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BILIPSCHITZ DETERMINACY OF QUASIHOMOGENEOUS GERMS

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Abstract. We obtain estimates for the degree of bilipschitz determinacy of quasihomogeneous function-germs.

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1. Introduction. A basic problem in Singularity Theory is the local classification of mappings module diffeomorphisms. In 1965, H. Whitney justified the rigidity of the classification problem by C^1-diffeomorphism giving the following example:

$$F_t(x, y) = xy(x - y)(x - ty); \quad 0 < t < 1 \tag{1}$$

which presents the following phenomenon: for any $t \neq s$ in $I = (0, 1)$ it is not possible to construct a C^1-diffeomorphism $\phi : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ such that $F_t = F_s \circ \phi$. This motivated the classification of mappings by "isomorphisms" weaker than diffeomorphisms.

There is an extensive literature related to C^r-equivalence ($1 \leq r < \infty$) of map-germs, among them [**5**], [**4**] and [**1**] which are more closely related to this work. However, only few recent works deal with the problem of bilipschitz classification of map-germs. This work is inspired in a recent paper by J.-P. Henry and A. Parusinski [**2**], where they show that the bilipschitz equivalence of analytic function-germs admits continuous moduli. We obtain estimates for the degree of bilipschitz determinacy of quasihomogeneous function-germs. Examples are given to show that the estimates are sharp.

2. Bilipschitz equivalence. A mapping $\phi : U \subset \mathbb{R}^n \to \mathbb{R}^p$ is called *Lipschitz* if there exists a constant $\lambda > 0$ such that:

$$\|\phi(x) - \phi(y)\| \leq \lambda \|x - y\| \; \forall \, x, y \in U.$$

When $n = p$ and ϕ has a Lipschitz inverse, we say that ϕ is *bilipschitz*.

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Two germs $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ are called *bilipschitz equivalent* if there exists a bilipschitz map-germ $\phi : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that $f = g \circ \phi$.

EXAMPLE 2.1. Let $f, g : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ be given by $f(x) = x$, $g(x) = x^3$. It is easy to show that f and g are not bilipschitz equivalent. On the other hand, there is a homeomorphism $\phi : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ such that $f = \phi \circ g$.

Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be the germ of an analytic function,

$$f(x) = f_m(x) + f_{m+1}(x) + \cdots,$$

with f_i a homogeneous form of degree i, and $f_m \neq 0$. We denote by $m_f := m$, the *multiplicity* of f. We say that f has *non-degenerate tangent cone* if $0 \in \mathbb{R}^n$ is the only point in $\mathbb{R}^n$ in which

$$\frac{\partial f_m}{\partial x_1} = \cdots = \frac{\partial f_m}{\partial x_n} = 0.$$

PROPOSITION 2.2. *Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be the germ of an analytic function. Then*

$$m_f = ord_r[sup|f_{|B(0,r)}|],$$

where $B(0, r)$ denotes the ball centered at the origin with radius r.

Proof. Let $\alpha = ord_r[sup|f_{|B(0,r)}|]$. Write

$$f(x) = f_m(x) + f_{m+1}(x) + \cdots$$

with f_i a homogeneous form of degree i, and $f_m \neq 0$. Let $x = (x_1, \ldots, x_n)$ be such that $f_m(x) \neq 0$. Then, for $r > 0$ small enough, we have

$$\begin{aligned} |f(rx)| &= r^m |f_m(x) + r f_{m+1}(x) + \cdots| \\ &\geq K r^m \end{aligned}$$

for some constant $K > 0$, hence $m \geq \alpha$.

On the other hand, from the Curve Selection Lemma, there exists an analytic arc $\gamma : [0, \epsilon) \to \mathbb{R}^n$, $\gamma(0) = 0$, such that

$$\alpha = ord_r |f(\gamma(r))|$$

and $|\gamma(r)| \leq r$ for each $r > 0$. Since $\gamma(0) = 0$, we can write $\gamma(r) = r\tilde{\gamma}(r)$ with $\lim_{r \to 0} \tilde{\gamma}(r) \leq 1$. Therefore,

$$\begin{aligned} |f(\gamma(r))| &= r^m |f_m(\tilde{\gamma}(r)) + r f_{m+1}(\tilde{\gamma}(r)) + \cdots| \\ &\leq L r^m \end{aligned}$$

for some constant $L > 0$. Hence, $m \leq \alpha$.

