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POSITIVE DEPENDENCE OF EXCHANGEABLE SEQUENCES

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ABSTRACT. Infinite sequences of exchangeable binary random variables have strong positive dependence properties; in particular, we show they are strong FKG. If the infinite exchangeable sequence is allowed to have multiple values this is no longer true. Positive dependence conditions such as association still have natural application in this context. We establish necessary and sufficient conditions for an infinite exchangeable sequence to be associated. This result shows that exchangeable Polyà urn processes are associated. We also establish necessary and sufficient conditions for finite exchangeable sequences to be weakly associated. The match set distribution of a random permutation has recently been shown to be associated by an extensive analysis of cases. Our result easily yields the weak association of such distributions.

1. **Introduction.** This paper examines the positive dependence properties of exchangeable sequences of real random variables. Infinite sequences of exchangeable binary random variables are shown to have strong positive dependence properties; in particular, we show they are strong FKG (Theorem 2.1). If an infinite exchangeable sequence is allowed to have multiple values this is no longer true. Weaker positive dependence conditions, such as association, still may be applied in this context. We establish necessary and sufficient conditions for an infinite exchangeable sequence to be associated (Theorem 3.2). This result shows that exchangeable Polyà urn processes are associated. Finite exchangeable sequences are distinctly different from infinite exchangeable sequences. The conditions of Theorem 3.2 are necessary and sufficient conditions for finite exchangeable sequences to be weakly associated. The match set distribution of a random permutation has recently been shown to be associated by an extensive analysis of cases. Our result easily yields the weak association of such distributions. We start by stating the basic definitions of exchangeability and of positive dependence.

DEFINITION 1.1. A stochastic process $\{X_i : i \in I\}$ (where I is either $\mathbb{N}$, the set of positive integers or the finite set $\{1, 2, \ldots, N\}$) is *exchangeable* if its distribution is invariant under all permutations of the index set. If $I = \mathbb{N}$ then the set of exchangeable distributions is a convex set and its extreme points are the $i.i.d.$ processes. Thus, according to deFinetti's Theorem [5], an infinite exchangeable stochastic process is a mixture of $i.i.d.$ processes. In particular, we can define a tail field random variable, Θ, so that, conditional on Θ, $\{X_i : i \geq 1\}$ is an $i.i.d.$ sequence of random variables.

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DEFINITION 1.2 (SEE VAN DEN BERG AND BURTON [2] OR KEMPERMAN [9]). A stochastic process $\{X_i : i \in I\}$ is *strong* FKG if the random variables have discrete distributions and for all n and $\alpha, \beta \in \mathbb{R}^n$ we have

$$P[\mathbf{X}_n > \alpha \vee \beta]P[\mathbf{X}_n > \alpha \wedge \beta] \geq P[\mathbf{X}_n \geq \alpha]P[\mathbf{X}_n \geq \beta].$$

Here $\mathbf{X}_n = \langle X_1, \ldots, X_n \rangle$ and all operations are taken coordinate-wise. Of course, we require that $n \in I$.

DEFINITION 1.3 (SEE BARLOW AND PROSCHAN [1]). A stochastic process $\{X_i : i \in I\}$ is *associated* if for all $n \geq 1$ and for all coordinate-wise non-decreasing functions $f, g \colon \mathbb{R}^n \longrightarrow \mathbb{R}$ we have

$$\operatorname{Cov}\bigl(f(X_1, X_2, \ldots, X_n), g(X_1, X_2, \ldots, X_n)\bigr) \geq 0.$$

DEFINITION 1.4 (BURTON, DABROWSKI AND DEHLING [4], SEE ALSO C. NEWMAN [10]). A process, $\{X_i : i \in I\}$, is *weakly associated*, if for every pair of coordinate-wise non-decreasing functions, $f_1 \colon \mathbb{R}^k \longrightarrow \mathbb{R}$ and $f_2 \colon \mathbb{R}^l \longrightarrow \mathbb{R}$, with $\operatorname{Var}\bigl(f_i(\mathbf{X}_1, \mathbf{X}_2)\bigr) < +\infty$ we have that $\operatorname{Cov}\bigl(f_1(\mathbf{X}_1), f_2(\mathbf{X}_2)\bigr) \geq 0$. Here $\mathbf{X}_1 = \langle X_{i(1)}, X_{i(2)}, \ldots, X_{i(k)} \rangle$ and $\mathbf{X}_2 = \langle X_{i(k+1)}, X_{i(k+2)}, \ldots, X_{i(k+l)} \rangle$ are defined over two disjoint sets of indices.

