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DELEEUW'S THEOREM ON LITTLEWOOD-PALEY FUNCTIONS

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Abstract. We establish certain deLeeuw type theorems for Littlewood-Paley functions. By these theorems, we know that the boundedness of a Littlewood-Paley function on $\mathbb{R}^n$ is equivalent to the boundedness of its corresponding Littlewood-Paley function on the torus $\mathbb{T}^n$.

§1. Introduction

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space and $\mathbb{T}^n$ be the n-dimensional torus. $\mathbb{T}^n$ can be identified with $\mathbb{R}^n/\Lambda$, where Λ is the unit lattice which is the additive group of points in $\mathbb{R}^n$ having integral coordinates. For an $L^1(\mathbb{R}^n)$ function Φ we define $\Phi_t(x) = 2^{-tn}\Phi(x/2^t)$, $t \in \mathbb{R}$. Then the Fourier transform of Φ_t is $\hat{\Phi}_t(\xi) = \hat{\Phi}(2^t\xi)$. The Littlewood-Paley g-function $g_\Phi(f)$ on $\mathbb{R}^n$ is defined by

$$g_\Phi f(x) = \left(\int_{\mathbb{R}} |\Phi_t * f(x)|^2 dt\right)^{1/2}, \tag{1.1}$$

initially, for f in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$.

The Littlewood-Paley g-function on $\mathbb{T}^n$ can be defined similarly. For $\tilde{f} \in C^\infty(\mathbb{T}^n)$, $\tilde{f}$ has the Fourier series

$$\tilde{f}(x) = \sum_{k\in\Lambda} a_k e^{2\pi i\langle k,x\rangle},$$

where $\langle x, \xi\rangle = x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n$ for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$. We let

$$G_\Phi \tilde{f}(x) = \left(\int_{\mathbb{R}} |\tilde{\Phi}_t * \tilde{f}(x)|^2 dt\right)^{1/2} \tag{1.1$'$}$$

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where

$$\tilde{\Phi}_t * \tilde{f}(x) = \sum_{k\in\Lambda} \hat{\Phi}(2^t k) a_k e^{2\pi i\langle k,x\rangle}.$$

For a nice function Φ, the following theorem is well-known.

Theorem A. *Suppose that $\Phi \in \mathcal{S}(\mathbb{R}^n)$ satisfies $\int_{\mathbb{R}^n} \Phi(x)dx = 0$. Then for any $p \in (1,\infty)$*

$$\|g_\Phi(f)\|_{L^p(\mathbb{R}^n)} \le A_1 \|f\|_{L^p(\mathbb{R}^n)}, \tag{1.2}$$

$$\|G_\Phi(\tilde{f})\|_{L^p(\mathbb{T}^n)} \le A_2 \|\tilde{f}\|_{L^p(\mathbb{T}^n)}. \tag{1.3}$$

In additional, if Φ is radial and non-zero, then

$$\|f\|_{L^p(\mathbb{R}^n)} \le B_1 \|g_\Phi(f)\|_{L^p(\mathbb{R}^n)}, \tag{1.2$'$}$$

$$\|\tilde{f}\|_{L^p(\mathbb{T}^n)} \le B_2 \|G_\Phi(\tilde{f})\|_{L^p(\mathbb{T}^n)} \tag{1.3$'$}$$

for any $\tilde{f}$ satisfying $\int_{\mathbb{T}^n} \tilde{f} = 0$, where A_1, A_2, B_1 and B_2 are positive constants independent of f and $\tilde{f}$.