COROLLARY 2.3. *Let $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be germs of analytic functions. If f and g are bilipschitz equivalent, then $m_f = m_g$.*

The corollary above in the complex case was proved by J.-J. Risler and D. Trotman in [**3**]. It is obvious that the converse statement is false, but we can prove the following result.

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Proposition 2.4. *Let $F_t : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be a smooth family of smooth function-germs. If m_{F_t} is constant and F_t has non-degenerate tangent cone for each t, then for each $t \neq s$, F_t and F_s are bilipschitz equivalent.*

The result above will follow as consequence of Theorem 3.3.

Corollary 2.5. *The family* (1) *satisfies: F_t and F_s are bilipschitz equivalent $\forall\, t, s \in (0, 1)$.*

It is valuable to observe that the Proposition 2.4 does not guarantee the non-rigidity of the bilipschitz classification problem for analytic functions. In fact, J.-P. Henry and A. Parusinski ([**2**]) presented the family $F_t : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ given by $F_t(x, y) = x^3 - 3t^2xy^4 + y^6$ which satisfies: for any $t \neq s \in (0, \frac{1}{2})$ there is no bilipschitz map-germ $\phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ such that $F_t = F_s \circ \phi$. The proof is based on the analysis of the expansion of the germs of the family along each arc of their polar curves. The argument in [**2**] also holds in the real case, that is, the following holds:

Proposition 2.6. *The family $F_t : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ given by $F_t(x, y) = x^3 - 3t^2xy^4 + y^6$ satisfies: for any $t \neq s \in (0, \frac{1}{2})$ there is no bilipschitz map $\phi : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ such that $F_t = F_s \circ \phi$.*

Note that F_t is a deformation of the quasihomogeneous germ $f = x^3 + y^6$ which has an isolated singularity at the origin. Therefore, it is natural to ask for which $\theta(x, y)$ the family $f + t\theta$ is bilipschitz trivial.

3. Bilipschitz determinacy of quasihomogeneous germs. Let $f_t : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ $t \in I$ (an interval in $\mathbb{R}$) be a smooth family of smooth function-germs. That is, there is a neighborhood U of 0 in $\mathbb{R}^n$ and a smooth function $F : U \times I \to \mathbb{R}$ such that $F(0, t) = 0$ and $f_t(x) = F(x, t)$ $\forall t \in I, \forall x \in U$. We call f_t *strongly bilipschitz trivial* when there is a continuous family of λ-Lipschitz map-germs (vector field) $v_t : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that

$$\frac{\partial f_t}{\partial t}(x) = (df_t)_x(v_t(x))$$

$\forall t \in I$ and $\forall x$ near 0 in $\mathbb{R}^n$.

Theorem 3.1. *If f_t is strongly bilipschitz trivial, then for each $t \neq s \in I$ there is a bilipschitz map-germ $\phi : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that $f_t = f_s \circ \phi$.*

The theorem above is known as a result of Thom-Levine type and its proof is immediate, since the flow of a Lipschitz vector field is bilipschitz.

Let $\mathcal{E}_n$ be the space of smooth function-germs $(\mathbb{R}^n, 0) \to \mathbb{R}$. Given $f \in \mathcal{E}_n$, we denote $Nf(x) = \sum[\frac{\partial f}{\partial x_i}(x)]^2$.

Let $(r_1, \dots, r_n : d)$; $r_1, \dots, r_n, d \in \mathbb{Q}^+$. We recall that a function f is called *quasihomogeous* of type $(r_1, \dots, r_n : d)$ if f satisfies the following equation:

$$f(\lambda \cdot x) = \lambda^d f(x_1, \dots, x_n)$$

$\forall\, \lambda \in \mathbb{R} - \{0\}$ and $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, where $\lambda \cdot x = (\lambda^{r_1}x_1, \dots, \lambda^{r_n}x_n)$. With respect to the given weights $(r_1, \dots, r_n)$, for each monomial $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, we define $\mathrm{fil}(x^\alpha) = \sum_{i=1}^n \alpha_i r_i$. We define a filtration in the ring $\mathcal{E}_n$ via the function $\mathrm{fil}(f) = \min\{\mathrm{fil}(x^\alpha)\ :$

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$(\frac{\partial^\alpha f}{\partial x^\alpha})(0) \neq 0\}$, for each $f \in \mathcal{E}_n$. We can extend this definition to $\mathcal{E}_{n+1}$, the ring of 1-parameter families of smooth function-germs in $\mathcal{E}_n$, by defining $\mathrm{fil}(x^\alpha t^\beta) = \mathrm{fil}(x^\alpha)$.

