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NOWHERE-ZERO 3-FLOWS IN TWO FAMILIES OF VERTEX-TRANSITIVE GRAPHS

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Abstract

Let Γ be a graph of valency at least four whose automorphism group contains a minimally vertex-transitive subgroup G. It is proved that Γ admits a nowhere-zero 3-flow if one of the following two conditions holds: (i) Γ is of order twice an odd number and G contains a central involution; (ii) G is a direct product of a 2-subgroup and a subgroup of odd order.

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1. Introduction

Graphs considered in this paper are finite, undirected, loopless, but allowed to have multiple edges. Let Γ be a graph. As usual, we use $V(\Gamma)$ and $E(\Gamma)$ to denote the vertex set and edge set of Γ, respectively. An *orientation* $\mathcal{D}$ of Γ is an assignment of one of the two possible orientations for every $e \in E(\Gamma)$. Let φ be a mapping from $E(\Gamma)$ to the set of integers and k a positive integer. For every $v \in V(G)$, we use $\varphi^+(v)$ to denote the sum of values $\varphi(e)$ of edges e with orientation originating from v and $\varphi^-(v)$ the sum of values $\varphi(e)$ of edges e with orientation pointing to v. If $-k < \varphi(e) < k$ for every $e \in E(\Gamma)$ and $\varphi^+(v) = \varphi^-(v)$ for every $v \in V(\Gamma)$, then we call the ordered pair $(\mathcal{D}, \varphi)$ a *k-flow* of Γ. If further $\varphi(e) \neq 0$ for every $e \in E(\Gamma)$, then we call $(\mathcal{D}, \varphi)$ a *nowhere-zero k-flow* of Γ. For convenience, we use $\mathcal{NZ}_k$ to denote the family of graphs which admit a nowhere-zero k-flow.

Tutte proposed three conjectures in the middle of the last century on integer flows which are still unsolved, namely the 5-flow, 4-flow and 3-flow conjectures. The 3-flow conjecture (see, for example, [16, Conjecture 1.1.8]) is stated as follows: every 4-edge-connected graph is contained in $\mathcal{NZ}_3$. By the equivalent version of the 3-flow conjecture given by Kochol [6], it suffices to prove this conjecture for 5-edge-connected graphs. However, it was conjectured by Jaeger [5] that every

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k-edge-connected graph is contained in $\mathcal{NZ}_3$ for some given positive integer k. This so called weak 3-flow conjecture was solved by Thomassen [14] who proved that the statement holds for an 8-edge-connected graph. Lovász *et al.* [8] improved this breakthrough by proving that the statement of the weak 3-flow conjecture is true when $k = 6$. However, the 3-flow conjecture remains wide open for 5-edge-connected graphs. In this situation, it is natural to attempt to verify the conjecture for interesting families of graphs, for example, vertex-transitive graphs.

A graph Γ is *vertex-transitive* if its automorphism group $\mathrm{Aut}(\Gamma)$ acts transitively on $V(\Gamma)$. A subgroup G of $\mathrm{Aut}(\Gamma)$ is said to be *minimally vertex-transitive* if G is transitive on $V(\Gamma)$, but any proper subgroup of G is intransitive on $V(\Gamma)$. In particular, if G acts regularly (transitively and every nontrivial element fixes no vertex) on $V(\Gamma)$, then Γ is called a *Cayley graph* on G. A graph is said to be *k-regular* (or *regular* for short) if each of its vertices has valency k where k is a positive integer. It is obvious that every vertex-transitive graph is regular. In [9], it was proved that the edge connectivity of a connected vertex-transitive simple graph is equal to its valency. Thus, the 3-flow conjecture for vertex-transitive graphs asserts that every vertex-transitive simple graph of valency at least four is contained in $\mathcal{NZ}_3$. In this direction, the 3-flow conjecture was verified for Cayley graphs on abelian groups [11], nilpotent groups [10], dihedral groups [15], generalised dihedral groups [7], generalised quaternion groups [7], generalised dicyclic groups [1], groups of order pq^2 (p and q are two primes) (J. Zhang and Z. Zhang, 'Nowhere-zero 3-flows in Cayley graphs of order pq^2', submitted for publication) and two families of supersolvable groups [17]. Very recently, the first author and Zhou [18] proved that a graph of valency at least four is contained in $\mathcal{NZ}_3$ if its automorphism group has a vertex-transitive nilpotent subgroup.