The smoothness condition on Φ in Theorem A can be replaced by some weaker conditions (see [DFP], [Sa1]). One of the results in [DFP] is the following

Theorem B. *Let $m, n \in \mathbb{N}$ and $A : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.*

Suppose that $\Phi \in L^1(\mathbb{R}^n)$ satisfies

(i) $$\|\sup_{t\in\mathbb{R}} |\Phi_t| * f\|_{L^p(\mathbb{R}^n)} \le C_p \|f\|_{L^p(\mathbb{R}^n)} \quad \text{for all} \quad p \in (1,\infty),$$

(ii) $$|\hat{\Phi}(\xi)| \le C \min(|A\xi|^\alpha, |A\xi|^{-\beta})$$

for some $\alpha, \beta > 0$ all $\xi \in \mathbb{R}^n$. Then for every $p \in (1,\infty)$, there exist constants $C = C(p) > 0$ and $C' = C'(p) > 0$ such that

$$\|g_\Phi(f)\|_{L^p(\mathbb{R}^n)} \le C \|f\|_{L^p(\mathbb{R}^n)} \tag{1.4}$$

and

$$\|G_\Phi(\tilde{f})\|_{L^p(\mathbb{T}^n)} \le C'\|\tilde{f}\|_{L^p(\mathbb{T}^n)}, \tag{1.5}$$

for $f \in \mathcal{S}(\mathbb{R}^n)$ and $\tilde{f} \in C^\infty(\mathbb{T}^n)$.

When $\Phi \in \mathcal{S}(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} \Phi(x)dx = 0$, one sees easily that (i) is satisfied and $|\hat{\Phi}(\xi)| \le C \min\{|\xi|, |\xi|^{-1}\}$ always holds.

After submitting the paper, the second author of [DFP] noticed that in early 80's, Kaneko [K] already established several deLeeuw type theorems and proved that, merely assuming $\Phi \in L^1(\mathbb{R}^n)$ (without conditions (i) and (ii)), inequalities (1.4) and (1.5) are equivalent (see [L][K][KS][KT][SW] for the history of deLeeuw's Theorem). The first purpose in this paper is to give a different proof from that in [K] and we will show that the constants C and C' in (1.4), (1.5) are the same. More precisely, let

$$B = \sup\{\|g_\Phi(f)\|_{L^p(\mathbb{R}^n)}, \|f\|_{L^p(\mathbb{R}^n)} = 1\}$$

and

$$\tilde{B} = \sup\{\|G_\Phi(\tilde{f})\|_{L^p(\mathbb{T}^n)}, \|\tilde{f}\|_{L^p(\mathbb{T}^n)} = 1\}.$$

THEOREM 1. *Suppose $\Phi \in L^1(\mathbb{R}^n)$ and $1 < p < \infty$. Then $B = \tilde{B}$.*

The proof of $B \le \tilde{B}$ is essentially contained in Theorem 2 of [K]. We will only prove $\tilde{B} \le B$. To this end, we will invoke the following lemma in [F].

LEMMA 1. ([F] Lemma 3.7) *Suppose that $\Phi \in L^1(\mathbb{R}^n)$. Let $\Psi(x)$ be a continuous function with compact support, and set $\Psi^{1/N}(\xi) = \Psi(\xi/N)$. If Ψ satisfies $\Psi(0) = 1$ and $\hat{\Psi} \in L^1(\mathbb{R}^n)$, then for any $\tilde{f} = \sum_{k\in\Lambda} c_k e^{2\pi i\langle x,k\rangle} \in C^\infty(\mathbb{T}^n)$ and any positive integer N,*

$$\Psi(y/N)(\tilde{\Phi}_t * \tilde{f})(y) = \Phi_t * (\tilde{f}\Psi^{1/N})(y) + J_{N,2^t}(y) \tag{1.6}$$

for all $y \in \mathbb{R}^n$, where

$$J_{N,2^t}(y) = -\sum_{k\in\Lambda} c_k e^{2\pi i\langle y,k\rangle} \int_{\mathbb{R}^n} \hat{\Psi}(x) e^{2\pi i\langle x/N,y\rangle} \{\hat{\Phi}(2^t x/N + 2^t k) - \hat{\Phi}(2^t k)\} dx$$

tends to zero uniformly for $y \in \mathbb{R}^n$ and $-R \le t \le R$ ($R > 0$ is any fixed number), as $N \to \infty$.

