On certain expansions involving Bessel functions and Whittaker's M-functions

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1. Adopting the notation of Barnes[1] and Fox[2], let us write

$$ {}_pf_q(a_1, a_2, \ldots a_p; \rho_1, \rho_2, \ldots \rho_q; x) = \sum_{r=0}^{\infty} \frac{\Gamma(a_1 + r) \ldots \Gamma(a_p + r)}{\Gamma(\rho_1 + r) \ldots \Gamma(\rho_q + r)} \frac{x^r}{\Gamma(r+1)} \tag{1} $$

$$ = \frac{\Gamma(a_1) \ldots \Gamma(a_p)}{\Gamma(\rho_1) \ldots \Gamma(\rho_q)} {}_pF_q(a_1, a_2, \ldots a_p; \rho_1, \rho_2, \ldots \rho_q; x). \tag{2} $$

If $q > p - 1$, the series on the right of (1) represents an integral function, while if $q = p - 1$, the series converges only inside or on the circle $|x| = 1$.

Now Barnes has proved that

$$ {}_pf_q(a_1, a_2, \ldots a_p; \rho_1 + m, \rho_2, \ldots \rho_q; x) = $$
$$ \frac{1}{2\pi i} \int_C \Gamma(-s) \frac{\Gamma(a_1 + s) \ldots \Gamma(a_p + s)(-x)^s}{\Gamma(\rho_1 + m + s)\Gamma(\rho_2 + s) \ldots \Gamma(\rho_q + s)} ds, \tag{3} $$

the integral being taken along a contour C, which encloses all the poles of $\Gamma(-s)$ but none of the other poles of the integrand. The contour C, as in Fox's paper, may be taken to be a rectangle except for necessary loops, having infinite sides parallel to the x-axis and sides of finite length parallel to the y-axis.

2. The following result is well-known[3].

$$ {}_4F_3\left(\begin{matrix} a, 1 + \tfrac{1}{2}a, & c, & d; -1 \\ \tfrac{1}{2}a, 1 + a - c, & 1 + a - d & \end{matrix}\right) = \frac{\Gamma(1 + a - c)\Gamma(1 + a - d)}{\Gamma(1 + a)\Gamma(1 + a - c - d)}. \tag{4} $$

In the above, let us put $a = \rho - 1$, $c = -m$ and $d = -s$. We get

$$ {}_4F_3\left(\begin{matrix} \rho - 1, \tfrac{1}{2}\rho + \tfrac{1}{2}, & -m, & -s; -1 \\ \tfrac{1}{2}\rho - \tfrac{1}{2}, & \rho + m, & \rho + s \end{matrix}\right) = \frac{\Gamma(\rho + m)\Gamma(\rho + s)}{\Gamma(\rho)\Gamma(\rho + m + s)}. \tag{5} $$

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[1] E. W. Barnes, *Proc. London Math. Soc.* (2), **5** (1906), 59-116.

[2] C. Fox, *Proc. London Math. Soc.* (2), **26** (1927), 201.

[3] W. N. Bailey, *Generalized Hypergeometric Series* (Cambridge Tract No. 32, 1935), p. 28.

F. J. W. Whipple, *Proc. London Math. Soc.* (2), **25** (1926), 247-263.

Hence

$$\tfrac{1}{2}\frac{\Gamma(-m)\,\Gamma(-s)}{\Gamma(\rho+m+s)} =$$

$$\sum_{r=0}^{\infty} \frac{(-1)^r\,\Gamma(\rho-1+r)\,\Gamma(\tfrac{1}{2}\rho+\tfrac{1}{2}+r)\,(\Gamma(r-m)\,\Gamma(r-s)}{\Gamma(r+1)\,\Gamma(\tfrac{1}{2}\rho-\tfrac{1}{2}+r)\,\Gamma(\rho+m+r)\,\Gamma(\rho+s+r)}. \quad (6)$$