In [10], to study nowhere-zero 3-flows in Cayley graphs, Nánásiová and Škoviera introduced the method of decomposing a graph into a union of closed ladders. They proved that a Cayley graph of valency at least four on a group G is contained in $\mathcal{NZ}_3$ if its connected set contains a central involution of G. They also proved that every Cayley graph of valency at least four on a group which is a direct product of a 2-subgroup and a subgroup of odd order is contained in $\mathcal{NZ}_3$. In this paper, we attempt to generalise the above two results to vertex-transitive graphs. We obtain the following two theorems.

THEOREM 1.1. *Let Γ be a vertex-transitive graph of order twice an odd number and valency at least 4. Let G be a minimally vertex-transitive subgroup of the automorphism group of Γ. If G contains a central involution, then $\Gamma \in \mathcal{NZ}_3$.*

THEOREM 1.2. *Let Γ be a graph of valency at least four. If there exists a subgroup of* $\mathrm{Aut}(\Gamma)$ *which acts transitively on $V(\Gamma)$ and is a direct product of a 2-subgroup and a subgroup of odd order, then $\Gamma \in \mathcal{NZ}_3$.*

The proof of Theorem 1.2 relies on Theorem 1.1 and the main result of [18].

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2. Preparations

Let Γ_1 and Γ_2 be two graphs. The *Cartesian product* $\Gamma := \Gamma_1 \Box \Gamma_2$ of Γ_1 and Γ_2 is the graph defined as follows:

- $V(\Gamma) = V(\Gamma_1) \times V(\Gamma_2)$;
- $E(\Gamma) = V(\Gamma_1) \times E(\Gamma_2) \cup E(\Gamma_1) \times V(\Gamma_2)$;
- each $(u_1, e_2) \in V(\Gamma_1) \times E(\Gamma_2)$ is an edge with ends (u_1, u_2) and (u_1, v_2), where e_2 is an edge in $E(\Gamma_2)$ with ends u_2 and v_2;
- each $(e_1, u_2) \in E(\Gamma_1) \times V(\Gamma_2)$ is an edge with ends (u_1, u_2) and (v_1, u_2), where e_1 is an edge in $E(\Gamma_1)$ with ends u_1 and v_1.

Let C_n be the cycle of length n which has vertex set $\{1, \ldots, n\}$ and edge set $\{\{1, 2\}, \ldots, \{n-1, n\}, \{n, 1\}\}$. Let K_2 be the complete graph of order two with vertex set $\{0, 1\}$. The graph $CL_n := C_n \Box K_2$ is called a *circular ladder*. The *Möbius ladder* M_n is a graph obtained from CL_n by replacing the edges $\{(1, 0), (n, 0)\}$ and $\{(1, 1), (n, 1)\}$ with $\{(1, 0), (n, 1)\}$ and $\{(1, 1), (n, 0)\}$, respectively (see Figure 1). A graph is called a *closed ladder* if it is a circular ladder or a Möbius ladder. In a closed ladder Γ, every edge of the form $\{(i, 0), (i, 1)\}$ is called a rung of Γ. We use $R(\Gamma)$ to denote the set of all rungs of Γ.

In [10], the method of decomposing a graph into a union of closed ladders was introduced to study nowhere-zero 3-flows in Cayley graphs. The following lemma, given in [7], is derived from the proof of [10, Theorem 3.3].

LEMMA 2.1. *Let* $\Gamma := \bigcup_{i=1}^{s} \Theta_i$ *be a connected graph where every subgraph* Θ_i *is a closed ladder. If* $E(\Theta_i) \cap E(\Theta_j) = R(\Theta_i) \cap R(\Theta_j)$ *for any pair of distinct* $i, j \in \{1, \ldots, s\}$ *and each edge of* $\bigcup_{i=1}^{s} R(\Theta_i)$ *is contained in at least two closed ladders in* $\{\Theta_1, \ldots, \Theta_s\}$*, then* Γ *is contained in* $\mathcal{NZ}_3$.