DEFINITION 1.5 (SEE JOAG-DEV, SHEPP AND VITALI [7]). A stochastic process $\{X_i : i \in I\}$ is *positively quadrant dependent* (PQD) if we restrict f and g in the definition of weak association to the indicators of events of the form $\{X_{i(1)} > u_1, X_{i(2)} > u_2, \ldots, X_{i(k)} > u_k\}$.

The above definitions are listed in decreasing order of stringency. A large number of positive dependence conditions are discussed in the literature. For further background we suggest the book by Barlow and Proschan [1] and the monograph edited by Tong [11]. We will show in Theorem 2.1 that an infinite exchangeable binary sequence is strong FKG. The following example shows that a 3-valued exchangeable sequence is not necessarily associated, nor even PQD.

EXAMPLE 1.6. Let Y be a Bernoulli random variable with parameter $p = .5$. Let $\{X_i : i \in I\}$ be a sequence of exchangeable real random variables such that, conditional on Y, $\{X_i : i \geq 1\}$ is a sequence of independent and identically distributed random variables with $P[X = 0] = P[X = 2] = .5$ if $Y = 0$ and $P[X = 1] = 1$ if $Y = 1$. Let $I\{A\}$ be the indicator of the event A, and define $f_1 = I\bigl\{X_1 \in \{1,2\}\bigr\}$ and $f_2 = I\bigl\{X_2 \in \{2\}\bigr\}$. These are non-decreasing functions in X_1 and X_2. Then X_1 and X_2 are not even PQD because $\operatorname{Cov}(f_1, f_2) = P[X_1 \in \{1,2\}, X_2 \in \{2\}] - P[X_1 \in \{1,2\}]P[X_2 \in \{2\}] = -1/16 < 0$.

Theorem 3.2 states that association of an infinite exchangeable sequence is equivalent to the PQD property for all non-decreasing $\{0,1,2\}$-valued images of the original sequence. We can check this condition for an important class of processes, exchangeable Polyà urn models.

DEFINITION 1.7. Consider an urn containing a finite number of balls of various colors. Each color is assigned a distinct real value. Let h be a fixed positive integer. Mix the balls, remove one, note the value of its color, replace it in the urn, and add h more of this color. By continuing this process we generate a sequence, $\{X_i : i \geq 1\}$, of random variables, where X_i is the value assigned to the ith ball drawn from the urn. These random variables are exchangeable by a straightforward calculation. Such a process will be called an *exchangeable Polyà process*.

EXAMPLE 1.8. Let the urn have three balls labelled 0, 1, 2 and let $h = 1$. If $i \neq j$ then $P[X_1 = i, X_2 = j] = 1/12$ and $P[X_1 = X_2 = i] = 1/6$. Thus $P[X_1 = 2, X_2 = 1]P[X_1 = 1, X_2 = 0] < P[X_1 = X_2 = 1]P[X_1 = 2, X_2 = 0]$. So generally exchangeable Polyà processes with more than two colors are not strong FKG. Theorem 5.1, below, states that such processes are associated.

SUMMARY OF RESULTS AND GUIDE TO PAPER. Section 2 proves a theorem stating that $\{0, 1\}$-valued infinite exchangeable sequences are strong FKG. Given such a strong positive dependence property in the 2-valued case, similar properties in the general exchangeable case are to be expected. In Section 3 we show that association of an infinite exchangeable sequence is equivalent to verification of PQD for $\{g(X_i) : i \geq 1\}$, for all $g: \mathbb{R} \longrightarrow \{0, 1, 2\}$ where g is a non-decreasing function. The same proof yields similar necessary and sufficient conditions for a finite exchangeable sequence to be weakly associated. In Section 4 we examine $\{0, 1, 2\}$-valued infinite exchangeable sequences in more detail. Such a sequence is associated if its structural deFinetti measure belongs to a particular subset of measures, $\mathcal{A}$. We state a few properties of that set which are useful in computations. In Section 5 we apply our results to exchangeable Polyà urn models to show that all such models are associated. As an application of our results to finite exchangeable sequences, we provide a short demonstration that match set distributions are weakly associated. Fishburn, Doyle and Shepp [6] have shown that match set distributions are associated by coupling arguments and a lengthy examination of cases. Our technique provides a large part of this result with much less work.