Combining this with (3), we get

$${}_pf_q\,(\alpha_1, \alpha_2, \ldots \alpha_p;\ \rho_1+m, \rho_2, \ldots \rho_q;\ x) =$$

$$\frac{1}{2\pi i}\ \frac{2}{\Gamma(-m)} \int_C \sum_{r=0}^{\infty} \frac{(-1)^r\,\Gamma(\rho_1-1+r)\,\Gamma(\tfrac{1}{2}\rho_1+\tfrac{1}{2}+r)\,\Gamma(r-m)\,\Gamma(r-s)}{\Gamma(\tfrac{1}{2}\rho_1-\tfrac{1}{2}+r)\,\Gamma(\rho_1+m+r)\Gamma(\rho_1+s+r)\Gamma(r+1)} \times$$

$$\frac{\Gamma(\alpha_1+s)\,\ldots\,\Gamma(\alpha_p+s)}{\Gamma(\rho_2+s)\,\ldots\,\Gamma(\rho_q+s)}\,(-x)^s\,ds. \quad (7)$$

The right hand side can be proved without difficulty to be equal to

$$\frac{2}{\Gamma(-m)} \sum_{r=0}^{\infty} \frac{\Gamma(\rho_1-1+r)\,\Gamma(\tfrac{1}{2}\rho_1+\tfrac{1}{2}+r)\,\Gamma(r-m)}{\Gamma(r+1)\,\Gamma(\tfrac{1}{2}\rho_1-\tfrac{1}{2}+r)\,\Gamma(\rho_1+m+r)} \times$$

$$x^r\,{}_pf_q\begin{pmatrix} \alpha_1+r, & \ldots\ldots\ldots & \alpha_p+r & ;\ x \\ & \rho_1+2r,\ \rho_2+r, & \ldots\ \rho_q+r & \end{pmatrix}. \quad (8)$$

It is easily seen that

$$\sum_{r=0}^{\infty} (-1)^r \frac{\Gamma(\rho_1-1+r)\,\Gamma(\tfrac{1}{2}\rho_1+\tfrac{1}{2}+r)\,\Gamma(r-m)\,\Gamma(r-s)}{\Gamma(r+1)\,\Gamma(\tfrac{1}{2}\rho_1-\tfrac{1}{2}+r)\,\Gamma(\rho_1+m+r)\,\Gamma(\rho_1+s+r)}$$

is uniformly convergent with regard to s, provided that

$$R\,(\rho_1+2m+2s+1) > 0.$$

Again

$$\Gamma(\alpha_1+s) \sim \exp\left\{(\alpha_1+s-\tfrac{1}{2})\log s - s + \tfrac{1}{2}\log 2\pi + O\left(\frac{1}{s}\right)\right\},$$

when $|\arg s| < \pi$. As $s \to \infty$ along C, $I(s)$ remains finite. We have

$$\frac{\Gamma(r-s)\,\Gamma(\alpha_1+s)\,\ldots\,\Gamma(\alpha_p+s)\,(-x)^s}{\Gamma(\rho_1+s+r)\,\Gamma(\rho_2+s)\,\ldots\,\Gamma(\rho_q+s)} \sim$$

$$\exp\{(q-p+1)(s-s\log s) + s\log|x| + O(\log s)\}.$$

If $p < q+1$ or $p = q+1$ and $|x| < 1$, the integrand tends to zero with exponential rapidity as $R(s) \to +\infty$. It follows that the order of integration and summation may be interchanged even when the range of integration is infinite[1].

[1] I am indebted to a referee for this suggestion.

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3. In (8) let us put $p = 1$, $q = 2$, $\rho_1 = 2a$ and $\rho_2 = a + \frac{1}{2}$. Then

$$ {}_1F_2(a;\ 2a+m,\ a+\tfrac{1}{2};\ x) = $$

$$ \frac{2\,\Gamma(a+\frac{1}{2})\,\Gamma(2a+m)}{\Gamma(a)\,\Gamma(-m)} \sum_{r=0}^{\infty} \frac{\Gamma(a+r)\,\Gamma(2a-1+r)\,\Gamma(r-m)}{\Gamma(r+1)\,\Gamma(a-\frac{1}{2}+r)\,\Gamma(2a+m+r)\,\Gamma(2a+2r)}\, x^r \times $$