Let $(r_1, \ldots, r_n : 2k)$ be fixed. The standard control function of type $(r_1, \ldots, r_n : 2k)$ is $\rho(x) = x_1^{2\alpha_1} + \cdots + x_n^{2\alpha_n}$, where the $\alpha_i = \frac{k}{r_i}$ are chosen such that the function ρ is quasihomogeneous of type $(r_1, \ldots, r_n : 2k)$.

LEMMA 3.2. *Let $(r_1, \ldots, r_n : 2k)$; $r_1, \ldots, r_n, k \in \mathbb{Q}^+$, with $r_1 \leq \cdots \leq r_n$, and ρ the standard control function of type $(r_1, \ldots, r_n : 2k)$. If $h_t(x)$ is a continuous family of polynomial function-germs in $0 \in \mathbb{R}^n$ such that:*

$$\mathrm{fil}(h_t) \geq 2k + r_n, \quad t \in [0, 1]. \tag{2}$$

Then $\frac{h_t(x)}{\rho(x)}$ is the germ of a c-Lipschitz function in $0 \in \mathbb{R}^n$, with c independent of t.

Proof. Let $h_t(x)$ be a polynomial function such that $\mathrm{fil}(h_t) \geq 2k + r_n$.

Let $f(x) = \frac{h_t(x)}{\rho(x)}$, then

$$Grad(f(x)) = \frac{1}{\rho(x)^2}(\rho.Grad(h_t) - h_t.Grad(\rho))$$

and

$$\begin{aligned} \mathrm{fil}\left(\rho.\frac{\partial h_t}{\partial x_i} - h_t.\frac{\partial \rho}{\partial x_i}\right) &\geq \mathrm{fil}(h_t) + \mathrm{fil}(\rho) - r_n \\ &\geq 2k + \mathrm{fil}(\rho) \\ &= \mathrm{fil}(\rho^2) \end{aligned}$$

Therefore $Grad(f)$ is bounded and f is Lipschitz.

THEOREM 3.3. *Let $f : \mathbb{R}^n, 0 \to \mathbb{R}, 0$ be the germ of a quasihomogeneous polynomial function of type $(r_1, \ldots, r_n : d)$, $r_1 \leq \cdots \leq r_n$ with isolated singularity. Let $f_t(x) = f(x) + t\theta(x, t)$, $t \in [0, 1]$, be a smooth deformation of f. If $\mathrm{fil}(\theta) \geq d + r_n - r_1$, then f_t admits a strong bilipschitz trivialization along $I = [0, 1]$.*

Proof. We can see that for each i there exists a s_i such that $\frac{\partial f}{\partial x_i}$ is quasihomogeneous of the type $(r_1, \ldots, r_n : s_i)$, $s_i = d - r_i$.

Let N^*f be defined by

$$N^*f = \sum \left[\frac{\partial f}{\partial x_i}\right]^{2\alpha_i},$$

where $\alpha_i = \frac{k}{s_i}$ and $k = l.c.m.(s_i)$. Therefore N^*f is a quasihomogeneous function of the type $(r_1, \ldots, r_n : 2k)$.

The lemma bellow is proved in **[4]**.

LEMMA 3.4. *There exist constants $0 < c_2 < c_1$ such that*

$$c_2\rho(x) \leq N^*f_t(x) \leq c_1\rho(x).$$

We have the following equation;

$$\frac{\partial f_t}{\partial t} N^*f_t = df_t(W),$$

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where W is given by

$$W = \sum W_i \frac{\partial}{\partial x_i} \text{ where } W_i = \frac{\partial f_t}{\partial t}\left[\frac{\partial f}{\partial x_i}\right]^{2\alpha_i - 1}.$$

Since $\mathrm{fil}(\frac{\partial f_t}{\partial t}) \geq d + r_n - r_1$ and

$$\begin{aligned}
\mathrm{fil}\left(\left[\frac{\partial f_t}{\partial x_i}\right]^{2\alpha_i-1}\right) &= (2\alpha_i - 1)\mathrm{fil}\left(\frac{\partial f_t}{\partial x_i}\right)\\
&= (2\alpha_i - 1)(d - r_i)\\
&= 2k - d + r_i\\
&\geq 2k - d + r_1
\end{aligned}$$

we have that $\min \mathrm{fil}(W_i) \geq \mathrm{fil}(\theta) + 2k - d + r_1 \geq 2k + r_n$.