2. $\{0, 1\}$-valued infinite exchangeable sequences.

Infinite exchangeable sequences taking on just two values are discussed in Hill, Lane and Sudderth [8]. They show that such sequences are essentially 2-color Polyà urn models. Here we show that these sequences have the strongest positive dependence property in common use.

THEOREM 2.1. *A two-valued infinite exchangeable sequence is strong* FKG.

PROOF. Let $\{X_1, X_2, \ldots\}$ be exchangeable and without loss of generality we assume that X_i has values in $\{0, 1\}$. It follows from exchangeability, and from van den Berg and Burton [2] or Kemperman [9], that $\{X_1, X_2, \ldots\}$ is strong FKG if and only if for all N and all k, $0 \leq k \leq N$, the conditional distribution of $\{X_{N+1}, X_{N+2}, \ldots\}$ given $\sum_{i=1}^{N} X_i = k$ is associated. We must verify this last condition.

In this case the sequence of conditional random variables is itself $\{0, 1\}$-valued and exchangeable. DeFinetti's representation shows there is a tail field random variable Θ

with values in $[0,1]$ so that conditioned on Θ, $\mathbb{E}[X_N = 1 \mid \sum_{i=1}^N X_i = k]$ are *i.i.d.* Bernoulli (Θ) random variables. The distribution function for Θ is given by

$$\int_0^x p^k(1-p)^{N-k}\,dF(p) \Big/ \int_0^1 p^k(1-p)^{N-k}\,dF(p)$$

where F is the distribution function of the tail field random variable of the original $\{X_i : i \geq 1\}$. Let $\{Y_1, Y_2, \ldots\}$ be *i.i.d.* uniform random variables on $[0,1]$ that are independent of Θ. Then we have the conditional sequence $\{\mathbb{E}[X_{N+i} = 1 \mid \sum_{i=1}^N X_i = k] : i \geq 1\}$ has the same distribution as $\{I\{Y_1 + \Theta > 1\}, I\{Y_2 + \Theta > 1\}, \ldots\}$ and so must be associated because these variables are increasing functions of $\{\Theta, Y_1, Y_2, \ldots\}$, an associated sequence of random variables. ∎

3. **Association in terms of 3-valued processes.** Let $\{X_i : i \geq 1\}$ be an infinite exchangeable sequence. We have seen in Section 2 that if the X_i are $\{0,1\}$-valued, then this sequence has the strong FKG property. Further, Examples 1.6 and 1.8 show that for a $\{0,1,2\}$-valued $\{X_i : i \geq 1\}$, we need no longer have the strong FKG property, nor even the PQD property. Simple conditions for the strong FKG property are difficult, as this condition is not easily linked to exchangeability. The condition of association, however, does permit further analysis. That is the topic of this section.

Some of our conditions for association will be expressed in terms of a specific class of functions. Let π be a permutation of $\{1,2,\ldots,k\}$, and denote by $\mathbf{X}_1$ and $\mathbf{X}_{\pi(1)}$ the vectors $\langle X_1, X_2, \ldots, X_k\rangle$, and $\langle X_{\pi(1)}, X_{\pi(2)}, \ldots, X_{\pi(k)}\rangle$.

DEFINITION 3.1. A function $g\colon \mathbb{R}^k \longrightarrow \mathbb{R}$ is *permutation-symmetric* if $g(\mathbf{X}) = g(\mathbf{X}_\pi)$ for all $\mathbf{X} \in \mathbb{R}^k$ and permutation π of $\{1,2,\ldots,k\}$. A set $\mathbb{A}$ is permutation-symmetric if $I_{\mathbb{A}}(x)$ is a permutation-symmetric function.

THEOREM 3.2. *Consider an infinite exchangeable sequence $\{X_i : i \geq 1\}$ as above. Let $\mathbf{X}_1$ and $\mathbf{X}_2$ denote arbitrary vectors defined on disjoint sets of random variables, say on $\{X_1, \ldots, X_k\}$ and $\{X_{k+1}, \ldots, X_{k+l}\}$. The following are equivalent:*

(a) $\operatorname{Cov}\big(I_{A_1}(\mathbf{X}_1), I_{A_2}(\mathbf{X}_2)\big) \geq 0$ *for all permutation-invariant sets A_1 and A_2 such that I_{A_1} and I_{A_2} are non-decreasing in each coordinate.*