$$ {}_1F_2(a+r;\ a+\tfrac{1}{2}+r,\ 2a+2r;\ x). \tag{9} $$

Writing $-x^2$ for x and taking m to be a positive integer, we get[1]

$$ x^{2a-1}\,{}_1F_2(a;\ a+\tfrac{1}{2},\ 2a+m;\ -x^2) = $$

$$ 2\sqrt{\pi}\,\frac{\Gamma(a+\frac{1}{2})\,\Gamma(2a+m)}{\Gamma(a)} \sum_{r=0}^{m} \binom{m}{r} \frac{\Gamma(2a-1+r)}{\Gamma(a-\frac{1}{2}+r)}\,\frac{\Gamma(a+\frac{1}{2}+r)}{\Gamma(2a+m+r)}\, J^2_{a-\frac{1}{2}+r}(x). \tag{10} $$

We can also easily deduce the following results

$$ x^{-2\rho-2m}\,J^2_{\rho+m}(x) = $$

$$ \frac{1}{\sqrt{\pi}} \sum_{r=0}^{2m} \binom{2m}{r} \frac{\Gamma(2\rho+r)\,\Gamma(\rho+m+\frac{1}{2}+r)}{\Gamma(\rho+m+1+r)\,\Gamma(2\rho+2m+1+r)\,\Gamma(2\rho+2r)}\, x^{2r} \times $$

$$ {}_1F_2(\rho+m+\tfrac{1}{2}+r;\ \rho+m+1+r,\ 2\rho+2r+1;\ -x^2) \tag{11} $$

$$ {}_1F_1(a;\ \rho+m;\ x) = $$

$$ \frac{\Gamma(\rho+m)}{\Gamma(a)} \sum_{r=0}^{m} \binom{m}{r} \frac{\Gamma(\rho-1+r)\,\Gamma(a+r)}{\Gamma(\rho+m+r)\,\Gamma(\rho-1+2r)}\,(-x)^r\,{}_1F_1(a+r;\ \rho+2r;\ x) \tag{12} $$

whence we get, on writing $2\rho+1$ for ρ, $2m$ for m and $a+\frac{1}{2}$ for a,

$$ x^{-m}\,M_{\rho+m-a,\,\rho+m}(x) = $$

$$ \frac{\Gamma(2\rho+2m+1)}{\Gamma(a+\frac{1}{2})} \sum_{r=0}^{2m} (-1)^r \binom{2m}{r} \frac{\Gamma(a+\frac{1}{2}+r)\,\Gamma(2\rho+r)}{\Gamma(2\rho+2m+1+r)\,\Gamma(2\rho+2r)}\, M_{\rho-a,\,\rho+r}(x), \tag{13} $$

where[2]

$$ M_{k,m}(x) = e^{-\frac{1}{2}x}\,x^{m+\frac{1}{2}}\,{}_1F_1(m+\tfrac{1}{2}-k;\ 2m+1;\ x), \tag{14} $$

$2m$ not being a negative integer.

Since[3]

$$ J_n(x) = \frac{x^{-\frac{1}{2}}}{2^{2n+\frac{1}{2}}\,i^{n+\frac{1}{2}}\,\Gamma(n+1)}\, M_{0,n}(2ix), \tag{15} $$

and

$$ I_n(x) = i^{-n}\,J_n(ix), \tag{16} $$

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[1] $\{J_\nu(x)\}^2 = \{\Gamma(\nu+1)\}^{-2}(\frac{1}{2}x)^{2\nu}\,{}_1F_2(\nu+\frac{1}{2};\ \nu+1,\ 2\nu+1;\ -x^2)$.

[2] E. T. Whittaker and G. N. Watson, *Modern Analysis* (Cambridge, 1920), p. 338.