The proof of $\tilde{B} \leq B$ can be found in Section 2. As mentioned before, we use a different proof from that in [K], which also allows us to treat the case $0 < p \leq 1$, namely the Hardy spaces H^p. Let $H^p(\mathbb{R}^n)$ and $H^p(\mathbb{T}^n)$, $0 < p \leq 1$ be the Hardy spaces on $\mathbb{R}^n$ and $\mathbb{T}^n$, respectively. We have the following result.

THEOREM 2. *Suppose that $\Phi \in L^1(\mathbb{R}^n)$, $0 < p \leq 1$. Then*

$$\|g_\Phi(f)\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{H^p(\mathbb{R}^n)} \tag{1.7}$$

for all $f \in H^p(\mathbb{R}^n) \cap \mathcal{S}(\mathbb{R}^n)$ if and only if

$$\|G_\Phi(\tilde{f})\|_{L^p(\mathbb{T}^n)} \leq C'\|\tilde{f}\|_{H^p(\mathbb{T}^n)} \tag{1.8}$$

for all $\tilde{f} \in C^\infty(\mathbb{T}^n)$. Here C and C' are two positive constants.

Let $|E|$ denote the Lebesgue measure of a measurable set E. We also can establish a weak type theorem.

THEOREM 3. *Suppose that $\Phi \in L^1(\mathbb{R}^n)$.*

(i)
$$|\{x \in \mathbb{R}^n, g_\Phi(f)(x) > \lambda\}| \leq B\|f\|^p_{L^p(\mathbb{R}^n)}/\lambda^p$$

for all $f \in \mathcal{S}(\mathbb{R}^n)$ and all $\lambda > 0$ if and only if

$$|\{x \in Q, G_\Phi(\tilde{f})(x) > \lambda\}| \leq \tilde{B}\|\tilde{f}\|^p_{L^p(\mathbb{T}^n)}/\lambda^p$$

for all $\tilde{f} \in C^\infty(\mathbb{T}^n)$ and all $\lambda > 0$, where $1 \leq p < \infty$ and $\tilde{B} = B$.

(ii)
$$|\{x \in \mathbb{R}^n, g_\Phi(f)(x) > \lambda\}| \leq C\|f\|^p_{H^p(\mathbb{R}^n)}/\lambda^p$$

for all $f \in \mathcal{S}(\mathbb{R}^n) \cap H^p(\mathbb{R}^n)$ and all $\lambda > 0$ if and only if

$$|\{x \in Q, G_\Phi(\tilde{f})(x) > \lambda\}| \leq C'\|\tilde{f}\|^p_{H^p(\mathbb{T}^n)}/\lambda^p$$

for all $\tilde{f} \in C^\infty(\mathbb{T}^n)$ and all $\lambda > 0$, where $0 < p \leq 1$. Here $Q = [-1/2, 1/2]^n$ is the fundamental cube on which

$$\int_{\mathbb{T}^n} \tilde{f}(x)dx = \int_Q \tilde{f}(x)dx.$$

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Theorem 2 and Theorem 3 will be proved in Section 3 and Section 4, respectively. But here we want to remark that the proof of $B \leq \tilde{B}$ in (i) of Theorem 3 was obtained in [K] already, while the "only if" part of (i) in Theorem 3 is a significant improvement over Theorem 1 of [K]. In [K], Kaneko obtained $\tilde{B} \leq pB/(p-1)$ so that his result works only for $p > 1$.