(b) $\operatorname{Cov}\big(g_1(\mathbf{X}_1), g_2(\mathbf{X}_2)\big) \geq 0$ *for all pairs of functions g_1 and g_2 with representation*

$$g_i(X_i) = \sum_{j=1}^{n_i} a_{ji} I_{A_{ji}}(\mathbf{X}_i) \text{ for } i = 1,2$$

where $a_{ji} \geq 0$ for $i = 1,2$ and $j = 1,2,\ldots, A_{1i}, A_{2i}, \ldots, A_{n_i i}$ is a decreasing sequence of sets, $A_{11} = \mathbb{R}^k$, $A_{12} = \mathbb{R}^l$, and the A_{ji} are permutation-invariant sets.

(c) $\operatorname{Cov}\big(g_1(\mathbf{X}_1), g_2(\mathbf{X}_2)\big) \geq 0$ *for all pairs of functions g_1, g_2 which are applied to disjoint sets of random variables, and where each g has a representation as in (3.2b), but where the a_{ji} may take on any real value.*

(d) For all coordinate-wise non-decreasing functions $g_1\colon \mathbb{R}^k \longrightarrow \mathbb{R}$, $g_2\colon \mathbb{R}^l \longrightarrow \mathbb{R}$, which are permutation symmetric, which are applied to disjoint sets of random variables, and which then have finite variance, we have that $\operatorname{Cov}\big(g_1(\mathbf{X}_1), g_2(\mathbf{X}_2)\big) \geq 0$.

(e) The sequence $\{X_i : i \geq 1\}$ is weakly associated.

(f) The sequence $\{X_i : i \geq 1\}$ is associated.

PROOF. That (a) $\Leftrightarrow$ (b) is simple. Let g be as in (c). Set $\tilde{g} = g + |\min\{g(x)\}|$ and note that $\mathrm{Cov}\big(g_1(\mathbf{X}_1), g_2(\mathbf{X}_2)\big) = \mathrm{Cov}\big(\tilde{g}_1(\mathbf{X}_1), \tilde{g}_2(\mathbf{X}_2)\big)$. This shows (b) $\Leftrightarrow$ (c). Clearly (d) $\Rightarrow$ (c). For (c) $\Rightarrow$ (d), note that the measurable coordinate-wise non-decreasing function on $\mathbb{R}^m$ are limits of the functions described in (c). In fact, the functions in (c) may be taken to converge monotonically (either increasing or decreasing) to the appropriate limit. If the limit is L_2, then the approximation of (c) can also can be taken in L_2. Clearly (e) implies (d).

Assume (d). Exchangeability requires that $(\mathbf{X}_1, \mathbf{X}_2)$ have the same joint distribution as $(\mathbf{X}_{\pi(1)}, \mathbf{X}_{\pi(2)})$. Here the permutations π are taken over $\{X_1, \ldots, X_k\}$ and $\{X_{k+1}, \ldots, X_{k+l}\}$ separately. Hence for any permutation $\pi_{1,2}$, we must have that

$$\mathrm{Cov}\big(f_1(\mathbf{X}_{\pi(1)}), f_2(\mathbf{X}_{\pi(2)})\big) = \mathrm{Cov}\big(f_1(\mathbf{X}_1), f_2(\mathbf{X}_2)\big) \geq 0.$$

Let $\sum_1$ ($\sum_2$) denote a sum over all permutations of the indices $\{1, 2, \ldots, k\}$ (respectively $\{k+1, k+2, \ldots, k+l\}$) and define $g_i(\mathbf{X}_i) = \sum_i f_i(\mathbf{X}_{\pi(i)})$, for $i = 1$ or 2. These g_i: $\mathbb{R}^m \longrightarrow \mathbb{R}$ are non-decreasing. By exchangeability,

$$\begin{aligned}\mathrm{Cov}\big(g_1(\mathbf{X}_1), g_2(\mathbf{X}_2)\big) &= \textstyle\sum_1 \sum_2 \mathrm{Cov}\big(f_1(\mathbf{X}_{\pi(1)}), f_2(\mathbf{X}_{\pi(2)})\big) \\ &= k!\, l!\, \mathrm{Cov}\big(f_1(\mathbf{X}_1), f_2(\mathbf{X}_2)\big).\end{aligned}$$

