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Expansion of the Riemann Ξ Function in Meixner–Pollaczek Polynomials

Alexey Kuznetsov

Abstract. In this article we study in detail the expansion of the Riemann Ξ function in Meixner–Pollaczek polynomials. We obtain explicit formulas, recurrence relation and asymptotic expansion for the coefficients and investigate the zeros of the partial sums.

1 Introduction

Riemann Ξ function is defined as

$$\Xi(t) = \frac{1}{2} s(s-1)\pi^{-\frac{s}{2}}\Gamma\Big(\frac{s}{2}\Big)\zeta(s), \quad s = \tfrac{1}{2} + it.$$

It is known that $\Xi(t)$ is an even entire function, which is real for real t, and its zeros coincide with the nontrivial zeros of the Riemann zeta function $\zeta(\frac{1}{2} + it)$ (see [11]).

In this article we study the expansion of the Ξ-function in the Meixner–Pollaczek polynomials. The expansions in polynomials are important for the following reasons. First, the location of the zeros of partial sums $S_n(t)$ can provide information about the zeros of the Ξ-function. For example, if the polynomials $S_n(t)$ have only real zeros and converge uniformly on compact subsets of $\mathbb{C}$ to $\Xi(t)$, then $\Xi(t)$ itself has only real zeros. Secondly, the zeros of the partial sums $S_n(t)$ can be quite easily computed numerically (compared to the Dirichlet series or expansions in special functions). Thirdly, there are many results in the theory of multiplier sequences, biorthogonal polynomials and linear operators on polynomials that can potentially be applied to study the roots of the partial sums $S_n(t)$.

In order to be useful the expansion must be simple and natural. Thus it is important to choose the right basis functions. Depending on the choice of the basis, the coefficients of the expansion can be given by explicit formulas and be easily computable, or they can be completely intractable. For example, the well known Stieltjes constants γ_n are defined (up to a plus/minus sign) as the derivatives of $\zeta(s) - \frac{1}{s-1}$ at $s = 1$. These numbers have been studied extensively [12]. Here we just note the following two formulae:

$$\gamma_n = \lim_{m\to\infty}\Big[\sum_{j=1}^{m}\frac{\ln(j)^n}{j} - \frac{\ln(m)^{n+1}}{n+1}\Big] = \frac{\ln(2)^n}{n+1}\sum_{j\geq 1}\frac{(-1)^j}{j}B_{j+1}\Big(\frac{\ln(j)}{\ln(2)}\Big).$$

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One can see that the above formulae are very complicated; it is hard to extract any information about γ_n from either of them. This indicates that $(s-1)^n$ is not a very convenient basis for expansion.

The fact that power functions do not form a convenient basis for expansion is not surprising; the zeta function is not a solution of any simple ODE. The close connection of the zeta function with the Mellin transform indicates that falling factorials $\binom{s-1}{j}$ might be a better choice for basis functions. The following expansion of $\zeta(s)$ can be obtained with the help of the Gauss–Kuzmin–Wirsing operator (see [3]):

$$\zeta(s) = \frac{s}{s-1} - \frac{1}{2} - s\sum_{n\geq 0}(-1)^n \binom{s-1}{n} a_n.$$

Coefficients a_n are given in terms of values of $\zeta(s)$ at integers:

$$a_n = 1 - \frac{1}{2(n+1)} - \gamma + \sum_{k=1}^{n}(-1)^k \binom{n}{k}\left[\frac{1}{k} - \frac{\zeta(k+1)}{k+1}\right].$$

Even though a_n are not given in closed form, they can be computed with high precision (at least for small n).

The above expansions have one major drawback: their partial sums do not respect the functional equation. If one wants the zeros of the partial sums to be real, the partial sums $S_n(t)$ themselves must be real for real t (up to a constant multiple), which means that they must satisfy the functional equation. It turns out that a natural basis for expansion of the Ξ-function is given by the Meixner–Pollaczek polynomials. In this article we show that the coefficients of this expansion have many important and useful properties: closed form expression, explicit formula for the exponential generating function, recurrence relation, integral representation, and asymptotic expansion.