[3] E. T. Whittaker and G. N. Watson, *Modern Analysis* (Cambridge, 1920), p. 360.

it easily follows from the above that

$$x^{-m-\frac{1}{2}} M_{m,\,a+m}(x) =$$

$$2\sqrt{\pi}\,\frac{\Gamma(2a+2m+1)}{\Gamma(a+\frac{1}{2})}\sum_{r=0}^{2m}(-1)^r\binom{2m}{r}\frac{\Gamma(2a+r)\,\Gamma(a+1+r)}{\Gamma(a+r)\Gamma(2a+2m+1+r)}\,I_{a+r}(\tfrac{1}{2}x). \quad (17)$$

Taking $a = \rho + m$, we find that

$$x^{-m+\frac{1}{2}} I_{\rho+m}(\tfrac{1}{2}x) =$$

$$\frac{1}{\sqrt{\pi}}\sum_{r=0}^{2m}(-1)^r\binom{2m}{r}\frac{\Gamma(2\rho+r)\,\Gamma(\rho+m+\frac{1}{2}+r)}{\Gamma(2\rho+2m+1+r)\,\Gamma(2\rho+2r)}\,M_{-m,\,\rho+r}(x). \quad (18)$$

4. In the formula (4) let us put $d = \frac{1}{2}a + \frac{1}{2}$. We get

$${}_3F_2\left(\begin{matrix}a,\ 1+\frac{1}{2}a, & c;\ -1\\ \frac{1}{2}a,\ 1+a-c & \end{matrix}\right) = \frac{\Gamma(\frac{1}{2}+\frac{1}{2}a)\,\Gamma(1+a-c)}{\Gamma(1+a)\,\Gamma(\frac{1}{2}+\frac{1}{2}a-c)}. \quad (19)$$

Let us put $c = -s$, $a + 1 = 2\rho_1$ and combine with (3). We get, on proceeding as in Art. 2,

$${}_pf_q(a_1, a_2, \ldots a_p;\ \rho_1, \rho_2, \ldots \rho_q;\ x) = \frac{2}{\Gamma(\rho_1)}\sum_{r=0}^{\infty}\frac{\Gamma(2\rho_1-1+r)\,\Gamma(\rho_1+\frac{1}{2}+r)}{\Gamma(r+1)\,\Gamma(\rho_1-\frac{1}{2}+r)}\,x^r \times$$

$${}_pf_q(a_1+r,\ a_2+r,\ \ldots\ a_p+r;\ 2\rho_1+2r,\ \rho_2+r,\ \ldots\ \rho_q+r;\ x) \quad (20)$$

Taking $p = 1$, $q = 2$, $\rho_1 = \alpha$ and $\rho_2 = \alpha + \frac{1}{2}$, we get after a little simplification[1]

$$x^{\alpha-\frac{1}{2}} J_{\alpha-\frac{1}{2}}(2x) = \frac{2\sqrt{\pi}}{\Gamma(\alpha)}\sum_{r=0}^{\infty}(-1)^r\frac{\Gamma(2\alpha-1+r)\,\Gamma(\alpha+\frac{1}{2}+r)}{\Gamma(r+1)\,\Gamma(\alpha-\frac{1}{2}+r)}\,J^2_{\alpha-\frac{1}{2}+r}(x), \quad (21)$$

where $\alpha \geqq \frac{1}{2}$.

In a similar manner we can prove that

$$x^{\frac{1}{2}\rho} M_{\frac{1}{2}\rho-\alpha,\,\frac{1}{2}\rho-\frac{1}{2}}(x) = \frac{1}{\Gamma(\alpha)}\sum_{r=0}^{\infty}\frac{\Gamma(2\rho-1+r)\,\Gamma(\alpha+r)}{\Gamma(r+1)\,\Gamma(2\rho-1+2r)}\,M_{\rho-\alpha,\,\rho-\frac{1}{2}+r}(x). \quad (22)$$

5. Again in (4) let us write $c = -s$, $d = \alpha + s$. We get

$${}_4F_3\left(\begin{matrix}a,\ 1+\frac{1}{2}a, & -s, & \alpha+s;\ -1\\ \frac{1}{2}a,\ 1+a+s,\ 1+a-\alpha-s & & \end{matrix}\right) = \frac{\Gamma(1+a+s)\Gamma(1+a-\alpha-s)}{\Gamma(1+a)\,\Gamma(1+a-\alpha)}. \quad (23)$$

