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Going-Down Results for C_i-Fields

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Abstract. We search for theorems that, given a C_i-field K and a subfield k of K, allow us to conclude that k is a C_j-field for some j. We give appropriate theorems in the case $K = k(t)$ and $K = k((t))$. We then consider the more difficult case where K/k is an algebraic extension. Here we are able to prove some results, and make conjectures. We also point out the connection between these questions and Lang's conjecture on nonreal function fields over a real closed field.

A field k is called a C_i*-field* if every homogeneous form of degree d in $n > d^i$ variables has a nontrivial zero in k. This idea was introduced in [Ts] and rediscovered in [L]. There are a number of "going-up" theorems for C_i-fields. That is, given a C_i-field k and an extension K/k, such a theorem allows us to conclude that K is a C_j-field for some $j \geq i$. We are concerned with finding corresponding "going-down" theorems. That is, we are given a C_i-field K and a subfield $k \subseteq K$, and we want to conclude that k is a C_j-field for some $j \leq i$. We provide several theorems of this type, in analogy to known going-up theorems, and suggest some directions for further research.

We now list the basic "going-up" results for C_i-fields. For proofs, and a discussion of the history of these results, see [P] or [G].

Theorem 1 *If k is a C_i-field and K/k is an algebraic extension, then K is a C_i-field.*

Theorem 2 *If k is a C_i-field, then the rational function field $k(t)$ is a C_{i+1}-field.*

Theorem 3 *If k is a C_i-field, then the field of formal Laurent series $k((t))$ is a C_{i+1}-field.*

We note that a field k is a C_0-field if and only if k is algebraically closed. By a well-known theorem of Chevalley, every finite field is C_1.

First we establish going-down versions of Theorems 2 and 3.

Theorem 4 *If $k(t)$ is a C_i-field, then k is a C_{i-1}-field.*

Proof Note that since $k(t)$ is not algebraically closed, $i \geq 1$. Suppose that k is not a C_{i-1}-field. Then there exists a form f of degree d in $n > d^{i-1}$ variables with coefficients in k and no nontrivial zero over k. Now consider the form

$$F = f(X_0) + tf(X_1) + \cdots + t^{d-1}f(X_{d-1})$$

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where each $X_i = (X_{i1}, X_{i2}, \dots, X_{in})$. The total number of variables is now $dn > d^i$. Thus there exists x, a nontrivial zero over $k(t)$, where $x = (x_0, x_1, \dots, x_{d-1})$ and each $x_i = (x_{i1}, x_{i2}, \dots, x_{in})$. We may assume that all $x_{ij} \in k[t]$ and at least one of the x_{ij} is not divisible by t. Now set $t = 0$, and the result is $f\big(x_{01}(0), x_{02}(0), \dots, x_{0n}(0)\big) = 0$. Since f has no nontrivial zero over k, each $x_{0j}(0) = 0$. Thus t divides x_{0j} for all j. Write $y_{0j} = \frac{x_{0j}}{t}$ and set $y = (y_{01}, y_{02}, \dots, y_{0n})$. Then we may write

$$t^d f(y) + t f(x_1) + \cdots + t^{d-1} f(x_{d-1}) = 0$$

and dividing by t gives

$$f(x_1) + \cdots + t^{d-2} f(x_{d-1}) + t^{d-1} f(y) = 0.$$

By repeating this argument we conclude that t divides x_{ij} for all i, j, which is a contradiction. ∎

By making the appropriate changes to this proof, we also get

Theorem 5 *If $k((t))$ is a C_i-field, then k is a C_{i-1}-field.*

We would like to have a going-down version of Theorem 1 as well. However it is clear that there must be some restrictions on the type of algebraic extensions considered. For example, let F be any field, $k = F(S)$ where S is an infinite set of independent indeterminates, and K the algebraic closure of k. Then K is C_0, k is not C_i for any i, and K/k is an infinite dimensional algebraic extension. For another example, let R be a real closed field and $C = R(\sqrt{-1})$. Then R is not C_i for any i, C is C_0, and C/R has dimension 2. Thus, at the least, we should restrict our attention to nonreal subfields of finite codimension. We are led to the following definitions.

Definition 6 A field K is said to be an *S_i-field* if all subfields L of K such that $[K:L] < \infty$ are C_i-fields.