In Section 5 we will study deLeeuw's theorem on (1.2′) implies (1.3′). Precisely, we will prove

THEOREM 4. *Suppose that Φ is a nonzero function which satisfies $|\Phi(x)| \leq C(1+|x|)^{-n-\delta}$ with some $\delta > 0$, and $\int_{\mathbb{R}^n} \Phi(x)dx = 0$. Then (1.2′) implies (1.3′) for $p \in (1,\infty)$.*

In the case $0 < p \leq 1$, we have

THEOREM 5. *Suppose Φ is the function as in Theorem 4. If*

$$\|f\|_{H^p(\mathbb{R}^n)} \leq C\|g_\Phi(f)\|_{L^p(\mathbb{R}^n)} \tag{1.9}$$

for all $f \in H^p(\mathbb{R}^n) \cap \mathcal{S}(\mathbb{R}^n)$, then

$$\|\tilde{f}\|_{H^p(\mathbb{T}^n)} \leq C'\|G_\Phi(\tilde{f})\|_{L^p(\mathbb{T}^n)} \tag{1.10}$$

for all $\tilde{f} \in C^\infty(\mathbb{T}^n)$ satisfying $\int_{\mathbb{T}^n} \tilde{f}dx = 0$.

Remark. It is easy to see that the condition $\int_{\mathbb{T}^n} \tilde{f}dx = 0$ is necessary in Theorems 4 and 5.

In this paper, we will adopt some ideas in our previous paper [FS1]. Also, the letter C and C' will denote positive constants that may vary at each occurrence but are independent of the essential variables. We also denote $f(x) \cong g(x)$ if there exist positive constants C_1 and C_2 independent of x such that $C_1 f(x) \leq g(x) \leq C_2 f(x)$.

§2. The "$\tilde{B} \leq B$" part of Theorem 1

Fix $R > 0$, we define

$$\Delta_R \tilde{f}(x) = \left\{ \int_{|t| \leq R} |\tilde{\Phi}_t * \tilde{f}(x)|^2 dt \right\}^{1/2}. \tag{2.1}$$

Since $\Delta_R \tilde{f}$ increasingly tends to $\{\int_{\mathbb{R}} |\tilde{\Phi}_t * \tilde{f}(x)|^2 dt\}^{1/2}$ and $C^\infty(\mathbb{T}^n)$ is dense in $L^p(\mathbb{T}^n)$, it suffices to show that

$$\|\Delta_R \tilde{f}\|_{L^p(\mathbb{T}^n)} \leq B\|\tilde{f}\|_{L^p(\mathbb{T}^n)} \tag{2.2}$$

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uniformly for $\tilde{f} \in C^\infty(\mathbb{T}^n)$ and $R > 0$. Fix a positive integer K, define the set Ω_K by

$$\Omega_K = [-1/2 - 1/K, 1/2 + 1/K]^n.$$

Let Ψ be a function in $\mathcal{S}(\mathbb{R}^n)$ satisfying $\operatorname{supp}\Psi \subseteq \Omega_K$, $0 \le \Psi(x) \le 1$, and $\Psi(x) \equiv 1$ on Q. Noting $\Delta_R \tilde{f}$ is a periodic function, we have

$$\begin{aligned}
\|\Delta_R \tilde{f}\|_{L^p(\mathbb{T}^n)} &= \left\{ N^{-n} \int_{NQ} \Psi(x/N) |\Delta_R \tilde{f}(x)|^p dx \right\}^{1/p} \\
&= \left\{ N^{-n} \int_{NQ} \left(\int_{|t|\le R} |\Psi(x/N) \tilde{\Phi}_t * \tilde{f}(x)|^2 dt \right)^{p/2} dx \right\}^{1/p}.
\end{aligned}$$