A graph is said to be *even* if each of its vertices is of even valency. It is well known [2, Theorem 21.4] that every even graph is contained in $\mathcal{NZ}_k$ for all integers $k \geq 2$. Let Γ be a graph and E a subset of $E(\Gamma)$. We use $\Gamma - E$ to denote the subgraph of Γ with vertex set $V(\Gamma)$ and edge set $E(\Gamma) - E$. A subgraph Γ' of Γ is called a *parity subgraph* of Γ if $\Gamma - E(\Gamma')$ is even. It is also well known [2, Theorem 21.5] that a cubic bipartite graph is contained in $\mathcal{NZ}_3$. All that leads to the following obvious lemma.

LEMMA 2.2. *Let* Γ *be a graph and* Γ' *a parity subgraph of* Γ*. If* $\Gamma' \in \mathcal{NZ}_3$*, then* $\Gamma \in \mathcal{NZ}_3$*. In particular, if every vertex of* Γ *is of odd valency and* Γ' *is a spanning cubic bipartite subgraph of* Γ*, then* $\Gamma \in \mathcal{NZ}_3$.

In [4], it was proved that the Cartesian product of two nontrivial connected bipartite graphs is contained in $\mathcal{NZ}_3$. This result was generalised in [13] by proving that the Cartesian product of every pair of graphs is contained in $\mathcal{NZ}_3$ except when one factor has a cut edge and every block of another factor is a circuit of odd length. By using Lemma 2.1, we prove the following lemma.

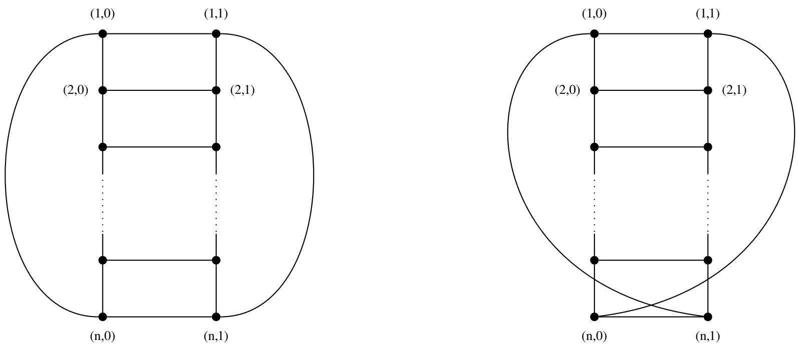

FIGURE 1. CL_n and M_n.

LEMMA 2.3. *Let Γ_1 be an even graph with minimum valency at least four and Γ_2 be an arbitrary graph. Then $\Gamma_1 \square \Gamma_2$ is contained in $\mathcal{NZ}_3$.*

PROOF. Let $\Gamma_{11}, \ldots, \Gamma_{1m}$ be all the connected components of Γ_1. Then $\Gamma_1 \square \Gamma_2$ is an edge-disjoint union of $\Gamma_{11} \square \Gamma_2, \ldots, \Gamma_{1m} \square \Gamma_2$. Therefore, $\Gamma_1 \square \Gamma_2 \in \mathcal{NZ}_3$ if and only if $\Gamma_{1i} \square \Gamma_2 \in \mathcal{NZ}_3$ for every $1 \le i \le m$. Since Γ_1 is an even graph with minimum valency at least four, Γ_{1i} is an even graph with minimum valency at least four for every $1 \le i \le m$. Therefore, we assume that Γ_1 is connected (for otherwise, we consider its components). By Veblen's theorem [2, Theorem 2.7], every even graph is an edge disjoint union of cycles. Therefore, there is a family $\mathcal{F}_1$ of edge disjoint cycles of Γ_1 such that $\bigcup_{\Sigma \in \mathcal{F}_1} \Sigma = \Gamma_1$. Let $\mathcal{F}_2$ be the decomposition of Γ_2 such that every member of $\mathcal{F}_2$ is either a complete graph of order two or a trivial graph with just one isolated vertex in Γ_2 (note that every graph has such a decomposition).