Definition 7 A field K is said to be an *S_i^{nr}-field* if K is a C_i-field and any nonreal subfield L of K such that $[K:L] < \infty$ is a C_i-field.

We note that a field of characteristic zero is S_i only if it has no real subfield of finite codimension. A field of positive characteristic is S_i if and only if it is S_i^{nr}.

We will consider these properties in an attempt to understand when C_i goes-down an algebraic extension. In the next section, we will see that the property S_i goes-up "polite" extensions and that many natural extensions are "polite." Using these ideas we will show that these properties have a connection to the following outstanding conjecture.

Conjecture 8 (Lang's Conjecture) *If R is a real closed field and E is a nonreal function field of transcendence degree j over R, then E is a C_j-field.*

We will show, among other things, that Lang's Conjecture is equivalent to the statement: If F is an algebraically closed field of characteristic 0 and E is a function field of transcendence degree j over F, then E is S_j^{nr}. This statement, unlike Lang's Conjecture itself, can be cast in positive characteristic. We are able to prove the truth of the following positive characteristic version of Lang's Conjecture: If F is an algebraically closed field of positive characteristic and E is a function field of transcendence degree j over F, then E is S_j.

In the last section we study the property S_i for fields of Laurent series.

1 Polite Algebraic and Function Field Extensions

Theorems 1 and 2 imply that if F is a C_i-field and E/F has tr deg j, then E is C_{i+j}. We consider the corresponding questions for S_i or S_i^{nr}. We are able to answer this question in the cases where F is an absolutely algebraic or algebraically closed field. In order to do this, we need the following idea.

Definition 9 A field extension E/F is called *polite* if every finite codimension subfield of E restricts to a finite codimension subfield of F. That is, E/F is polite if for every field $L \subseteq E$ such that $[E : L]$ is finite, $[F : F \cap L]$ is finite.

Let Q be a prime subfield; that is, Q is either $\mathbb{Q}$ or one of the $\mathbb{F}_p$. If F is any finite extension of Q and E/F is any extension, then E/F is (almost trivially) polite.

The following simple Theorem explains the usefulness of the concept.

Theorem 10 *If F is S_i and E/F is a polite extension of* tr deg *j, then E is S_{i+j}.*

Proof Let L be a finite codimension subfield of E. Since E/F is polite, $F \cap L$ is a finite codimension subfield of F. Since F is S_i, $F \cap L$ is C_i. Now $E/(F \cap L)$ has tr deg j and so $L/(F \cap L)$ has tr deg j. Thus L is C_{i+j}. Hence E is S_{i+j}. ■

Recall that a field is called *absolutely algebraic* if it is an algebraic extension of its prime subfield.

Corollary 11 *An absolutely algebraic field of characteristic $p > 0$ is S_1.*

Proof The absolutely algebraic field of characteristic $p > 0$ is a polite extension of transcendence degree 0 of the S_1 field $\mathbb{F}_p$. ■

Algebraic extensions are not, in general, polite: Let C be an algebraically closed field of characteristic 0 and let R_1 and R_2 be distinct real closed subfields of C. Then C/R_1 is not polite as $[C : R_2] = 2$ while $[R_1 : R_1 \cap R_2]$ is infinite. Function field extensions are also not, in general, polite. Here are two examples: First, let t be an indeterminate. Then $C(t)/R_1$ is not polite as $[C(t) : R_2(t)] = 2$ while $[R_1 : R_1 \cap R_2(t)]$ is infinite. Another example is provided below.

Example 12 If k is any field and s, t are algebraically independent indeterminates, then the extension $k(s,t)/k(s)$ is not polite.

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Proof $L = k(s+t, st)$ is a subfield of $k(s,t)$ of codimension two as s and t are roots of $T^2 - (s+t)T + st \in L[T]$. Since L consists of symmetric rational functions in s and t with coefficients in k, $L \cap k(s) = k$. Thus $k(s)/L \cap k(s)$ is not finite. ■

On the other hand, we do have the following positive results.

Theorem 13 *If E is any field and F is the algebraic closure of Q, the prime subfield, in E, then E/F is polite.*

Proof Let L be a finite codimension subfield of E.