Thus by Lemma 1, we have that

$$\begin{aligned}
\|\Delta_R \tilde{f}\|_{L^p(\mathbb{T}^n)} \le& \left\{ N^{-n} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}} |\Phi_t * (\Psi^{1/N} \tilde{f})(x)|^2 dt \right)^{p/2} dx \right\}^{1/p} \\
&+ \left\{ N^{-n} \int_{NQ} \left(\int_{|t|\le R} |J_{N,2^t}(x)|^2 dt \right)^{p/2} dx \right\}^{1/p},
\end{aligned} \tag{2.3}$$

and that the second integral on the right side of the above inequality goes to zero as $N \to \infty$. On the other hand, the first integral on the right side of the above inequality is equal to

$$N^{-n/p} \|g_\Phi(\Psi^{1/N} \tilde{f})\|_{L^p(\mathbb{R}^n)}. \tag{2.4}$$

Thus by the assumption and the choice of Ψ, it is bounded by

$$N^{-n/p} B \|\Psi^{1/N} \tilde{f}\|_{L^p(\mathbb{R}^n)} \le B N^{-n/p} \left\{ \int_{N\Omega_K} |\tilde{f}(x)|^p dx \right\}^{1/p} \tag{2.5}$$

where $N\Omega_K = [-N/2 - N/K, N/2 + N/K]^n$. Choose N such that N/K are integers. Then as $N \to \infty$ we have, since $\tilde{f}$ is a periodic function, that

$$\begin{aligned}
\|\Delta_R(\tilde{f})\|_{L^p(\mathbb{R}^n)} &\le B \left\{ N^{-n} (N + 2N/K)^n \int_Q |\tilde{f}(x)|^p dx \right\}^{1/p} + o(1) \\
&= B(1 + 2/K)^{n/p} \|\tilde{f}\|_{L^p(\mathbb{T}^n)} + o(1).
\end{aligned}$$

Letting $N \to \infty$, then $K \to \infty$, finally $R \to \infty$, we obtain $\|G_\Phi(\tilde{f})\|_{L^p(\mathbb{T}^n)} \le B\|\tilde{f}\|_{L^p(\mathbb{T}^n)}$. The proof is complete.

We now present several applications.

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Corollary 1. *Suppose* $\Phi \in L^1(\mathbb{R}^n)$ *and satisfies* $\int \Phi = 0$. *If*

$$\sup_{|\xi|=1} \iint_{\mathbb{R}^n \times \mathbb{R}^n} |\ \Phi(x)\Phi(y) \log |\langle \xi, x-y \rangle||\ dxdy < \infty$$

then

$$\|G_\Phi(\tilde{f})\|_{L^p(\mathbb{T}^n)} \leq C\|\tilde{f}\|_{L^p(\mathbb{T}^n)}.$$

Proof. We obtain this corollary by Theorem 1 and Proposition 3 of [Sa1]. Similarly by Theorem 1 and Theorem 1 in [Sa1], we have

Corollary 2. *Let* $\Phi \in L^1(\mathbb{R}^n)$ *satisfy* $\int \Phi = 0$. *If* Φ *satisfies the following conditions*

$$B_\varepsilon(\Phi) = \int_{|x|>1} |\Phi(x)|\, |x|^\varepsilon dx < \infty \quad \textit{for some} \quad \varepsilon > 0, \tag{1}$$

$$D_u(\Phi) = \Big(\int_{|x|<1} |\Phi(x)|^u dx \Big)^{1/u} < \infty \quad \textit{for some} \quad u > 1, \tag{2}$$

$$H_\Phi(x) = \sup_{|y|\geq|x|} |\Phi(y)| \in L^1(\mathbb{R}^n), \tag{3}$$

then

$$\|G_\Phi(\tilde{f})\|_{L^p(\mathbb{T}^n)} \leq C\|\tilde{f}\|_{L^p(\mathbb{T}^n)} \quad \textit{for all} \quad p \in (1, \infty).$$

In the one dimensional case, we let $\Phi(x) = \operatorname{sign} x(1-|x|)^{\alpha-1}$ with $\alpha > 0$ if $|x| < 1$ and $\Phi(x) = 0$ otherwise; and denote $G_\Phi(\tilde{f}) = \mu_\alpha(\tilde{f})$. The square function μ_1 coincides with the ordinary Marcinkiewicz integral on $\mathbb{T}^1$. By Theorem 1 again together with Theorem 4 in [Sa2], we now have

Corollary 3. *For* $p \in (1, \infty)$ *and* $\alpha > 0$ *we have*

$$\|\mu_\alpha(\tilde{f})\|_{L^p(\mathbb{T}^1)} \leq C\|\tilde{f}\|_{L^p(\mathbb{T}^1)}$$

provided $\alpha > (2-p)/2p$.