Consider an arbitrary member $\Lambda \in \mathcal{F}_2$. If Λ is a trivial graph, then $\Gamma_1 \square \Lambda$ is isomorphic to Γ_1 and therefore an even graph. Since every even graph is contained in $\mathcal{NZ}_3$, we have $\Gamma_1 \square \Lambda \in \mathcal{NZ}_3$. Now consider the case that Λ is the complete graph of order two. Set $\mathcal{F}_1 = \{\Sigma_1, \ldots, \Sigma_s\}$ and $\Theta_i = \Sigma_i \square \Lambda$ for every $\Sigma_i \in \mathcal{F}_1$. Then $\mathcal{F} := \{\Theta_1, \ldots, \Theta_s\}$ is a family of circular ladders. It is obvious that $\Gamma_1 \square \Lambda = \bigcup_{i=1}^{s} \Theta_i$. Moreover, $\Gamma_1 \square \Lambda$ is connected as both Γ_1 and Λ are connected. Let $1 \le i < j \le s$. Since Σ_i and Σ_j have no common edge, $E(\Theta_i) \cap E(\Theta_j) = R(\Theta_i) \cap R(\Theta_j)$. Since the minimum valency of Γ_1 is at least four, every vertex of Γ_1 is contained in at least two cycles in $\mathcal{F}_1$. It follows that each edge of $\bigcup_{i=1}^{s} R(\Theta_i)$ is contained in at least two members of $\mathcal{F}$. By Lemma 2.1, $\Gamma_1 \square \Lambda \in \mathcal{NZ}_3$.

Set $\mathcal{F}_2 = \{\Lambda_1, \ldots, \Lambda_t\}$. By the above discussion, $\Gamma_1 \square \Lambda_i \in \mathcal{NZ}_3$ for every $\Lambda_i \in \mathcal{F}_2$. Since $\Gamma_1 \square \Gamma_2$ is the edge disjoint union of $\Gamma_1 \square \Lambda_1, \ldots, \Gamma_1 \square \Lambda_t$, we have $\Gamma_1 \square \Gamma_2 \in \mathcal{NZ}_3$. □

The following lemma is extracted from [18, Lemma 4.8].

LEMMA 2.4. *Let Γ be a graph of valency five whose automorphism group contains a vertex-transitive subgroup G having a central involution z. Suppose that Γ has a*

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perfect matching M of which every edge is of the form $\{v, z(v)\}$, $v \in V(\Gamma)$. If Γ is an edge-disjoint union of M, Γ_1 and Γ_2, where Γ_1 and Γ_2 are both spanning 2-regular subgraphs of Γ preserved by G, then $\Gamma \in \mathcal{NZ}_3$.

3. Proof of Theorems 1.1 and 1.2

PROOF OF THEOREM 1.1. Let Γ be a graph of order $2n$, where n is an odd number. We assume that Γ is of odd valency as $\Gamma \in \mathcal{NZ}_3$ if Γ is even. Let G be a minimally vertex-transitive subgroup of the automorphism group of Γ and z a central involution of G.

We first prove that z does not fix any vertex of Γ. Otherwise, if $z(v) = v$ for some $v \in V(\Gamma)$, then $z(g(v)) = zg(v) = gz(v) = g(v)$ for all $g \in G$. Since G acts transitively on $V(\Gamma)$, it follows that z fixes all vertices of Γ. This contradicts the fact that z is not the identity automorphism of Γ.

Since z does not fix any vertex of Γ and $|V(\Gamma)| = 2n$, we conclude that z is a permutation factorising into n disjoint transpositions. Therefore, z is an odd permutation on $V(\Gamma)$ as n is an odd number. Let H be a subset of G consisting of all even permutations of G. Then $z \notin H$ and H is a normal subgroup of G of index 2. Since both $\langle z \rangle$ and H are normal in G and $\langle z \rangle \cap H = 1$, we get $G = \langle z \rangle \times H$.