Suppose first that E/L is Galois. Then L is the fixed field of G, a finite group of automorphisms of E. Since F/Q is algebraic we have $\sigma(F) = F$ for each $\sigma \in G$. If we denote the restriction of $\sigma \in G$ to F by $\bar{\sigma}$, then the correspondence $\sigma \mapsto \bar{\sigma}$ is a homomorphism of G to $\operatorname{Aut}(F)$. Let F^G be the fixed field of the group $\{\bar{\sigma} \mid \sigma \in G\}$. Then $F^G \subseteq F \cap L$. Since F/F^G is a finite dimensional extension, so is $F/(F \cap L)$.

Now suppose that E/L is purely inseparable. If $L = E$, then, of course, $F \cap L = F$. If $L \neq E$, then the characteristic is $p > 0$ and there exists $n \geq 1$ such that $E^{p^n} \subseteq L$. Since F is perfect, we have $F = F^{p^n} \subseteq E^{p^n} \subseteq L$. Thus $F \cap L = F$.

Finally suppose that E/L is arbitrary. Let N/L be the normal closure of E/L. Let K be the purely inseparable closure of L in N. Thus N/K is Galois and K/L is purely inseparable. Let F_1 be the algebraic closure of Q in N. We note that $[F_1 : F] \leq [N : E]$ is finite. Applying the first paragraph to N/F_1 and the subfield K, we see that $[F_1 : F_1 \cap K]$ is finite. Applying the second paragraph to $K/(F_1 \cap K)$ and the subfield L, we get $(F_1 \cap K) \cap L = F_1 \cap K$. Thus $F_1 \cap L = (F_1 \cap K) \cap L$ is a finite codimension subfield of F_1.

We claim $F_1 \cap L = F \cap L$. Since $F \subseteq F_1$ we have $F \cap L \subseteq F_1 \cap L$. Now, let $\alpha \in F_1 \cap L$. Then α is algebraic over Q and $\alpha \in L \subseteq E$. Thus $\alpha \in F$ and hence $\alpha \in F \cap L$.

Since $[F_1 : F \cap L] = [F_1 : F_1 \cap L]$ is finite and $[F_1 : F]$ is finite, we have $[F : F \cap L]$ is finite. ■

Recall that E is a function field of transcendence degree j over F means that E/F is finitely generated of tr deg j and F is the algebraic closure of F in E.

Corollary 14 *If F is an absolutely algebraic field and E is a function field of transcendence degree j over F, then E/F is polite.*

Corollary 15 *If F is an absolutely algebraic field of characteristic $p > 0$ and E is a function field of transcendence degree j over F, then E is S_{j+1}.*

Using similar techniques we derive the next result.

Theorem 16 *If F is an algebraically closed field and E is a finite dimensional extension of a purely transcendental extension of F, then E/F is polite.*

We first need a Lemma.

Lemma 17 *If σ is an automorphism of E, then $\sigma(F) = F$.*

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Proof By hypothesis there is a transcendence basis B for E/F such that $E/F(B)$ is a finite dimensional extension. We have $Q^a \subseteq F$ and $\sigma(Q^a) = Q^a$ where Q^a is the algebraic closure of Q. If $F = Q^a$ then we're done. If not, we have some $\alpha \in F - Q^a$. Now $X^n - \alpha$ splits in F for all $n \geq 1$. Thus $\sigma(\alpha) \in E$ is an n-th power in E for all $n \geq 1$. Assume $\sigma(\alpha) \notin F$. Then $\sigma(\alpha)$ is transcendental over F. Now, $F(B)\big(\sigma(\alpha)\big)$ is algebraic over $F(B)$. Thus there exists a $t \in B$ such that $B' = B \cup \{\sigma(\alpha)\} - \{t\}$ is a transcendence basis for E/F. In addition, $E/F(B')$ is finite dimensional because it is algebraic and finitely generated (by t and basis for $E/F(B)$). Now $F[B']$ is a UFD and $\sigma(\alpha)$ is a prime element. Hence $X^n - \sigma(\alpha)$ is irreducible over $F(B')$. Since $X^n - \sigma(\alpha)$ has a root in E, we must have $[E\colon F(B')] \geq n$. Since n is arbitrary, we have a contradiction. Thus $\sigma(\alpha) \in F$. Therefore $\sigma(F) \subseteq F$. Suppose $\sigma(F) \neq F$. Choose $\beta \in F \setminus \sigma(F)$. σ is an automorphism, so $\beta = \sigma(\gamma)$ for some $\gamma \in E \setminus F$. Now since $\gamma = \sigma^{-1}(\beta)$, we have $\sigma^{-1}(F)$ is not a subset of F. This is a contradiction, and therefore $F = \sigma(F)$. ■

Proof of the Theorem Let L be a finite codimension subfield of E.