This yields (e). Since (f) implies (e), it remains to show the reverse implication. Consider f, and g non-decreasing in each coordinate, and consider

$$\mathbb{E}\{f(X_1, \ldots, X_n) \cdot g(X_1, \ldots, X_n)\} - \mathbb{E}\{f(X_1, \ldots, X_n)\} \cdot \mathbb{E}\{g(X_1, \ldots, X_n)\}.$$

We need only look at simple functions f, and g. We proceed to general functions by standard limiting arguments as in (c) $\Rightarrow$ (d). Suppose $f(X_1, \ldots, X_n) = I[X_1 > f_1, \ldots, X_n > f_n]$ and $g(X_1, \ldots, X_n) = I[X_1 > g_1, \ldots, X_n > g_n]$ for some constants $f_j, g_j \in \mathbb{R} \cup \{-\infty\}$. Let us denote $I[X_i > f_i]$ by ϕ_i and $I[X_i > g_i]$ by γ_i. By exchangeability, wthout loss of generality, we can write $f(X_1, \ldots, X_n) = \prod_{i=1}^n \phi_i$ and $g(X_1, \ldots, X_n) = \prod_{i=1}^n \gamma_i$. Then if the variables $\{X_i\}$ are conditionally independent and indentically distributed with conditional distribution function F, we have that

$$\begin{aligned}\mathbb{E}f(X_1, \ldots, X_n)g(X_1, \ldots, . X_n) &= \mathbb{E}\left[\prod_{i=1}^n \phi_i\gamma_i\right] = \int \mathbb{E}_F\left[\prod_{i=1}^n \phi_i\gamma_i\right] dM(F) \\ &= \int \prod_{i=1}^n \mathbb{E}_F[\phi_i\gamma_i]\, dM(F) \\ &\geq \int \prod_{i=1}^n \mathbb{E}_F[\phi_i]\mathbb{E}_F[\gamma_i]\, dM(F) \\ &= \int \prod_{i=1}^n \mathbb{E}_F[\phi_i]\mathbb{E}_F[\gamma_{i+n}]\, dM(F) \\ &= \mathbb{E}f(X_1, \ldots, X_n)g(X_{1+n}, \ldots, X_{2n}).\end{aligned}$$

The desired inequality now follows from the assumed weak association. ∎

REMARK 3.3. Note that the condition (3.2a) is equivalent to the following simple statement. For any choice of positive integers l and k, define $\mathbf{X}_1$, $\mathbf{X}_2$ as before, and for any 'non-decreasing' choice of permutation-symmetric sets A_1, A_2 we have

$$\mathbb{P}[\langle \mathbf{X}_1, \mathbf{X}_2 \rangle \in A_1 \times A_2] \geq \mathbb{P}[\mathbf{X}_1 \in A_1]\mathbb{P}[\mathbf{X}_2 \in A_2].$$

Moreover, the sequence $\{X_i : i \geq 1\}$ is associated iff $\{g(X_i) : i \geq 1\}$ is associated for any non-decreasing function $g: \mathbb{R} \longrightarrow \mathbb{R}$. Consequently if we look only at $\{0, 1, 2\}$-valued functions g, then this suffices for the association of $\{X_i : i \geq 1\}$. Furthermore, the transform $g' = 2 - g$ allows us to consider only those sets A_i which count the numbers of variables $g(X_i)$ which exceed 0 and exceed 1. If we consider finite exchangeable sequences, then the arguments proving the equivalence of conditions (a) through (e) of Theorem 3.2 still hold. The only place where we used an infinite index set was to show the equivalence of (e) and (f). We summarize the conclusions of this paragraph in the following theorem.

THEOREM 3.4. *Let $\{X_i : i \geq 1\}$ be an infinite (finite) exchangeable sequence of real-valued random variables. Then $\{X_i : i \geq 1\}$ is (weakly) associated iff for any non-decreasing function $g: \mathbb{R} \longrightarrow \{0, 1, 2\}$, and for any negative integers k, l, m and n, $f(k+m, l+n) \geq f(k,l) \cdot f(m,n)$. Here $f(i,j) = P[g(X_1) \geq 1, \ldots, g(X_i) \geq 1, g(X_{i+1}) \geq 2, g(X_{i+j}) \geq 2,]$.*

REMARK 3.5. The equivalence of weak association and association as given in the proof of Theorem 3.2 depends on being able to 'slide' a finite piece of the sequence into the future. If the sequence is finite, this will not be possible in all circumstances. We conjecture that for a finite exchangeable sequence which cannot be embedded into a longer exchangeable sequence, weak association does not necessarily imply association.