§3. Proof of Theorem 2

First we prove the "only if" part. Let Δ_R be defined in Section 2. It sufficies to show

$$\|\Delta_R(\tilde{f})\|_{L^p(\mathbb{T}^n)} \le C\|\tilde{f}\|_{H^p(\mathbb{T}^n)} \tag{3.1}$$

with C independent of $\tilde{f}$ and $R > 0$. By the discussion in [FS1], we may assume $\int_Q \tilde{f}(x)dx = 0$. Take

$$\Psi(x) = \prod_{j=1}^{n}(1-4x_j^2)_+. \tag{3.2}$$

Then from (2.3) and (2.4) we have, as $N \to \infty$,

$$\|\Delta_R(\tilde{f})\|^p_{L^p(\mathbb{T}^n)} \le N^{-n}\|g_\Phi(\Psi^{1/N}\tilde{f})\|^p_{L^p(\mathbb{R}^n)} + o(1). \tag{3.3}$$

Thus by the assumption,

$$\|\Delta_R(\tilde{f})\|^p_{L^p(\mathbb{T}^n)} \le CN^{-n}\|\Psi^{1/N}\tilde{f}\|^p_{H^p(\mathbb{R}^n)} + o(1).$$

By checking the proof of (4.7) in [FS1], we have

$$N^{-n}\|\Psi^{1/N}\tilde{f}\|^p_{H^p(\mathbb{R}^n)} \le C\|\tilde{f}\|^p_{H^p(\mathbb{T}^n)} \tag{3.4}$$

with C being independent of $\tilde{f}$ and N. Thus we prove the "only if" part by letting $N \to \infty$.

Next, we turn to prove the "if" part. Let $\mathcal{D}(\mathbb{R}^n) = \{f \in \mathcal{S}(\mathbb{R}^n) : f$ has compact support$\}$. Since $\mathcal{D}(\mathbb{R}^n) \cap H^p(\mathbb{R}^n)$ is dense in $H^p(\mathbb{R}^n)$, it is enough to prove the theorem when f is in $\mathcal{D}(\mathbb{R}^n) \cap H^p(\mathbb{R}^n)$. In order to do so, we follow the idea in [SW] to define $\tilde{f}_\varepsilon$, for $\varepsilon > 0$, to be the dilated and periodized version of f, viz

$$\tilde{f}_\varepsilon(x) = \varepsilon^{-n}\sum_{m\in\Lambda} f(\varepsilon^{-1}(x+m)).$$

Then by the Poisson summation formula

$$\tilde{f}_\varepsilon(x) = \sum_{k\in\Lambda}\hat{f}(\varepsilon k)e^{2\pi i\langle k,x\rangle}.$$

By the definition of the Riemann integral, we know that

$$\Phi_t * f(x) = \lim_{\varepsilon\to 0+}\varepsilon^n\sum_{m\in\Lambda}\hat{\Phi}(2^t\varepsilon m)\hat{f}(\varepsilon m)e^{2\pi i\varepsilon\langle m,x\rangle}.$$

Thus by the Fatou Lemma, we have

$$\int_{\mathbb{R}} |\Phi_t * f(x)|^2 dt \leq \liminf_{\varepsilon\to 0} \int_{\mathbb{R}} |\varepsilon^n \sum_{m\in\Lambda} \hat{\Phi}(2^t \varepsilon m)\hat{f}(\varepsilon m)e^{2\pi i\varepsilon\langle m,x\rangle}|^2 dt.$$