Let $v : \mathbb{R}^n \times \mathbb{R}, 0 \to \mathbb{R}^n \times \mathbb{R}, 0$ be the vector field given by $\frac{W}{N^* f_t}$. From Lemma 3.2, it follows that v is a Lipschitz vector field.

Finally the equation $(\frac{\partial f_t}{\partial t})(x) = (df_t)_x(v(x, t))$ gives the strong bilipschitz triviality of the family $f_t(x)$ along a small open interval around $t = 0$. Since the same argument is true for each $t \in I$, the proof is complete.

The following result shows that the estimate given in Theorem 3.3 is sharp.

PROPOSITION 3.5. *Let* $f_t : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0);\ t \in I = (-\delta, \delta) \subset \mathbb{R}$ *be given by*

$$f_t(x, y) = \frac{1}{3}x^3 - t^2 x y^{3n-2} + y^{3n}.$$

Then f_t *is not strongly bilipschitz trivial.*

REMARK 3.6. Let $f(x, y) = \frac{1}{3}x^3 + y^{3n}$. Note that f is quasihomogeneous of type $(n, 1 : 3n)$. From Theorem 3.3 it follows that $f + t\theta$ is strongly bilipschitz trivial for each $\theta(x, t)$ such that $fil(\theta) \geq 4n - 1$. On the other hand, $fil(xy^{3n-2}) = 4n - 2$, therefore the proposition above proves the sharpness of the Theorem 3.3.

Proof (of the Proposition 3.5) Let $m = 3n - 2$. Here we repeat the argument-proof from Theorem 1.1 in [**2**]. Suppose that $v(x, y, t) = v_1(x, y, t)\frac{\partial}{\partial x} + v_2(x, y, t)\frac{\partial}{\partial y}$ is a vector field such that:

$$\left(\frac{\partial f_t}{\partial t}\right)(x, y) = (df_t)_x(v(x, y, t))$$

The polar curve $\{(x, y) \in \mathbb{R}^2 : \frac{\partial f_t}{\partial x}(x, y) = 0\}$ is equal to the set $\{(x, y) \in \mathbb{R}^2 : x^2 = t^2 y^m\}$. Thus, $v_2(ty^{m/2}, y, t)$ and $v_2(-ty^{m/2}, y, t)$ satisfy:

$$v_2(ty^{m/2}, y, t)\frac{\partial f_t}{\partial y}(ty^{m/2}, y, t) = -\frac{\partial f_t}{\partial t}(ty^{m/2}, y, t) \tag{3}$$

$$v_2(-ty^{m/2}, y, t)\frac{\partial f_t}{\partial y}(-ty^{m/2}, y, t) = -\frac{\partial f_t}{\partial t}(-ty^{m/2}, y, t). \tag{4}$$

From equations (3) and (4) we have:

$$v_2\big(ty^{m/2}, y, t\big) = \frac{2t^2y^{m/2-1}}{-mt^3y^{m/2-2} + 3n}$$

$$v_2\big(-ty^{m/2}, y, t\big) = \frac{-2t^2y^{m/2-1}}{mt^3y^{m/2-2} + 3n}$$

Thus,

$$v_2\big(ty^{m/2}, y, t\big) - v_2\big(-ty^{m/2}, y, t\big) \sim y^{m/2-1} \tag{5}$$

On the other hand,

$$\big\|\big(ty^{m/2}, y, t\big) - \big(-ty^{m/2}, y, t\big)\big\| \sim y^{m/2} \tag{6}$$

But, (5) and (6) show that v_2 is not Lipschitz. Hence f is not strongly bilipschitz trivial.

The invariant for bilipschitz equivalence $\mathrm{Inv}(f_t)$ presented in **[2]** does not distinguish the elements of the family $f_t(x, y) = \frac{1}{3}x^3 - t^2xy^{3n-2} + y^{3n}$; $\forall n > 2$.

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