First suppose that E/L is Galois. Then L is the fixed field of G, a finite group of automorphisms of E. By our Lemma, we have $\sigma(F) = F$ for all $\sigma \in G$. Thus $F^G \subseteq F \cap L$ and F/F^G is finite. Hence $[F\colon F \cap L]$ is finite.

Now suppose E/L is arbitrary. Let N/L be the normal closure of E/L. Let K be the purely inseparable closure of L in N. Thus N/K is Galois and K/L is purely inseparable. By the last paragraph, $F \cap K$ is a finite codimension subfield of F. If $K = L$, then we're done. Otherwise the characteristic of E is $p > 0$ and there is an $n \geq 1$ such that $K^{p^n} \subset L$. Now $F = F^{p^n} \subseteq K^{p^n}$. Thus $F = F \cap K^{p^n} \subseteq F \cap L$. Therefore $F = F \cap L$. ■

Corollary 18 *If R is a real closed field and E is a finite dimensional extension of a purely transcendental extension of F such that R is algebraically closed in E, then E/R is polite.*

Proof Let L be a finite codimension subfield of E. We apply the last theorem to $E(i)/R(i)$, where $i^2 = -1$, and the subfield L, to conclude that $k = R(i) \cap L$ is a finite codimension subfield of $R(i)$. Since each element of k is algebraic over R and $Rk \subseteq E$, we have $Rk = R$ and hence $k \subseteq R$. Since $[R(i)\colon k]$ is finite, $k = R$. ■

Corollary 19 *If F is an algebraically closed or real closed field and E is a function field of transcendence degree $j > 0$ over F, then E/F is polite.*

Corollary 20 *If F is an algebraically closed field of characteristic $p > 0$ and E is a function field of transcendence degree j over F, then E is S_j.*

The best we can do to extend the last result to characteristic 0 is to make the following connection to Lang's conjecture, see 2.4 of Chapter 5 in [P] .

Theorem 21 *The following statements are equivalent.*

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(1) *If F is an algebraically closed field of characteristic 0 and E is a function field of transcendence degree j over F, then E is S_j^{nr}.*
(2) *(Lang's Conjecture) If R is a real closed field and E is a nonreal function field of transcendence degree j over R,then E is C_j.*

Proof (1) $\Rightarrow$ (2): Let R be a real closed field and E a nonreal function field of transcendence degree j over R. Since $R(i)$ is algebraically closed, where $i^2 = -1$, $E(i)$ is S_j^{nr} and hence E is C_j.

(2) $\Rightarrow$ (1): Let F be an algebraically closed field of characteristic 0 and E a function field of transcendence degree j over F. Let L be a nonreal finite codimension subfield of E. Since E/F is polite, $F \cap L$ is a finite codimension subfield of F. Thus, by Artin–Schreier, $F \cap L = F$ or $F \cap L = R$, a real closed subfield of codimension 2. In the first case, L is a function field of transcendence degree j over F and thus L is C_j by an earlier result. In the second, L is a function field of transcendence degree j over R and hence L is C_j by Lang's Conjecture. ■

We note that in light of Theorem 21, and since S_j^{nr} is equivalent to S_j in characteristic $p > 0$, Corollary 20 can be thought of as the (much easier) positive characteristic version of Lang's Conjecture. In the case of transcendence degree one, we make some further observations.

Lemma 22 *Lang's Conjecture holds for transcendence degree one if and only if $R(u, \sqrt{-1-u^2})$ is a C_1-field for all real closed fields R.*

Proof $\Rightarrow$ This is just a special case of Lang's Conjecture.

