$, and finally

$$x = \phi^{-1}u = a_1 u + \frac{(a_1 u)^2}{\lfloor 2} + \frac{(a_1 u)^3}{\lfloor 3} + \frac{(a_1 u)^4}{\lfloor 4} + \frac{(a_1 u)^5}{\lfloor 5} + \cdots$$

$$\therefore \quad 1 + x = 1 + a_1 u + \frac{(a_1 u)^2}{\lfloor 2} + \frac{(a_1 u)^3}{\lfloor 3} + \cdots$$

Now $\log\overline{1+x} = u$ and if a is the base

$$a^u = 1 + x$$

$$\therefore \qquad a^u = 1 + \frac{(a_1 u)}{\lfloor 1} + \frac{(a_1 u)^2}{\lfloor 2} + \frac{(a_1 u)}{\lfloor 3} + \cdots$$

and a_1 must be otherwise determined.

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3. Required the development of $\sin^{-1}x$. By similar treatment of the identity

$$\sin^{-1}x + \sin^{-1}y = \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2})$$

we get

$$\sin^{-1}x = a_1\left(x + \frac{1}{2}\frac{x^3}{3} + \frac{3}{2^3}\frac{x^5}{5} + \frac{5}{2^4}\frac{x^7}{7} + \frac{35}{2^7}\frac{x^9}{9} + \frac{63x^{11}}{2^8\,11} + \frac{231x^{13}}{2^{10}\,13} + \cdots\right)$$

$$= a_1\left(x + \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7} + \cdots\right)$$

and a_1 is otherwise found to be 1.

$\sin^{-1}x$ is thus found to be an odd function of x.

4. The development of $\sin u$, $\cos u$. This is got from the identity

$$\sin^{-1}x + \sin^{-1}y = \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2})$$

or

$$u + v = \sin^{-1}(\sin u\cos v + \cos u\sin v.$$

It is shown, first, that $\sin u$ is an odd function of u, and $\cos u$ an even function of u. The series are then assumed, and the coefficients evaluated as above.

5. The development of $\tan^{-1}x$. This is got from the identity

$$\tan^{-1} + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}$$

It is first established that $\tan^{-1}x$ is an odd function of x, and then the series is assumed and the coefficients evaluated in the usual way.

6. The development of $\tan u$. Here we have

$$\tan u + \tan v = \tan(u+v) - \tan u\tan v\tan(u+v).$$

Assume the series involving odd powers, and proceed as above.

The paper ended with an expansion in terms of arcs of small tangents, for calculating π.

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