Following the proof on page 265 in [SW] we let $\eta(x) \geq 0$ be a function in $\mathcal{D}(\mathbb{R}^n)$ satisfying $\eta(0) = 1$ and $\sum_{m\in\Lambda} \eta(x+m) = 1$. By Fatou's lemma again, we have

$$\begin{aligned}&\|g_\Phi(f)\|^p_{L^p(\mathbb{R}^n)}\\ &\quad\leq \liminf_{\varepsilon\to 0} \int_{\mathbb{R}^n} \eta(\varepsilon x)\Big\{ \int_{\mathbb{R}} |\varepsilon^n \sum_{m\in\Lambda} \hat{\Phi}(2^t\varepsilon m)\hat{f}(\varepsilon m)e^{2\pi i\varepsilon\langle m,x\rangle}|^2 dt\Big\}^{p/2} dx.\end{aligned}$$

By changing variables on x, it is easy to see that

$$\begin{aligned}&\|g_\Phi(f)\|^p_{L^p(\mathbb{R}^n)}\\ &\quad\leq \liminf_{\varepsilon\to 0} \varepsilon^{np-n} \int_{\mathbb{R}^n} \eta(x)\Big\{ \int_{\mathbb{R}} | \sum_{m\in\Lambda} \hat{\Phi}(2^t\varepsilon m)\hat{f}(\varepsilon m)e^{2\pi i\langle m,x\rangle}|^2 dt\Big\}^{p/2} dx.\end{aligned}$$

After changing variables $2^t\varepsilon \to 2^t$ and using the fact $\sum_{m\in\Lambda} \eta(x+m) = 1$, we now have

$$\begin{aligned}&\|g_\Phi(f)\|^p_{L^p(\mathbb{R}^n)}\\ &\quad\leq \liminf_{\varepsilon\to 0} \varepsilon^{np-n} \int_{Q} \Big\{ \int_{\mathbb{R}} | \sum_{m\in\Lambda} \hat{\Phi}(2^t m)\hat{f}(\varepsilon m)e^{2\pi i\langle m,x\rangle}|^2 dt\Big\}^{p/2} dx\\ &\quad= \liminf_{\varepsilon\to 0} \varepsilon^{np-n} \|G_\Phi(\tilde{f}_\varepsilon)\|^p_{L^p(\mathbb{T}^n)}.\end{aligned}$$

By the assumption, we have that

$$\|g_\Phi(f)\|^p_{L^p(\mathbb{R}^n)} \leq C' \liminf_{\varepsilon\to 0} \varepsilon^{np-n} \|\tilde{f}_\varepsilon\|^p_{H^p(\mathbb{T}^n)}.$$

From Lemma 3 in [LL], $\liminf_{\varepsilon\to 0} \varepsilon^{n(1-1/p)} \|\tilde{f}_\varepsilon\|_{H^p(\mathbb{T}^n)} = \|f\|_{H^p(\mathbb{R}^n)}$. Thus we have

$$\|g_\Phi(f)\|_{L^p(\mathbb{R}^n)} \leq C'\|f\|_{H^p(\mathbb{R}^n)}.$$

Let μ_α be defined as in Corollary 3. By Theorem 2 and Theorem 4 in [Sa2] we have

COROLLARY 4. *For $\alpha > 0$ and $1 \geq p > 2/(2\alpha+1)$ we have*

$$\|\mu_\alpha(\tilde{f})\|_{L^p(\mathbb{T}^1)} \leq C\|\tilde{f}\|_{H^p(\mathbb{T}^1)}.$$