4. **Infinite exchangeable sequences.** The proof of Theorem 3.2 used only the permutation invariance property of exchangeable sequences. In the case of infinite exchangeable sequences, we will exploit DeFinetti's expression of an infinite exchangeable sequence as a conditionally *i.i.d.* sequence, and explore further the question as to when an exchangeable sequence is associated. We will maintain the following notation. The random variable M takes its values in the space, $\mathcal{M}$, of probability measures on $\mathbb{R}$. The space $\mathcal{M}$ has the topology induced by the Prohorov metric, and is partially ordered by the stochastic ordering, $\ll$, of probability measures. Given $M = m$, the sequence $\{X_i : i \geq 1\}$ is independent and identically distributed with common law m.

Section 4.1 looks at the case where M is concentrated on a totally ordered subset of $\mathcal{M}$. Theorem 3.2 indicates that the association of an infinite exchangeable sequence reduces to the question of the association of $\{0, 1, 2\}$-valued images of that sequence. Section 4.2 will examine those cases.

4.1 *Totally ordered case.* A function $f\colon \mathcal{M} \mapsto \mathbb{R}$ is *non-decreasing* if for $\xi, \eta \in \mathcal{M}$, $\xi \ll \eta$ implies $f(\xi) \le f(\eta)$. Suppose M is concentrated on a totally ordered subset of $\mathcal{M}$. For $\mathbf{X}_1, \mathbf{X}_2, A_1$, and A_2 as in Theorem 3.2,

$$\begin{aligned}\mathbb{P}[\mathbf{X}_1 \in A_1, \mathbf{X}_2 \in A_2] &= \int_{\mathcal{M}} \mathbb{P}[\mathbf{X}_1 \in A_1 \mid M] \cdot \mathbb{P}[\mathbf{X}_2 \in A_2 \mid M]\, d\mathcal{L}(M) \\ &= \int_{\mathcal{M}} f_1(M) f_2(M)\, d\mathcal{L}(M) \\ &\ge \int_{\mathcal{M}} f_1(M)\, d\mathcal{L}(M) \int_{\mathcal{M}} f_2(M)\, d\mathcal{L}(M) \\ &= \mathbb{P}[\mathbf{X}_1 \in A_1]\mathbb{P}[\mathbf{X}_2 \in A_2].\end{aligned}$$

The inequality above follows from the fact that a singleton, *e.g.* $\{M\}$, is always associated. Association follows by Theorem 3.2 and we have the following.

THEOREM 4.1.1. *Let an infinite real exchangeable sequence be represented as conditionally independent sequences given M, where M is a random variable taking values in the space of probability measures, $\mathcal{M}$. If M takes values (with probability one) in a totally ordered subset of $\mathcal{M}$, then the sequence is associated.*

Since the set of probability measures on $\{0,1\}$ is totally stochastically ordered, $\{0,1\}$-valued exchangeable sequences are automatically associated (even strong FKG by Theorem 2.1). However, there are $\{0,1,2\}$-valued exchangeable sequences which are associated but for which M is not concentrated on a totally ordered subset of $\mathcal{M}$.

DEFINITION 4.1.2. Define $m(p,q)$ to be the probability measure that puts mass p on 0, q on 1 and $1-p-q$ on 2.

EXAMPLE 4.1.3. Let M take on values $m(.5,.5)$, $m(.5,0)$, $m(0,.5)$ and $m(\frac{1}{3},\frac{1}{3})$ with probability .25 each. By an induction argument on k, l, m, and n in Theorem 3.4, we can show that this defines an associated $\{0,1,2\}$-valued exchangeable sequence. M is not concentrated on a totally ordered set.

EXAMPLE 4.1.4. Let M take on values $m(p, 1-2p)$ uniformly for $p \in [0,.5]$. By direct calculations in Theorem 3.4, we can show that this defines an associated $\{0,1,2\}$-valued exchangeable sequence. M is not concentrated on a totally ordered set.

EXAMPLE 4.1.5. Let M take on values m_1 and m_2 with probability .5 each. Suppose these two measures are incomparable. By direct calculations as in Example 1.6, this defines a $\{0,1,2\}$-valued exchangeable sequence which is not associated.

