Part III. Groups of matrices: K_1

Chapter 9. Definition of K_1 (elementary matrices; commutators and $K_1(R)$; determinants; the Bass K_1 of a category)

Chapter 10. Stability for $K_1(R)$ (surjective stability; injective stability)

Chapter 11. Relative K_1 (congruence subgroups of $GL_n(R)$; congruence subgroups of $SL_n(R)$; Mennicke symbols)

Part IV. Relations among matrices: K_2

Chapter 12. $K_2(R)$ and Steinberg Symbols (elementary matrices; commutators and $K_1(R)$; determinants; the Bass K_1 of a category)

Chapter 13. Exact sequences (the relative sequence; excision and the Mayer-Vietoris sequence; the localization sequence)

Chapter 14. Universal algebras (presentations of algebras; graded rings; the tensor algebra; symmetric and exterior algebras; the Milnor ring; tame symbols; norms on Milnor K-theory; Matsumoto's theorem)

Part V. Sources of K_2

Chapter 15. Symbols in arithmetic (Hilbert symbols; metric completion of fields; the p-adic numbers and quadratic reciprocity; local fields and norm residue symbols)

Chapter 16. Brauer groups (the Brauer group of a field; splitting fields; twisted group rings; the K_2 connection)

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MACLACHLAN, C. AND REID, A. W. *The arithmetic of hyperbolic* 3*-manifolds* (Springer, 2003), 0 387 98386 4 (hardback), £45.50.

The study of Kleinian groups and hyperbolic 3-manifolds involves the interplay of many different mathematical ideas. Much has been written on the subject from the points of view of complex analysis, topology, geometry and group theory. In this book Maclachlan and Reid give a comprehensive treatment of hyperbolic 3-manifolds and Kleinian groups from the viewpoint of algebraic number theory. Both authors have very successfully exploited arithmetic techniques in Kleinian groups and it is extremely useful to have a definitive account of the techniques and ideas they use and have developed.

As is well known, a hyperbolic 3-manifold M may be written as the quotient of hyperbolic 3-space $\boldsymbol{H}^3$ by a discrete, torsion-free subgroup Γ of $\mathrm{PSL}(2,\mathbb{C})$. If we drop the hypothesis that Γ is torsion-free, then the quotient $\boldsymbol{H}^3/\Gamma$ becomes an orbifold. In both cases the group Γ is said to be *Kleinian*. The hyperbolic manifolds and orbifolds discussed in this book are generally of finite volume (though they may have cusps). In this case the associated Kleinian group is said to have finite covolume. It is completely standard to switch between thinking of Γ as a group of Möbius transformations in $\mathrm{PSL}(2,\mathbb{C})$ and a group of matrices in $\mathrm{SL}(2,\mathbb{C})$.

If we are given Γ, a possibly torsion-free subgroup of $\mathrm{PSL}(2,\mathbb{C})$, then it is not an easy matter to decide whether or not it is discrete. Roughly speaking, there are three main techniques for showing the discreteness of Γ. Firstly, following Klein and Maskit, one may be able to show that Γ can be assembled from smaller groups by the operations of free product (possibly with amalgamation) and HNN extension. Secondly, we may use the geometry of Γ acting on $\boldsymbol{H}^3$

by constructing a fundamental polyhedron and using Poincaré's polyhedron theorem to show discreteness. Thirdly, we can use arithmetic techniques. This is the focus of Maclachlan and Reid's book.

The starting point for use of number theory in Kleinian groups is Mostow's rigidity theorem. A consequence of this theorem is that the matrix entries in $\mathrm{SL}(2,\mathbb{C})$ of a finite covolume Kleinian group Γ may be taken to lie in a number field, that is, a finite extension of $\mathbb{Q}$. For certain Kleinian groups we may exploit a further link with number theory. This arises as a special case of the connection between lattices in Lie groups and integral points in algebraic groups defined over the rational numbers, that is arithmetic groups. A celebrated theorem of Borel and Harish-Chandra says that, under mild hypotheses, arithmetic subgroups of Lie groups are discrete. The converse is true for lattices in Lie groups of rank at least 2, using a theorem of Margulis. For $\mathrm{SL}(2,\mathbb{C})$ (and certain other groups of rank 1) this is not true and there exist non-arithmetic lattices.

We may describe arithmetic subgroups of $\mathrm{SL}(2,\mathbb{C})$ using quaternion algebras. These are defined as follows. If K is a field, let A be a four-dimensional vector space over K with basis $\{1,i,j,k\}$. Define multiplication on A by letting 1 be the identity element, defining $i^2=a.1$, $j^2=b.1$, $ij=-ji=k$ and extending linearly. Denote A by the *Hilbert symbol* $((a,b)/K)$. For example, Hamilton's quaternions are $\mathbb{H}=((-1,-1)/\mathbb{R})$ and $M_2(K)=((1,1)/K)$. Let $\sigma : K \longmapsto \mathbb{R}$ be a real embedding of a number field K. Then the quaternion algebra $((a,b)/K)$ is said to be *ramified* at σ if $((\sigma(a),\sigma(b))/\mathbb{R})$ is isomorphic to $\mathbb{H}$. That is $\sigma(a)$ and $\sigma(b)$ are both negative.

Arithmetic subgroups of Lie groups are analogous to *orders* in quaternion algebras. An *order* $\mathcal{O}$ is a finitely generated submodule of A with a 1. Let K be a number field with exactly one complex place and A a quaternion algebra over K ramified at all real places. Let ρ be an embedding of A into $M_2(\mathbb{C})$. In this setting, a subgroup Γ of $\mathrm{SL}(2,\mathbb{C})$ is *arithmetic* if it is commensurable with the image under ρ of the elements of $\mathcal{O}$ with unit norm, for some order $\mathcal{O}$ of A.

Maclachlan and Reid's book gives a clear account of the algebraic machinery behind the constructions outlined above without assuming very much background knowledge from their readers. The reader is supposed to be familiar with some parts of algebraic number theory; such material would typically be contained in a standard final year undergraduate course in the subject. For readers who have not taken such a course, there is an accessible introductory chapter with background material and references. After this chapter, the authors introduce Kleinian groups and hyperbolic manifolds before going on to develop quaternion algebra and invariant trace fields. At this point, there are very useful chapters on examples and applications. Thereafter quaternion algebras are developed further, so that the notion of an arithmetic Kleinian group may be introduced. The remainder of the book is dedicated to giving a comprehensive treatment of arithmetic Kleinian groups and their quotient orbifolds.

Throughout the book, Maclachlan and Reid use examples to motivate and illustrate the ideas they develop. This helps the treatment to be very readable. Furthermore, at the end of each chapter is a section called 'Further Reading'. This extremely useful section both anchors the preceding material in the literature and points the reader to results that go beyond the scope of the book. Finally, in a series of appendices, the authors gather together useful reference data in the form of tables and lists.

This book is a welcome addition to the literature on Kleinian groups and hyperbolic geometry. It is both an accessible introduction to the number theoretic side of the field and a convenient source of reference material for the expert.

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