$\Leftarrow$ Suppose $R(u, \sqrt{-1-u^2})$ is C_1 for a real closed field R, and let F be any nonreal field of transcendence degree one over R. By Corollary 3.4 in Chapter 6 of [P], -1 is a sum of two squares in F, so we can write $-1 = u^2 + v^2$. If u is not transcendental over R, then $R(i) \subset F$ and we are done. Otherwise the function field $R(u, \sqrt{-1-u^2}) \subset F$, and $F/R(u, \sqrt{-1-u^2})$ is algebraic. Since $R(u, \sqrt{-1-u^2})$ is a C_1-field, it follows that F is a C_1-field, and so Lang's conjecture holds in this case. ■

Theorem 23 *Lang's Conjecture holds for transcendence degree one if and only if $k(x)$ is an S_1^{nr}-field for all algebraically closed fields k of characteristic zero.*

Proof $\Rightarrow$ Suppose Lang's Conjecture holds, and let $L \subseteq k(x)$ be a nonreal subfield with $[k(x){:}L] < \infty$. Since k is algebraically closed, $k(x)/k$ is a polite extension, and so $[k{:}k \cap L] < \infty$ which implies $[k{:}k \cap L] = 1$ or 2. Therefore L is a nonreal function field of transcendence degree one over R, where R is some real closed subfield of k. By Lang's Conjecture, L is a C_1-field, and thus $k(x)$ is an S_1^{nr}-field.

$\Leftarrow$ Now suppose $k(x)$ is an S_1^{nr}-field for all k which are algebraically closed of characteristic zero. By the previous Lemma, to prove Lang's Conjecture for transcendence degree one, it is enough to show that $R(u, \sqrt{-1-u^2})$ is a C_1-field. Consider $R(u, \sqrt{-1-u^2})(i)$. This is the function field of a conic over an algebraically closed field, and so is rational. In other words,

$$R(u, \sqrt{-1-u^2})(i) \cong k(x)$$

where $k = R(i)$ is algebraically closed. Now $k(x)$ is an S_1^{nr}-field and $R(u, \sqrt{-1-u^2})$ is a nonreal subfield of codimension 2. Therefore $R(u, \sqrt{-1-u^2})$ is a C_1-field and Lang's Conjecture holds for transcendence degree one. ■

We will end this section with another interesting example of a polite field extension.

Theorem 24 *If k is an algebraically closed field, $k((s))(t)/k((s))$ is a polite extension.*

Proof Suppose $L \subseteq k((s))(t)$ with $[k((s))(t):L] < \infty$. Choose a normal closure F of $k((s))(t)/L$, and let K be the purely inseparable closure of L in F. Note that F/K is a finite Galois extension and K/L is finite purely inseparable extension. Then the algebraic closure of $k((s))$ in F is a finite extension of $k((s))$, and so is of the form $k((s'))$. Thus F is an algebraic function field of tr deg 1 over $k((s'))$.

Let $\sigma \in \operatorname{Aut}(F/L)$ and $f \in k((s'))$. We will show that $\sigma(f) \in k((s'))$ as well. First suppose that $f \in k[[s']]$ is a unit. Then f is an n-th power in $k((s'))$ for all n prime to the characteristic of k. Assume that $\sigma(f) \notin k((s'))$. Then $\sigma(f)$ is transcendental over $k((s'))$. Now $T^n - \sigma(f)$ is irreducible in $k((s'))(\sigma(f))[T]$ but has a root in F for any n prime to the characteristic of k. Thus $[F:k((s'))(\sigma(f))] > n$ for all such n, which is a contradiction. Hence $\sigma(f) \in k((s'))$. If $f \in k[[s']]$ is not a unit, then $f + 1 \in k[[s']]$ is a unit. Now repeating the last argument we have $\sigma(f) + 1 = \sigma(f + 1) \in k((s'))$ and so $\sigma(f) \in k((s'))$. Finally, if $f \notin k[[s']]$, then $f^{-1} \in k[[s']]$ and we can conclude that $\sigma(f^{-1}) \in k((s'))$ and hence $\sigma(f) \in k((s'))$. Therefore every $\sigma \in \operatorname{Aut}(F/L)$ restricts to a field embedding $k((s')) \to k((s'))$ over $k((s)) \subset L$. Since $k((s'))/k((s))$ is algebraic, $\sigma\big(k((s'))\big) = k((s'))$ for every $\sigma \in \operatorname{Aut}(F/L)$.