2 Expansion in Meixner–Pollaczek Polynomials

In this article we consider only a special case of the Meixner–Pollaczek polynomials, defined as

$$p_n(t) = P_n^{(\frac{3}{4})}(\tfrac{t}{2};\tfrac{\pi}{2}) = \frac{(2n+1)!!}{(2n)!!} i^n {}_2F_1(-n, \tfrac{3}{4} + \tfrac{it}{2}; \tfrac{3}{2}; 2).$$

For the definition and properties of the general Meixner–Pollaczek polynomials $P_n^{(\lambda)}(t;\phi)$, see [8].

Polynomials $p_n(t)$ arise as the Mellin transform of the odd Hermite orthogonal functions $f_n(x) = e^{-\frac{x^2}{2}} H_{2n+1}(x)$:

$$\int_0^\infty f_n(x) x^{s-1} dx = n! 2^{2n+\frac{s+1}{2}} \Gamma\left(\frac{1+s}{2}\right) i^n p_n(t), \quad s = \frac{1}{2} + it. \tag{1}$$

Since Hermite functions $f_n(x)$ are invariant under the Fourier transform, polynomials $p_n(t)$ satisfy the “functional equation” $p_n(-t) = (-1)^n p_n(t)$. From (1) and the

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Parseval identity for the Mellin transform it follows that polynomials $p_n(t)$ are orthogonal on $\mathbb{R}$ with respect to the measure $\mu(dt) = |\Gamma(\frac{3}{4} + \frac{it}{2})|^2 dt$, which also implies that $p_n(t)$ have only real zeros (see [1]). The three term recurrence relation for $p_n(t)$ is

$$(n+1)p_{n+1}(t) - t p_n(t) + \left(n + \tfrac{1}{2}\right) p_{n-1}(t) = 0, \quad n \geq 0. \tag{2}$$

The next theorem shows that Meixner–Pollaczek polynomials $p_n(t)$ form a natural basis for expansion of the Ξ-function:

Theorem 2.1 *Define the sequence of real numbers* $\{\tilde{b}_n\}_{n\geq 0}$ *as*

$$\tilde{b}_n = \frac{d^n}{dx^n}\Big[-\frac{1}{4}\sqrt{\frac{\pi}{x}} \sin(x + \tfrac{\pi}{8}) \tanh(\sqrt{2\pi x})\Big]_{x=0}. \tag{3}$$

Let the sequence $\{b_n\}_{n\geq 0}$ *be a linear transformation of* $\{\tilde{b}_n\}_{n\geq 0}$

$$b_n = 4n(n-1)\tilde{b}_{n-2} + (8n^2 + 12n + 7)\tilde{b}_n + (2n+3)(2n+5)\tilde{b}_{n+2}. \tag{4}$$

Then we have

$$\Xi(t)e^{-\frac{\pi t}{4}} = \sum_{n\geq 0} b_n p_n(t), \tag{5}$$

where the series converges uniformly on compact subsets of $\mathbb{C}$*.*

Proof To prove this theorem we use the following result, derived in [9]. For all $t \in \mathbb{C}$

$$\Xi(t)e^{-\frac{\pi t}{4}} = \frac{1}{2}\cos(\tfrac{\pi}{8}) - t\sin(\tfrac{\pi}{8}) + (1 + 4t^2)\sum_{n\geq 0} a_n p_n(t), \tag{6}$$

where the coefficients $\{a_n\}_{n\geq 0}$ have the integral representation

$$a_n = \frac{(-1)^n}{(2n+1)!!}\int_0^\infty \frac{\sin(\frac{y^2}{2} + \frac{\pi}{8})}{e^{2\sqrt{\pi}y} + 1} y^{2n+1} dy, \tag{7}$$

and the exponential generating function for $\{a_n\}_{n\geq 0}$ is

$$\sum_{n\geq 0} \frac{a_n}{n!}x^n = \frac{1}{4}\sqrt{\frac{\pi}{x}}\Big[-\sin(x + \tfrac{\pi}{8})\tanh(\sqrt{2\pi x}) + \mathrm{Im}(\Phi(e^{\frac{\pi i}{4}}\sqrt{x})e^{ix + \frac{\pi i}{8}})\Big]. \tag{8}$$

Here $\Phi(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2} dt$ is the probability integral function (see [4]).