Hence

$$\frac{1}{2}\frac{\Gamma(-s)\,\Gamma(\alpha+s)}{\Gamma(1+a-\alpha)} = \sum_{r=0}^{\infty}\frac{(-1)^r\,\Gamma(a+r)\,\Gamma(1+\frac{1}{2}a+r)\,\Gamma(r-s)\,\Gamma(\alpha+s+r)}{\Gamma(r+1)\,\Gamma(\frac{1}{2}a+r)\,\Gamma(1+a+s+r)\,\Gamma(1+a-\alpha-s+r)}. \quad (24)$$

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[1] This is a special case of equation (12) with $Z=z$, $C_\nu = J_\nu$, $\nu = \alpha - \frac{1}{2}$. See G. N. Watson, *Bessel Functions* (Cambridge, 1922), p. 366.

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Combining this with (3), we get

$$ {}_pf_q(\alpha_1, \alpha_2, \ldots \alpha_p;\ \rho_1, \rho_2, \ldots \rho_q;\ x) = $$

$$ \frac{1}{2\pi i} 2\,\Gamma(1 + a - \alpha_1) \sum_{r=0}^{\infty} \int_C (-1)^r \frac{\Gamma(a + r)\,\Gamma(\tfrac{1}{2}a + 1 + r)}{\Gamma(r + 1)\,\Gamma(\tfrac{1}{2}a + r)} \times $$

$$ \frac{\Gamma(\alpha_2+s) \ldots \Gamma(\alpha_p+s)\,\Gamma(r-s)\,\Gamma(\alpha_1+s+r)\,(-x)^s}{\Gamma(\rho_1+s)\,\Gamma(\rho_2+s) \ldots \Gamma(\rho_q+s)\,\Gamma(a+1+s+r)\,\Gamma(a+1-\alpha_1-s+r)}\, ds, \quad (25) $$

$$ = \frac{2}{\Gamma(\alpha_1 - a)} \sum_{r=0}^{\infty} \frac{\Gamma(a + r)\,\Gamma(\tfrac{1}{2}a + 1 + r)}{\Gamma(r + 1)\,\Gamma(\tfrac{1}{2}a + r)} x^r \times $$

$$ {}_{p+1}f_{q+1}\left(\begin{matrix}\alpha_1 - a,\ \alpha_1 + 2r,\ \alpha_2 + r,\ \ldots\ \alpha_p + r, \\ a+1 + 2r,\ \rho_1 + r,\ \rho_2+r,\ \ldots\ \rho_q + r\end{matrix};\ -x\right), \quad (26) $$

the change in the order of summation and integration being justifiable.

Taking $p = 2$ and $q = 1$, we have, on putting $a = \alpha_1 - 1$,

$$ {}_2F_1(\alpha_1, \alpha_2;\ \rho_1;\ x) = $$

$$ \frac{\Gamma(\rho_1)}{\Gamma(\alpha_1)\,\Gamma(\alpha_2)} \sum_{r=0}^{\infty} \frac{\Gamma(\alpha_1 - 1 + r)\,\Gamma(\alpha_2 + r)\,\Gamma(\alpha_1 + 2r)}{\Gamma(r + 1)\,\Gamma(\rho_1 + r)\,\Gamma(\alpha_1 - 1 + 2r)} x^r\, {}_2F_1(1, \alpha_2 + r;\ \rho_1 + r;\ -x). \quad (27) $$

Putting $x = -1$ in the above and writing $\alpha_1 + 1$ and $\rho_1 + 1$ for α_1 and ρ_1 respectively, we have, if $R(\rho_1 - \alpha_1 - \alpha_2) > 0$,

$$ {}_3F_2(\alpha_1, \tfrac{1}{2}\alpha_1 + 1, \alpha_2;\ \tfrac{1}{2}\alpha_1, \rho_1;\ -1) = \frac{\rho_1 - \alpha_2}{\rho_1}\, {}_2F_1(\alpha_1 + 1, \alpha_2;\ \rho_1 + 1;\ -1). \quad (28) $$

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