PURE-INJECTIVE MODULES

by SYED M. FAKHRUDDIN

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Abstract. It is proved that a pure-injective module over a commutative ring with unity is a summand of a product of duals of finitely presented modules, where duals are to be understood with reference to the circle group T, with induced module structures. Using similar techniques, it is also shown that an R-module has its underlying group pure-injective precisely when it is a submodule of a product of duals of cyclic modules and also a summand as abelian group of the same product.

All rings considered are commutative with unity and all modules are unitary. Let Mod-R be the category of modules over a ring R. An exact sequence $0 \to A \to B \to C \to 0$ in Mod-R is *pure-exact* if, for any N in Mod-R, $0 \to A \otimes N \to B \otimes N \to C \otimes N \to 0$ is exact. A module M is *pure-injective* if it has the injective property relative to the class of pure-exact sequences in Mod-R. A module P is FP (finitely presented) if it is the image of a finitely generated free module with a finitely generated kernel. A module M is *compact* if it carries a Hausdorff compact topology so that M is a topological R-module. Let T denote the circle group—the group of real numbers modulo the integers—and let X^* denote the dual module $\mathrm{Hom}_{\mathbb{Z}}(X, T)$ of the module X.

The following results are known.

PROPOSITION 1. *An R-module M is pure-injective precisely when M is a summand of a compact R-module.* [**4**, p. 704, Theorem 2.]

PROPOSITION 2. *An R-module M is compact if and only if $M = N^*$ for some N in* Mod-R. [**3**, p. 242, Satz 1.6.]

PROPOSITION 3. *If D is an injective module, then* $\mathrm{Hom}(-, D)$ *splits every pure-exact sequence in* Mod-R. [**1**, Proposition 3.3.]

Now we have

THEOREM 1. *Let R be a commutative ring with unity. Then a module M is pure-injective if and only if M is a summand of a product of duals of* FP *modules.*

Proof. The dual of an FP module is pure-injective (Propositions 1 and 2). The class of pure-injective modules is closed for products and summands. The " if " part of the theorem follows.

Conversely, let M be pure-injective. Then M is a summand of a dual module N^*. It is enough to show that N^* has the desired structure as given in the theorem. Consider a " pure-projective " resolution of N (See [4, p. 700, Proposition 1] for details); this is a pure-exact sequence $0 \to K \to \oplus P_i \to N \to 0$, where each P_i is FP. Now, for any X in Mod-R, $X^* \cong \mathrm{Hom}(X, R^*)$ and R^* is injective in Mod-R. Hence, by Proposition 3, we have $K^* \oplus N^* \cong \Pi P_i^*$ in Mod-R. Thus the proof of the theorem is complete.

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A ring is called a generalized valuation ring if it is a local ring and the divisibility relation among its elements is a total order. A ring is called an LGV (locally generalized valuation) ring if all its maximal localizations are generalized valuation rings.

Warfield [5, Theorem 3, p. 169] proved the following

THEOREM. *A commutative ring with unity is an* LGV *ring if and only if every* FP *module over it is a summand of a direct sum of cyclic modules.*

So it follows that, over an LGV, the dual of an FP module is a summand of a product of duals of cyclic modules. Hence we have the

COROLLARY. *Let R be an* LGV; *then a module M is pure-injective precisely when M is a summand of a product of duals of cyclic modules.*

It is obvious from Propositions 1 and 2 that an R-pure-injective module is $\mathbb{Z}$-pure-injective (See also [2], Remark on page 178 and Problem 29). It is interesting to determine the structure of those R-modules whose underlying groups are pure-injective.

THEOREM 2. *An R-module M over a commutative ring R with unity is $\mathbb{Z}$-pure-injective exactly when M is a submodule of a product of duals of cyclic modules and is a $\mathbb{Z}$-summand of the same product.*

Proof. Duals of cyclic modules are $\mathbb{Z}$-pure-injective. Hence a product of such modules is also $\mathbb{Z}$-pure-injective. Now any submodule of such a product which is also a $\mathbb{Z}$-summand is clearly $\mathbb{Z}$-pure-injective.

Conversely, let M be an R-module which is $\mathbb{Z}$-pure-injective. Let N be its pure-injective envelope in Mod-R [4, p. 709, Proposition 6]. Then M is R-pure, hence $\mathbb{Z}$-pure in N, and thus a $\mathbb{Z}$-summand of N. Now N is a summand of a dual module E^*, for some E in Mod-R. Thus it is enough to prove the theorem for E^*.

Consider the following exact sequence in Mod-R.

$$0 \to K \to \bigoplus_{\substack{x \neq 0 \\ x \in E}} \frac{R_x}{I_x} \xrightarrow{\pi} E \to 0,$$

where $\pi(\bar{1}_x) = x$ and $I_x = (0:x)_R$ for each $x \neq 0$ in E. The above exact sequence is $\mathbb{Z}$-pure-exact. Let $nz = y$ with $y \in K$. We may assume that $y \neq 0$ and hence $n \neq 0$ in R. If z is not in K, then $\pi(z) = x$ and x is not zero in E. Consequently $r = z - \bar{1}_x$ lies in K. Since $y \in K$, we have $0 = \pi(y) = \pi(nz) = n\pi(z) = nx$. Hence $n \in (0:x) = I_x = (0:\bar{1}_x)$ and thus $n \cdot \bar{1}_x = 0$. Finally, $nr = nz - n \cdot \bar{1}_x = nz = y$. Thus the sequence is $\mathbb{Z}$-pure exact.

Now applying $\mathrm{Hom}_{\mathbb{Z}}(\ , T) = (\ )^*$ we have, by Proposition 3, $0 \to E^* \to \Pi(R_x/I_x)^* \to K^* \to 0$ is R-exact as well as $\mathbb{Z}$-split exact. Thus the theorem is proved.

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UNIVERSITY OF HAWAII
HONOLULU, HAWAII 96822

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