In order to obtain (5) all that is left to do is to express the function

$$(1 + 4t^2)\sum_{n\geq 0} a_n p_n(t)$$

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in (6) as expansion in $p_n(t)$. This will be done using the three term recurrence relation for the Meixner–Pollaczek polynomials $p_n(t)$.

Applying the recurrence relation (2) twice, we find

$$(9)\quad (1+4t^2)p_n(t) = 4(n+1)(n+2)p_{n+2}(t) \\ + (8n^2+12n+7)p_n(t) + (2n-1)(2n+1)p_{n-2}(t).$$

Now, using (8) and (3) we find that for $n \geq 0$

$$a_n - \tilde{b}_n = \frac{d^n}{dx^n}\Big[\frac{1}{4}\sqrt{\frac{\pi}{x}}\,\mathrm{Im}\,(\Phi(e^{\frac{\pi i}{4}}\sqrt{x})e^{ix+\frac{\pi i}{8}})\Big],$$

thus

$$(10)\qquad a_n = \tilde{b}_n + \frac{1}{2}\frac{(2n)!!}{(2n+1)!!}\cos\big(\tfrac{\pi}{2}n - \tfrac{\pi}{8}\big), \quad n \geq 0.$$

Next we use (9), (4) and (10) to transform the infinite sum in equation (6)

$$\begin{aligned}(1+4t^2)\sum_{n\geq 0} a_n p_n(t) &= \sum_{n\geq 0} p_n(t)[4n(n-1)a_{n-2} + (8n^2+12n+7)a_n \\ &\quad + (2n+3)(2n+5)a_{n+2}] \\ &= -\frac{1}{2}\cos(\tfrac{\pi}{8}) + t\sin(\tfrac{\pi}{8}) + \sum_{n\geq 0} b_n p_n(t).\end{aligned}$$

Combining the above equation with (6) we obtain the required expansion (5). ■

Remark 1. In [9], Expansion (6) was derived from the integral representation for $\Xi(t)e^{-\frac{\pi t}{4}}$ by expanding the integral kernel (confluent hypergeometric function) in a series of Meixner–Pollaczek polynomials. A more intuitive way to derive (6) is to note that the fractional Fourier sine transform $F(y) = [\mathcal{F}_s^{1/2} f](y)$ of the function $f(x) = (\exp(\sqrt{2\pi}x) - 1)^{-1}$ can be computed in closed form (see [4, 9]), and $F(y)$ can be used to obtain the explicit expression for the coefficients c_n of the following expansion

$$f(e^{\frac{\pi i}{4}}x) = \sum_{n\geq 0} c_n e^{-\frac{x^2}{2}} H_{2n+1}(x).$$

To obtain (6) we apply the Mellin transform to both sides of the above equation and use (1).

Remark 2. Note that for $n \geq 2$ we have

$$(11)\qquad b_n = 4n(n-1)a_{n-2} + (8n^2+12n+7)a_n + (2n+3)(2n+5)a_{n+2},$$

and that linear transformation (4) eliminates the term

$$\frac{1}{2}\frac{(2n)!!}{(2n+1)!!}\cos(\tfrac{\pi}{2}n - \tfrac{\pi}{8})$$

in (10).

Remark 3. The asymptotic formula for the Meixner–Pollaczek polynomials

$$p_n(t) = P_n^{(\frac{3}{4})}(\frac{t}{2}; \frac{\pi}{2})$$

can be derived from the integral representation (see [9]):

$$P_n^{(\lambda)}(t; \tfrac{\pi}{2}) \sim 2^{1-2\lambda}(-i)^n \left[\frac{(2n)^{\lambda+it-1}}{\Gamma(\lambda+it)} + (-1)^n \frac{(2n)^{\lambda-it-1}}{\Gamma(\lambda-it)} \right], \quad n \to \infty.$$

We see that expansions in Meixner–Pollaczek polynomials converge in some strip $\mathrm{Im}(t) < c$, and since $\Xi(t)e^{-\frac{\pi t}{4}}$ is an entire function, this expansion should converge in the entire complex plane. Another proof can be given with the help of the asymptotic formula for a_n derived in section 3.