Since G is minimally transitive on $V(\Gamma)$, we deduce that H is intransitive on $V(\Gamma)$. Let u be an arbitrary vertex of Γ. Then $|G : G_u| = |V(\Gamma)| = 2n$ and $|H : H_u|$ is a nontrivial divisor of $2n$. Since $|H : H_u| = (1/2)|G : H_u| \geq (1/2)|G : G_u| = n$, it follows that $|H : H_u| = n$. Therefore, the action of H on $V(\Gamma)$ has two orbits. Let U be the orbit of u under the action of H on $V(\Gamma)$. Then $z(U)$ is the orbit of $z(u)$ under the action of H on $V(\Gamma)$ as $zh = hz$ for all $h \in H$. Since $G = \langle z \rangle \times H$ acts transitively on $V(\Gamma)$, we have $U \cap z(U) = \emptyset$ and $U \cup z(U) = V(\Gamma)$. Let $\Gamma[U]$ and $\Gamma[z(U)]$ be the subgraphs of Γ induced by U and $z(U)$, respectively. Then $\Gamma[U]$ and $\Gamma[z(U)]$ are isomorphic and have no common edges. Since U is the orbit of u under the action of H and H preserves $\Gamma[U]$, we conclude that $\Gamma[U]$ is a regular graph. Assume that $\Gamma[U]$ is of valency s. Since $\Gamma[U]$ and $\Gamma[z(U)]$ are isomorphic and have no common edges, $\Gamma[U] \cup \Gamma[z(U)]$ is an s-regular graph. Let Γ' be the graph obtained from Γ by removing all the edges of $\Gamma[U] \cup \Gamma[z(U)]$. Then Γ' is a regular bipartite graph with bipartition $\{U, z(U)\}$. Assume that Γ' is of valency t. Then Γ is of valency $s + t$. In particular, $s + t$ is an odd number at least five. Since $\Gamma[U]$ is an s-regular graph of odd order, s is an even number and therefore t is an odd number.

By [2, Corollary 16.5], every regular bipartite graph has a perfect matching. Therefore, Γ' has a perfect matching. If $t \geq 3$, then, by removing a number of perfect matchings, one can get a spanning cubic bipartite subgraph Γ'' of Γ' which is also a spanning cubic bipartite subgraph of Γ. By Lemma 2.2, $\Gamma \in \mathcal{NZ}_3$.

From now on, we assume that $t = 1$. Then there exists a permutation μ on U such that $z\mu(v)$ is the unique vertex in $z(U)$ adjacent to v for all $v \in U$. Since $z \in \mathrm{Aut}(\Gamma)$, we see that $z(v)$ is the unique vertex in $z(U)$ adjacent to $\mu(v)$. Therefore, $\mu^2(v) = v$. It follows that μ fixes at least one vertex in U as the number n of vertices of U is odd.

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Without loss of generality, assume $\mu(u) = u$. Then u is adjacent to $z(u)$. Since H is transitive on U and $zh = hz$ for all $h \in H$, we conclude that v is adjacent to $z(v)$ for all $v \in U$. In other words, μ is the identity permutation. Note that $\Gamma[U]$ is an even regular graph of valency at least four. Let $\Sigma = \Gamma[U] \square K_2$ be the Cartesian product of $\Gamma[U]$ and K_2, where K_2 is the complete graph of order two with vertex set $\{0, 1\}$. By Lemma 2.3, $\Sigma \in \mathcal{NZ}_3$. Define a mapping ψ from $V(\Gamma)$ to $V(\Sigma)$ as follows:

$$\psi(v) = \begin{cases} (v, 0) & \text{if } v \in U, \\ (z(v), 1) & \text{if } v \in z(U). \end{cases}$$

It is straightforward to check that ψ is a well-defined bijection from $V(\Gamma)$ to $V(\Sigma)$. We further prove that ψ is an isomorphism.

Let v_1 and v_2 be two vertices of Γ. Since $\Gamma[U]$ is an induced subgraph of Γ, every edge in Γ joining two vertices in U is contained in $\Gamma[U]$. Therefore, if $v_1, v_2 \in U$, then by the definition of a Cartesian product, the number of edges in Γ joining v_1 and v_2 is equal to the number of edges in Σ joining $(v_1, 0)$ and $(v_2, 0)$. If $v_1, v_2 \in z(U)$, then $z(v_1), z(v_2) \in U$. Since $z \in \mathrm{Aut}(\Gamma)$, the number of edges in Γ joining v_1 and v_2 is equal to the number joining $z(v_1)$ and $z(v_2)$. It follows that the number of edges in Γ joining v_1 and v_2 is equal to the number of edges in Σ joining $(z(v_1), 1)$ and $(z(v_2), 1)$. Now consider the case that one of the two vertices v_1 and v_2 is contained in U and another is contained in $z(U)$. Without loss of generality, assume that $v_1 \in U$ and $v_2 \in z(U)$. Then

$$\begin{aligned} v_1 \text{ is adjacent to } v_2 \text{ in } \Gamma &\iff v_2 = z(v_1) \\ &\iff v_1 = z(v_2) \\ &\iff (v_1, 0) \text{ is adjacent to } (z(v_2), 1) \text{ in } \Sigma. \end{aligned}$$