Now, just as in the argument of Theorem 16, $[k((s')):k((s')) \cap K] < \infty$. If $K = L$, then we're done. Otherwise the characteristic is $p > 0$ and there exists an $n \geq 1$ such that $K^{p^n} \subseteq L$ for some n. By Theorem 28 from the next section, we conclude $k((s')) \cap K = k_0((u))$ where $u \in k((s'))$ and k_0 is isomorphic to a finite codimension subfield of k. Since k is a perfect field, we can suppose $k_0 \subseteq k$ by the Corollary to Theorem 10 of [C], and, since k is algebraically closed of characteristic $p > 0$, we must have $k_0 = k$. Therefore $k((u^{p^n})) = (k((s')) \cap K)^{p^n} \subseteq k((s')) \cap L$. But $[k((s')):k((u^{p^n}))]$ is finite, and so $[k((s')):k((s')) \cap L]$ is finite. Since $k((s))$ is algebraically closed in $k((s))(t)$, then $k((s')) \cap L = k((s)) \cap L$ and so $k((s))(t)/k((s))$ is polite. ■

The following corollary is also of some interest.

Corollary 25 *If k is an algebraically closed field of characteristic $p > 0$ and $K = k((s))(t)$, then every finite codimension subfield $L \subseteq K$ is isomorphic to K.*

Proof Since $k((s))(t)/k((s))$ is polite, $[k((s)):k((s)) \cap L] < \infty$, and, as we saw in the last paragraph of the proof of Theorem 24, we can write $k((s)) \cap L = k((u))$ for some $u \in k((s))$. Thus L is a function field of genus 0 over $k((u))$, and so L is the function field of a conic. We now follow the argument of Example 2.10 in Chapter 5 of [P]. The homogeneous equation is of degree 2 in 3 variables, and since $k((u))$ is a C_1-field, there is a nontrivial zero. Then L contains a rational place, and therefore is a rational function field $L = k((u))(x)$ for some $x \in L$. ■

Corollary 26 *If k is an algebraically closed field of characteristic $p > 0$, then $k((s))(t)$ is S_2.*

2 Complete Fields

We are also able to establish going-down theorems in some cases for fields complete with respect to a discrete real valuation.

We recall the concept of the degree of imperfection of a field of positive characteristic; see, for example, [Be-Ma] and [Te]. If F is a field of characteristic $p > 0$, then F^p is a subfield of F and F/F^p is a purely inseparable extension. The degree of imperfection of F, denoted $\operatorname{doi}(F)$, is defined to be i where $p^i = [F:F^p]$, $i \geq 0$. (We allow $i = \infty$ when $[F:F^p]$ is infinite.) If F has characteristic 0, we set $\operatorname{doi}(F) = 0$. With this convention, F is perfect if and only if $\operatorname{doi}(F) = 0$. And, F has finite degree of imperfection if and only if the characteristic is 0 or the characteristic is $p > 0$ and $[F:F^p]$ is finite.

We rely on the following theorems, which appear in [Be-Mo].

Theorem 27 *If K is complete with respect to a real valuation, K is not algebraically closed, and $\operatorname{doi}(K)$ is finite, then every finite codimension subfield of K is closed and hence complete.*

Theorem 28 *Let $K = k((t))$ where $\operatorname{doi}(k) < \infty$. Then every finite codimension subfield of K is of the form $k_0((s))$ where k_0 is isomorphic to a finite codimension subfield of k and $s \in K$.*

Using this, we can prove that if k is an S_i-field, then $k((t))$ is an S_{i+1}-field. We require the following lemma.

Lemma 29 *If k is an S_i-field of characteristic $p > 0$, then $\operatorname{doi}(k) \leq i$.*

Proof Let $a_1, \dots, a_l$ be elements of k linearly independent over k^p. Then the form $\sum_{j=1}^{l} a_j X_j^p$ has degree p with l variables and only the trivial zero in k. Thus $l \leq p^i$. ■

Theorem 30 *If k is an S_i-field, then $K = k((t))$ is an S_{i+1}-field.*

Proof Since k is S_i, $\operatorname{doi}(k)$ is finite by the previous lemma. Let L be a finite codimension subfield of K. By our previous results, $L = k_0((s))$ where k_0 is a finite codimension subfield of k. Since k is S_i, k_0 is C_i and hence $L = k_0((s))$ is C_{i+1}. Therefore, K is S_{i+1}. ■