4.2 $\{0,1,2\}$*-valued exchangeable sequences.* Let $\mathcal{M} = \{m(p,q)\}$ now denote the set of probability measures on $\{0,1,2\}$, and let $\mathbb{M}$ denote the space of Borel probability measures on $\mathcal{M}$. We will often identify an $M \in \mathbb{M}$ with an $\mathcal{M}$-valued random variable, M whose distribution is M. Of course, this abuse is common in probability. We want to determine exactly when the $\{0,1,2\}$-valued sequence, $\{X_i : i \ge 1\}$, is associated. For this sequence define (for f_M defined by f as in Theorem 3.4)

$$\mathcal{A}(i,j,k,l) = [M \in \mathbb{M} : f_M(i+k, j+l) \ge f_M(i,j) \cdot f_M(k,l)], \text{ and}$$
$$\mathcal{A} = \bigcap_{ijkl} \mathcal{A}(i,j,k,l).$$

Clearly a sequence $\{X_i : i \geq 1\}$ is associated if and only if its structural measure, M, belongs to $\mathcal{A}$. The topology on $\mathcal{M}$ may be identified with the Euclidean topology on $\Delta = \{(p,q,r) \in [0,1] : p+q+r = 1\}$. The topology on $\mathbb{M}$ is generated by the Prohorov metric. Then each $\mathcal{A}(i,j,k,l)$ is a closed set. Consequently $\mathcal{A}$ is a closed set as well.

Since $M \equiv m(p,q)$ leads to independent, identically distributed random variables, each such 'pure' measure belongs to $\mathcal{A}$. If $f_M(i+k,j+l) = f_M(i,j)\cdot f_M(k,l)$ for all choices of the indices, the M must be a 'pure' measure. If M is concentrated on a totally ordered subset of $\mathcal{M}$, then Section 4.1 implies that it belongs to $\mathcal{A}$. Examples 4.1.3 and 4.1.4 show that $\mathcal{A}$ contains M which put mass on incomparable $m(p,q)$. If $\{X_i : i \geq 1\}$, is associated, then $\{(2-X_i) : i \geq 1\}$ is also associated. Consequently if $M \in \mathcal{A}$, then its skew-symmetric image defined by the above transform, M^*, also belongs to $\mathcal{A}$. However, not every skew-symmetric M is in $\mathcal{A}$. For example take M equal to $m(0,1)$ and $m(.5,0)$ with probability .5 each.

Consider M and N in $\mathcal{A}$, $\alpha \in [0,1]$, and define $P = M$ with probability α, and $P = N$ with probability $(1-\alpha)$. Then $L(\alpha) = f_P(i+k,j+l) - f_P(i,j)\cdot f_P(k,l)$ is a quadratic function in α, and we may easily specify whether or not $L(\alpha)$ is non-negative for all $\alpha \in [0,1]$ in terms of $L(0)$, $L(1)$, and $[f_M(i,j) - f_M(i,j)]\cdot[f_N(k,l) - f_N(k,l)]$. For example, if this last product is non-negative, then $L(\alpha)$ is non-negative for $\alpha \in [0,1]$. This type of calculation shows that no linear combination of $m(0,1)$ and $m(.5,0)$ is associated. Consequently, $\mathcal{A}$ is not convex. Since the measure of Example 4.1.3 can be expressed as the average of two totally ordered measures, each of which is the skew-symmetric image of the other, we obtain the association of that example from the above reasoning. It is not difficult to find other examples for other cases. A similar reasoning may also be applied to the complement of $\mathcal{A}$ to show that it has properties similar to those of $\mathcal{A}$.

5. **Applications.** Theorem 2.1 shows that 2-valued Polyà urns are strong FKG. Example 1.8 shows we cannot expect the strong FKG property to hold for more general Polyà processes. It is difficult to apply the geometric characterization of the structural measure given by Theorem 3.2 directly. However, the reduction to 3-valued processes in Theorem 3.4 is useful. We show the following result.