Theorem 2.1 allows us to obtain an interesting expression for the nontrivial zeros of the Riemann zeta function as the limit of eigenvalues of some $n \times n$ real (nonsymmetric) matrices $\mathbf{B}_n$, defined as

(12)

$$\mathbf{B}_n = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ \frac{3}{2} & 0 & 2 & \dots & 0 & 0 & 0 & 0 \\ 0 & \frac{5}{2} & 0 & \dots & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & n-3 & 0 & 0 \\ 0 & 0 & 0 & \dots & n-\frac{5}{2} & 0 & n-2 & 0 \\ 0 & 0 & 0 & \dots & 0 & n-\frac{3}{2} & 0 & n-1 \\ -n\frac{b_0}{b_n} & -n\frac{b_1}{b_n} & -n\frac{b_2}{b_n} & \dots & -n\frac{b_{n-4}}{b_n} & -n\frac{b_{n-3}}{b_n} & n-\frac{1}{2}-n\frac{b_{n-2}}{b_n} & -n\frac{b_{n-1}}{b_n} \end{pmatrix}.$$

Let $\sigma(\zeta)$ denote the nontrivial zeros of the Riemann zeta function $\zeta(\frac{1}{2}+it)$:

$$\sigma(\zeta) = \{\gamma \in \mathbb{C} : \zeta(\tfrac{1}{2}+i\gamma) = 0, \quad \mathrm{Im}[\gamma] < \tfrac{1}{2}\}.$$

Lemma 2.2 *As $n \to \infty$, the spectrum of matrices $\mathbf{B}_n$ converges to the set of nontrivial zeros of the Riemann zeta function $\zeta(\frac{1}{2}+it)$:*

$$\sigma(\mathbf{B}_n) \longrightarrow \sigma(\zeta).$$

The convergence is understood in the following sense: $\gamma \in \sigma(\zeta)$ if and only if $\gamma \neq \infty$ is a limit of some sequence $\{\gamma_n\}_{n\geq 0}$, such that $\gamma_n \in \sigma(\mathbf{B}_n)$.

Proof The proof is just an application of the Theorem 2.3 in [2] to the partial sums $S_n(t) = \sum_{k=0}^{n} b_k p_k(t)$. By Theorem 2.1 the partial sums $S_n(t)$ converge to $\Xi(t)e^{-\frac{\pi t}{4}}$ uniformly on compact subsets of $\mathbb{C}$, thus the zeros of $S_n(t)$ converge to the set of zeros of $\Xi(t)e^{-\frac{\pi t}{4}}$, which coincides with $\sigma(\zeta)$. Now, to finish the proof we only need to use the Theorem 2.3 in [2], which allows us to express the roots of the polynomials $S_n(t)$ as the eigenvalues of the matrix $\mathbf{B}_n$ defined by (12). Here is the main idea: one can check using the recurrence relation (2) that

$$\mathbf{B}_n \mathbf{f}_n(t) = t\mathbf{f}_n(t) - \frac{n}{b_n} S_n(t)\mathbf{e}_n,$$

where $\mathbf{f}_n(t) = [p_0(t), p_1(t), \dots, p_{n-1}(t)]^T$ and $\mathbf{e}_n = [0, 0, \dots, 0, 1]^T$. Thus $S_n(\gamma) = 0$ if and only if γ is an eigenvalue of $\mathbf{B}_n$ (and in this case $\mathbf{f}_n(\gamma)$ is the right eigenvector). ■