The discussion above implies that the number of edges joining v_1 and v_2 in Γ is equal to the number of edges in Σ joining $\psi(v_1)$ and $\psi(v_2)$. Therefore, ψ is an isomorphism from Γ to Σ. Since $\Sigma \in \mathcal{NZ}_3$, we have $\Gamma \in \mathcal{NZ}_3$. □

Proof of Theorem 1.2. Let Γ be a graph of valency at least four and G a subgroup of $\mathrm{Aut}(\Gamma)$ acting transitively on $V(\Gamma)$ and being a direct product of a 2-subgroup Q and a subgroup H of odd order. We assume that Γ is of odd valency as $\Gamma \in \mathcal{NZ}_3$ whenever the valency of Γ is even. Then Γ is of even order and it follows that Q is nontrivial.

We proceed by induction on the order $|Q|$ of Q. By Theorem 1.1, $\Gamma \in \mathcal{NZ}_3$ if $|Q| = 2$. Now assume $|Q| > 2$. Suppose that the theorem is true for all graphs whose automorphism groups have a vertex-transitive subgroup which is a direct product of a 2-subgroup of order less that $|Q|$ and a subgroup of odd order.

It is well known [12, Theorem 4.2] that every 2-group has a nontrivial centre. Let z be an involution contained in the centre of Q. Since $G = Q \times H$, we see that z is also contained in the centre of G. Therefore, $\langle z \rangle$ is a normal subgroup of G and z does not fix any vertex of Γ. Set $\tilde{v} := \{v, z(v)\}$ for every $v \in V(\Gamma)$ and $\tilde{V} := \{\tilde{v} \mid v \in V(\Gamma)\}$. Let $\Gamma[\tilde{v}]$ be the subgraph of Γ induced by $\tilde{v}$. Since G acts transitively on $V(\Gamma)$, it follows that $\Gamma[\tilde{u}]$ and $\Gamma[\tilde{v}]$ are isomorphic for every pair of vertices $u, v \in V(\Gamma)$. Set $\Gamma' = \bigcup_{\tilde{v} \in \tilde{V}} \Gamma[\tilde{v}]$

and $\Gamma'' = \Gamma - E(\Gamma')$. Then both Γ' and Γ'' are spanning subgraphs of Γ preserved by G. Therefore, Γ' and Γ'' are both vertex-transitive. Assume that the valency of Γ' and Γ'' are s and t, respectively. Then Γ is of valency $s+t$. In particular, $s+t$ is odd.

Case 1: $s \geq 2$. Note that every connected component of Γ' is a graph with two vertices joined by s edges. Therefore, Γ' is a bipartite graph. If $s \geq 3$, then Γ' has a spanning cubic bipartite graph which is also a spanning cubic bipartite graph of Γ. It follows from Lemma 2.2 that $\Gamma \in \mathcal{NZ}_3$. If $s = 2$, then t is odd. By [3, Theorem 3.51], every vertex-transitive graph of odd valency has a perfect matching. Therefore, Γ'' has a perfect matching M. Since every connected component of Γ' is a graph with two vertices joined by two edges, every connected component of $\Gamma' \cup M$ is a graph obtained from an even length cycle by adding a parallel edge to each edge of one of the two perfect matchings of this cycle. Therefore, $\Gamma' \cup M$ is a spanning cubic bipartite graph of Γ and it follows from Lemma 2.2 that $\Gamma \in \mathcal{NZ}_3$.