THEOREM 5.1. *Every exchangeable Polyà process is associated.*

PROOF. Because of Theorems 3.4 and 2.1 we may assume the process has three colors labelled 0, 1, 2. Suppose there are N balls labelled 2, M balls labelled 1 and L balls labelled 0. We let

$$\begin{aligned}\bar{f}(n,m) &= P[X_1 = X_2 = \cdots = X_n = 2; X_{n+1} = \cdots X_{n+m} = 1] \\ &= \frac{N(N+h)\cdots\big(N+(n-1)h\big)M(M+h)\cdots\big(M+(m-1)h\big)}{(N+M+L)(N+M+L+h)\cdots\big(N+M+L+(n+m-1)h\big)}.\end{aligned}$$

Let $x = N/h$, $y = M/h$ and $t = (N+M+L)/h$. Then $\bar{f}(n,m) = (x)_n(y)_m/(t)_{n+m}$, where we use the notation $(x)_n = x(x+1)\cdots(x+n-1)$. Let

$$f(n,m) = P[X_1 = \cdots X_n = 2; X_{n+1} \geq 1, \ldots, X_{n+m} \geq 1]$$
$$= \sum_{k=0}^{m} \binom{m}{k} \bar{f}(n+k, m-k) = \sum_{k=0}^{m} \binom{m}{k} \frac{(x)_{n+k}(y)_{m-k}}{(t)_{n+m}}.$$

To check association we need to check $f(n+s, m+r) \geq f(n,m)f(s,r)$ for all n, m, s, r. This amounts to showing

$$\frac{\sum_{k=0}^{m+r} \binom{m+r}{k} (x)_{n+s+k}(y)_{m+r-k}}{\sum_{i=0}^{m} \sum_{j=0}^{r} \binom{m}{i}\binom{r}{j} (x)_{n+i}(y)_{m-i}(x)_{s+j}(y)_{r-j}} \geq \frac{(t)_{n+m+s+r}}{(t)_{n+m}(t)_{s+r}}.$$

Now $t \geq x+y$ and the right hand side is equal to

$$\left(1 - \frac{n+m}{t}\right)\left(1 - \frac{n+m+1}{t+1}\right)\cdots\left(1 - \frac{n+m+s+r}{t+2}\right)$$

which clearly decreases as t increases. Thus the left hand side is largest when $t = x+y$. For $t = x+y$ the inequality is clearly true because this is the situation of a two color exchangeable process which is strong FKG and thus associated. Note that even though the process is not in general strong FKG this argument shows that it is some sense "more" associated than the two state process. ∎

Now we show that match set distributions are weakly associated. We let $N > 1$ and choose a premutation $\sigma: \{1, \ldots, N\} \longrightarrow \{1, \ldots, N\}$ with uniform probability. Consider the random variables $X_1, \ldots, X_N$ where $X_i = I\{\sigma(i) = i\}$ is the indicator of the event that i is fixed under the permutation. These random variables are clearly exchangeable. Fishburn, Doyle and Shepp [6] have shown by a lengthy examination of cases and coupling arguments that the above random variables are associated. Theorem 3.4 can be easily employed to show the random variables are weakly associated. The fact that the proof of association is so much more difficult in this case provides "evidence" that there are random variables that are exchangeable, weakly associated but not associated.

THEOREM 5.2. *The match set random variables* $(X_1, \ldots, X_N)$ *are weakly associated.*

PROOF. By Theorem 3.4 we need only check that for all non-decreasing functions $g: \{0,1\} \longrightarrow \{0,1,2\}$ and for any nonnegative integers k, l, m and n with $k+l+m+n \leq N$ that $f(k+m, l+n) \geq f(k,l) \cdot f(m,n)$ where

$$f(i,j) = P[g(X_1) \geq 1, \ldots, g(X_i) \geq 1, g(X_{i+1}) \geq 2, g(X_{i+j}) \geq 2,].$$

We take the case where g is the identity. The other cases are similar. In this case, the condition is vacuous unless $l = n = 0$. Now $f(i,0) = (N-i)!/N!$ and the condition easily follows. ∎

REMARK 5.3. Our final remark concerns the central limit theorem for associated and exchangeable infinite sequences. If $\{X_i : i \geq 1\}$ is an infinite exchangeable sequence, the results of Blum *et al.* [3] state that the central limit theorem (with a standard normal limit) holds if and only if $\mathbb{E}X_1 = 0, \mathbb{E}X_1^2 = 1, \mathbb{E}X_1X_j = 0$, and $\mathbb{E}(X_1^2 - 1)(X_j^2 - 1) = 1$. The first and third conditions imply that all the lag covariances, $\mathrm{Cov}(X_1, X_j)\, j \geq 1$, must be 0. For an associated sequence, this requirement implies that the sequence is, in fact, an *i. i. d.* sequence. In particular, the Polyà urn models discussed earlier cannot satisfy the central limit theorem.

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