3 Computation of the Coefficients

From equation (3) we can derive an explicit formula for the coefficients $\tilde{b}_n$:

$$\tilde{b}_n = -\frac{n!}{\sqrt{2}} \sum_{k=0}^{n} \frac{4^k(4^{k+1}-1)B_{2k+2}}{(n-k)!(2k+2)!} \sin\left(\tfrac{\pi}{2}(n-k)+\tfrac{\pi}{8}\right)(2\pi)^{k+1}, \tag{13}$$

where $\{B_n\}_{n\geq 0} = \{1, -\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, 0, \dots\}$ are the Bernoulli numbers. Coefficients $\tilde{b}_n$ also satisfy the recurrence relation

$$\sum_{k=0}^{n} \frac{\pi^{n-k}}{(2n-2k-1)!!}\binom{n}{k}\tilde{b}_k = \frac{\sqrt{\pi}2^{-n-2}}{(2n+1)!!}\operatorname{Im}\Big[e^{\frac{\pi i}{8}} i^n H_{2n+1}\Big(\sqrt{\frac{\pi}{2}}e^{-\frac{\pi i}{4}}\Big)\Big]. \tag{14}$$

This relation can be obtained from the following equation for the generating function of $\tilde{b}_n$:

$$\cosh(\sqrt{2\pi x})\sum_{n\geq 0}\frac{\tilde{b}_n}{n!}x^n = -\frac{1}{4}\sqrt{\frac{\pi}{x}}\sin(x+\tfrac{\pi}{8})\sinh(\sqrt{2\pi x}),$$

which is derived from (3).

Equations (13) and (14) can be used to compute coefficients b_n for small n; however, they are not efficient when n is large. The precision is lost in (13) because Bernoulli numbers and factorials grow very rapidly while the recurrence relation (14) is unstable. Fortunately, for large n one can use equation (11) combined with the integral representation (7) for a_n to compute the coefficients b_n. The idea is to use the saddle point method. First we rewrite (7) as

$$a_n = \operatorname{Im}\Big[\frac{(-1)^n e^{\frac{\pi i}{8}}}{(2n+1)!!}\int_0^\infty \frac{e^{i\frac{y^2}{2}}}{e^{2\sqrt{\pi}y}+1}y^{2n+1}dy\Big].$$

Note that the integral in the above equation grows as $2^n n!$ as $n \to \infty$, but by shifting the contour of integration to $e^{\pi i/4}\mathbb{R}_+$, changing the variable of integration $y \mapsto \sqrt{2nv}e^{\pi i/4}$ and rearranging the terms, we isolate the dominant term $\exp(-2\sqrt{\pi n})$ and obtain a more manageable expression:

$$\begin{aligned} a_n = {}& (-1)^n\frac{\sqrt{\pi}}{2}e^{-2\sqrt{\pi n}}e^{(n+1)\ln(n)-n-\ln(\Gamma(n+\frac{3}{2}))} \\ &\times \operatorname{Im}\Big[e^{\frac{\pi i}{8}} i^{n+1}e^{-2\sqrt{\pi n}i}\int_0^\infty \frac{e^{n(1-v+\ln(v))+\theta\sqrt{n}(\sqrt{v}-1)}}{1+e^{\theta\sqrt{n}\sqrt{v}}}dv\Big], \end{aligned} \tag{15}$$

where $\theta = -2\sqrt{\pi}(1+i)$. The integral in (15) has a saddle point essentially at $v=1$, so we introduce a new variable $v = 1+n^{-\frac{1}{2}}u$ and obtain:

$$a_n = (-1)^n\frac{\sqrt{\pi}}{2}e^{-2\sqrt{\pi n}}e^{(n+\frac{1}{2})\ln(n)-n-\ln(\Gamma(n+\frac{3}{2}))}\operatorname{Im}\Big[e^{\frac{\pi i}{8}} i^{n+1}e^{-2\sqrt{\pi n}i}I_n\Big], \tag{16}$$

where

$$(17) \qquad I_n = \int_{-\sqrt{n}}^{\infty} \frac{\exp[n(\ln(1+un^{-\frac{1}{2}}) - un^{-\frac{1}{2}}) + \theta n^{\frac{1}{2}}(\sqrt{1+un^{-\frac{1}{2}}} - 1)]}{1+\exp[\theta n^{\frac{1}{2}}\sqrt{1+un^{-\frac{1}{2}}}]} du.$$