Corollary 31 *If k is an S_i^{nr}-field, then $k((t))$ is an S_{i+1}^{nr}-field.*

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Proof Let $L \subseteq k((t))$ be a nonreal subfield with $[k((t)):L] < \infty$. From the previous theorem, we know that $L = k'((u))$, with $[k:k'] < \infty$. Further, since $L = k'((u))$ is nonreal, then k' is nonreal by Example 1.2 of Chapter 3 in [P]. Since k is an S_i^{nr}-field, k' is a C_i-field, and so $L = k'((u))$ is a C_{i+1}-field. Therefore $k((t))$ is an S_{i+1}^{nr}-field. ■

Corollary 32 *If k is an algebraically closed field, then $k((x_1))\cdots((x_n))$ is an S_n^{nr}-field.*

Proof Since k is algebraically closed, it satisfies the condition for being a S_0^{nr}-field vacuously by a theorem of Artin–Schreier; see Chapter VI, Theorem 17 in [J]. The theorem follows by induction on n. ■

In light of Theorem 23, we note that Corollary 32 can be thought of as a local version of Lang's Conjecture in transcendence degree one. We can go a little further. It was shown in [L] that if K is complete with respect to a discrete real valuation, and with algebraically closed residue class field k, then K is a C_1-field. We can extend this theorem.

Theorem 33 *If K is complete with respect to a discrete real valuation v, with algebraically closed residue class field k, then K is an S_1^{nr}-field.*

Proof If K and k have the same characteristic, then K is a Laurent series field by the Cohen structure theorem, and the result follows from Corollary 32. So we may assume K has characteristic zero and k has characteristic $p > 0$. Suppose $L \subseteq K$ is a nonreal subfield with $[K:L] < \infty$. Then, by Theorem 27, L is a closed subfield of K, and hence is a complete field. As k is an extension of degree at most 2 of the residue class field of L and k is an algebraically closed field of positive characteristic, the residue field of L must also be k. So we see that L is complete with respect to a discrete real valuation and has algebraically closed residue field as well. Thus L is a C_1-field by Theorem 10 of [L], and so K is an S_1^{nr}-field. ■

3 Conjecture

We end with a conjecture.

Conjecture 34 *Every C_i-field is S_i^{nr}.*

The only evidence we have for this conjecture is a lack of counterexamples. We have seen that it implies Lang's Conjecture (Theorem 21) and that many reasonable classes of C_i-fields are S_i^{nr} (Corollaries 15, 20, 26, 32 and Theorem 33).

References

[Be-Ma] M. F. Becker and S. MacLane, *The Minimum Number of Generators for Inseparable Algebraic Extensions*. Bull. Amer. Math. Soc. **46**(1940), 182-186.

[Be-Mo] A. Bevelacqua and M. Motley, *Finite Codimension Subfields of a Field Complete with Respect to a Real Valuation*. Comm. Algebra, to appear.

[C] I. S. Cohen, *On the Structure and Ideal Theory of Complete Local Rings*. Trans. Amer. Math. Soc. (1) **59**(1946), 54–106.

[G] Marvin J. Greenberg, *Lectures on Forms in Many Variables*. Benjamin, 1969.
[J] Nathan Jacobson, *Lectures in Abstract Algebra Volume 3*. Van Nostrand, San Francisco, 1964.
[L] S. Lang, *On Quasi-Algebraic Closure*. Ann. of Math. **55**(1952), 373–390.
[N] Jurgen Neukirch, *Algebraic Number Theory*. Grundlehren der Mathematischen Wissenschaften **322**, Springer, 1999.
[P] Albrecht Pfister, *Quadratic Forms with Applications to Algebraic Geometry and Topology*. Cambridge University Press, 1995.
[S] J. P. Serre, *Local Fields*. Springer-Verlag, New York, 1979.
[Te] O. Teichmuller, *p-Algebren*. Deutsche Math. **1**(1936), 362–388.
[Ts] C. Tsen, *Zur Stufentheorie der Quasi-algebraisch-Abgeschlossenheit kommutativer Körper*. J. Chinese Math. Soc. **171**(1936), 81–92.
[Z-S] Oscar Zariski and Pierre Samuel, *Commutative Algebra Volume 2*. Springer-Verlag, 1960.

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