Case 2: $s = 0$. In this case, $\tilde{v}$ is an independent set of Γ for every $\tilde{v} \in \tilde{V}$. Use $\tilde{\Gamma}$ to denote the graph with vertex set $\tilde{V}$ and every pair of vertices $\tilde{u}$ and $\tilde{v}$ being joined by ℓ edges if and only if the subgraph $\Gamma[\tilde{u} \cup \tilde{v}]$ of Γ induced by $\tilde{u} \cup \tilde{v}$ is ℓ-regular (we treat an independent set as a 0-regular graph). Then $\tilde{\Gamma}$ is a graph of odd valency at least five and Γ is a cover (see [11]) of $\tilde{\Gamma}$. Furthermore, $\mathrm{Aut}(\tilde{\Gamma})$ contains $G/\langle z\rangle$ as a subgroup acting transitively on the vertex set $\tilde{V}$ of $\tilde{\Gamma}$. Note that $G/\langle z\rangle = Q/\langle z\rangle \times H\langle z\rangle/\langle z\rangle$ and $Q/\langle z\rangle$ is of order less than Q. By the induction hypothesis, $\tilde{\Gamma} \in \mathcal{NZ}_3$. It is known [11, Proposition 2.3] that if a graph admits a nowhere-zero 3-flow, then each of its covers does too. Therefore, $\Gamma \in \mathcal{NZ}_3$.

Case 3: $s = 1$. In this case, Γ' is a perfect matching of Γ. Since every group of odd order is solvable, H is a solvable group. Let H' be the derived subgroup of H. Then H' is a proper subgroup of H and normal in G. If H is abelian, then G $(= Q \times H)$ is nilpotent. By [18, Theorem 1.1], $\Gamma \in \mathcal{NZ}_3$. Now assume that H is nonabelian. Then H' is nontrivial and $G/H' = QH'/H' \times H/H'$ is nilpotent. Use $\bar{v}$ to denote the orbit of v under the action of H' and $\Gamma[\bar{v}]$ the subgraph of Γ induced by $\bar{v}$. Then $\Gamma[\bar{v}]$ is a vertex-transitive graph of odd order as H' acts transitively on $\bar{v}$. Therefore, $\Gamma[\bar{v}]$ is a regular graph of even valency, say r. Set $\Sigma = \bigcup_{v \in V(\Gamma)} \Gamma[\bar{v}]$. Then $\Gamma' \cup \Sigma$ is of valency $r+1$ and the automorphism group of every connected component of $\Gamma' \cup \Sigma$ contains $\langle z\rangle \times H'$ as a subgroup acting transitively on the vertex set. If $r \geq 4$, then by Theorem 1.1, every connected component of $\Gamma' \cup \Sigma$ is contained in $\mathcal{NZ}_3$ and therefore, $\Gamma' \cup \Sigma \in \mathcal{NZ}_3$. Since $\Gamma' \cup \Sigma$ is a parity subgraph of Γ, it follows from Lemma 2.2 that $\Gamma \in \mathcal{NZ}_3$. If $t - r \geq 4$, then the subgraph $\Gamma^* := \Gamma - E(\Sigma)$ of Γ is of odd valency at least five. Let $\overline{\Gamma^*}$ be the graph with vertex set $\{\bar{v} \mid v \in V\}$ and every pair of vertices $\bar{u}$ and $\bar{v}$ being joined by ℓ-edges if and only if $\bar{u} \cup \bar{v}$ induces a ℓ-regular subgraph of Γ^*. Then Γ^* is a cover of $\overline{\Gamma^*}$. Note that $\mathrm{Aut}(\overline{\Gamma^*})$ contains $QH'/H' \times H/H'$ as a subgroup acting transitively on the vertex set. Since $QH'/H' \times H/H'$ is nilpotent, it follows from [18] that $\overline{\Gamma^*} \in \mathcal{NZ}_3$. By [11, Proposition 2.3], $\Gamma^* \in \mathcal{NZ}_3$. Since Γ^* is a parity subgraph of Γ, by Lemma 2.2, we have $\Gamma \in \mathcal{NZ}_3$. Now we assume that both r and $t - r$ are less than

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four. Since $t \geq 4$ and r is even, we have $r = t - r = 2$. Then Σ and $\Gamma'' - E(\Sigma)$ are both spanning 2-regular subgraphs of Γ preserved by G. Note that Γ' is a perfect matching of which every edge is of the form $\{v, z(v)\}$. Note also that Γ is an edge-disjoint union of Γ', Σ and $\Gamma'' - E(\Sigma)$. By Lemma 2.4, $\Gamma \in \mathcal{NZ}_3$. $\square$

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