Now we can compute I_n (and thus a_n and b_n through formulas (16) and (11)) with high precision using numerical integration to evaluate the integral in (17). We can even go one step further and derive an asymptotic formula for I_n. First we expand the integrand around $u = 0$:

$$\begin{aligned} n\big(\ln(1+un^{-\frac{1}{2}}) - un^{-\frac{1}{2}}\big) &+ \theta n^{\frac{1}{2}}(\sqrt{1+un^{-\frac{1}{2}}} - 1) \\ &= -\frac{u^2}{2} + \frac{\theta u}{2} + n^{-\frac{1}{2}}\Big(-\frac{\theta u^2}{8} + \frac{u^3}{3}\Big) + n^{-1}\Big(\frac{\theta u^3}{16} - \frac{u^4}{4}\Big) + O\big(n^{-\frac{3}{2}}\big), \end{aligned}$$

and $(1+\exp[\theta n^{\frac{1}{2}}\sqrt{1+un^{-\frac{1}{2}}}])^{-1} = 1+O(e^{-2\sqrt{n\pi}})$. Collecting all the terms we obtain

$$\begin{aligned} I_n &= \int_{-\infty}^{\infty} e^{-\frac{u^2}{2}+\frac{\theta u}{2}} \Big[1 + \tfrac{1}{\sqrt{n}}\big(-\tfrac{\theta u^2}{8} + \tfrac{u^3}{3}\big) + \tfrac{1}{n}\Big(\tfrac{u^6}{18} - \tfrac{\theta u^5}{24} \\ &\qquad\qquad + \big(\tfrac{\theta^2}{128} - \tfrac{1}{4}\big)u^4 + \tfrac{\theta u^3}{16}\Big) + O\big(n^{-\frac{3}{2}}\big)\Big] du \\ &= -\sqrt{2\pi}[1 + \sqrt{\tfrac{\pi}{n}}\big(\tfrac{(1-i)\pi}{6} - \tfrac{3(1+i)}{4}\big) + \tfrac{1}{n}\big(-\tfrac{\pi^3 i}{36} - \tfrac{\pi^2}{4} + \tfrac{7\pi i}{16} + \tfrac{1}{12}\big) \\ &\qquad + O(n^{-\frac{3}{2}})]. \end{aligned}$$

4 Zeros of Expansions in Orthogonal Polynomials

An interesting approach to the study of zeros of expansions in orthogonal polynomials developed in [5–7] is based on the theory of biorthogonal polynomials. The following statement is a particular case of proposition 6 in [7]:

Proposition 4.1 *If a polynomial $\hat{S}_n(x) = \sum_{k=0}^{n} \frac{(2k+1)!!}{(2k)!!} b_k x^k$ has only real zeros, then polynomial $S_n(t) = \sum_{k=0}^{n} b_k p_k(t)$ also has only real zeros.*

Using the above proposition one can easily prove the following statement:

Proposition 4.2 *The entire function $\tilde{\Xi}(t) = \sum_{n\geq 0} \frac{1}{(2n+1)!!} \tilde{b}_n p_n(t)$ has only real zeros.*

Note that $\tilde{\Xi}(t)$ is "almost" $\Xi(t)e^{-\frac{\pi t}{4}}$ (see equation (5)), except for the factor $(2n+1)!!$ and the linear transformation (4). To prove proposition 4.2 we use (3) to show that the exponential generating function for $\tilde{b}_n$

$$\sum_{n\geq 0} \frac{\tilde{b}_n}{n!} x^n = -\frac{1}{4}\sqrt{\frac{\pi}{x}} \sin\big(x + \tfrac{\pi}{8}\big) \tanh(\sqrt{2\pi x})$$

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is an entire function with only real zeros, and we can approximate it uniformly by polynomials with only real zeros (for example, we can expand $\sin(x+\frac{\pi}{8})$, $\sinh(\sqrt{2\pi x})$, and $\cosh(\sqrt{2\pi x})$ as infinite products of linear factors $x - x_n$ and truncate the products). Applying proposition 4.1 to these polynomials completes the proof.

Thus, if numerical computations indicate that the partial sums $S_n(t) = \sum_{k=0}^{n} b_k p_k(t)$ have only real zeros, proposition 4.1 and explicit formulas for b_n could provide a way to prove this. Note, however, that proposition 4.1 gives only a sufficient condition. Unfortunately, the polynomial $S_3(t)$ already has two complex roots, and the numerical computations indicate that the number of complex roots of $S_n(t)$ increases as n increases.

However, $S_n(t) = \sum_{k=0}^{n} b_k p_k(t)$ is not the only possible way to truncate the series (5). As we will see, the situation can be improved if we use a *multiplier sequence*. A multiplier sequence $\{m_k\}_{k\geq 0}$ is characterized by the following property: if a polynomial $Q(x) = q_0 + q_1 x + \cdots + q_n x^n$ has only real zeros, then $T[Q](x) = m_0 q_0 + m_1 q_1 x + \cdots + m_n q_n x^n$ also has only real zeros. A classical example of a multiplier sequence is given by the *Jensen sequence*

$$m_k^{(n)} = \frac{n!}{(n-k)!n^k} = \Big(1 - \frac{1}{n}\Big)\Big(1 - \frac{2}{n}\Big)\dots\Big(1 - \frac{n-k+1}{n}\Big).$$

Note that $m_k^{(n)} \to 1$ as $n \to \infty$ and that $m_k^{(n)} = 0$ if $k > n$. This sequence provides a "right" way to truncate the Taylor series. A theorem by Jensen (see [10]) says that (under some additional conditions) an entire function $f(x) = \sum_{k\geq 0} f_k x^k$ has only real zeros if and only if all polynomials $T_n[f](x) = \sum_{k\geq 0} m_k^{(n)} f_k x^k$ have only real zeros.

Let us consider for example the exponential function e^x. If we simply truncate the Taylor expansion $Q_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}$, then the $Q_n(x)$ converge to e^x uniformly on compact subsets of $\mathbb{C}$, but it can be proven that all the zeros of $Q_n(x)$ are complex (except for one real negative root when n is odd). However if we truncate the Taylor series in a different way, by using the Jensen multiplier sequence $m_k^{(n)}$, we obtain

$$\tilde{Q}_n(x) = T_n[e^x](x) = \sum_{k=0}^{n} m_k^{(n)} \frac{x^k}{k!} = (1 + \frac{x}{n})^n,$$

and $\tilde{Q}_n(x)$ has only real zeros. Note that since $m_k^{(n)} \to 1$ as $n \to \infty$, polynomials $Q_n(x)$ and $\tilde{Q}_n(x)$ both converge to the same function, but $\tilde{Q}_n(x)$ has more predictable behavior of its zeros. The roots of $\tilde{Q}_n(x)$ are real and converge to the root of e^x at $x = -\infty$.

If we naively apply the multiplier sequence $m_k^{(n)}$ to the expansion of $\Xi(t)e^{-\frac{\pi t}{4}}$ in Meixner–Pollaczek polynomials,

$$\tilde{S}_n(t) = T_n\Big[\Xi(t)e^{-\frac{\pi t}{4}}\Big](t) = \sum_{k=0}^{n} m_k^{(n)} b_k p_k(t),$$

then $\tilde{S}_n(t)$ also converges to $\Xi(t)e^{-\frac{\pi t}{4}}$ uniformly on compact subsets (probably at a much slower rate), but the roots of $\tilde{S}_n(t)$ seem to behave more predictably. Numerical

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computations show that all the roots of $\tilde{S}_n(t)$ are real for $n \leq 5$, and $\tilde{S}_n(t)$ has just two complex zeros for $6 \leq n \leq 20$. Of course the sequence $m_k^{(n)}$ was designed to work with Taylor expansions; there is no obvious reason why it should improve the behavior of roots for expansions in Meixner–Pollaczek polynomials. Therefore, we have two interesting questions for future research: (A) to study the multiplier sequences and more general linear operators on expansions of entire functions in Meixner–Pollaczek polynomials, and (B) to investigate the behaviour of zeros of partial sums under such transformations.

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Department of Mathematics and Statistics, York University, Toronto, ON, M3J 1P3
e-mail: kuznetsov@mathstat.yorku.ca