§4. Proof of Theorem 3

First we prove the "if" part. The proofs for (i) and (ii) are essentially the same. We prove (ii) only. For $f \in \mathcal{D}(\mathbb{R}^n)$ we define $\tilde{f}_\varepsilon$, for $\varepsilon > 0$ as in Section 3 and define

$$\tilde{\Phi}_{t,\varepsilon} * \tilde{f}_\varepsilon(x) = \sum_{m\in\Lambda} \hat{\Phi}(2^t \varepsilon m)\hat{f}(\varepsilon m)e^{2\pi i\langle m,x\rangle}.$$

Then we proved in Section 3 that $\lim_{\varepsilon\to 0} \varepsilon^n \tilde{\Phi}_{t,\varepsilon} * \tilde{f}_\varepsilon(\varepsilon x) = \Phi_t * f(x)$. We write

$$\Big(\int_{\mathbb{R}} |\tilde{\Phi}_{t,\varepsilon} * \tilde{f}_\varepsilon(x)|^2 dt\Big)^{1/2} = T_\varepsilon \tilde{f}_\varepsilon(x).$$

By changing variables $2^t\varepsilon \to 2^t$, it is easy to see that

$$T_\varepsilon \tilde{f}_\varepsilon(x) \cong G_\Phi(\tilde{f}_\varepsilon)(x). \tag{4.1}$$

Let $\eta(x) = \chi_Q(x)$. Then by the Fatou Lemma for each $x \in \mathbb{R}^n$, we have

$$g_\Phi(f)(x) \le \liminf_{\varepsilon\to 0} \eta(\varepsilon x)\varepsilon^n T_\varepsilon \tilde{f}_\varepsilon(\varepsilon x).$$

By Fatou's lemma again, for each $\lambda > 0$

$$\begin{aligned}&|\{x \in \mathbb{R}^n, g_\Phi(f)(x) > \lambda\}|\\ &\quad\le \liminf_{\varepsilon\to 0} |\{x \in \mathbb{R}^n, \eta(\varepsilon x)T_\varepsilon \tilde{f}_\varepsilon(x) > \lambda\varepsilon^{-n}\}|.\end{aligned}$$

(by changing variables on x)

$$\le \liminf_{\varepsilon\to 0} \varepsilon^{-n}|\{x \in Q, T_\varepsilon \tilde{f}_\varepsilon(x) > \lambda\varepsilon^{-n}\}|.$$

By (4.1) and the assumption of the theorem and Lemma 3 in [LL], the above limit is bounded by

$$\liminf_{\varepsilon\to 0} C\|\tilde{f}_\varepsilon\|^p_{H^p(\mathbb{T}^n)}\varepsilon^{np-n}/\lambda^p \le C\|f\|^p_{H^p(\mathbb{R}^n)}/\lambda^p.$$

The "if" part is proved. Now we turn to prove the "only if" part. For any $\tilde{f}$, without loss of generality, we assume $\int_Q \tilde{f}(x)dx = 0$. Let Δ_R be defined as in Section 2 and Ψ be as in (3.2). Then for any $\lambda > 0$ fixed,

$$\begin{aligned}|\{x \in Q : |\Delta_R \tilde{f}(x)| > \lambda\}| &= N^{-n}|\{x \in NQ : |\Delta_R \tilde{f}(x)| > \lambda\}|\\ &\le N^{-n}|\{x \in NQ : |\Psi(x/2N)\Delta_R \tilde{f}(x)| > (3/4)^n\lambda\}|.\end{aligned}$$

By Lemma 1 we know that $E_N(x) = (\int_{-R}^{R} |J_{N,2^t}(x)|^2 dt)^{1/2} \to 0$ uniformly on x, for any $\varepsilon \in (0, \lambda)$, so that we can choose N sufficiently large such that

$$|\{x \in Q : |\Delta_R \tilde{f}(x)| > \lambda\} \\ \leq N^{-n} |\{x \in \mathbb{R}^n : |g_\Phi(\Psi^{1/(2N)} \tilde{f})(x)| > (3/4)^n (\lambda - \varepsilon)\}|.$$

Thus in case (ii), by the assumption and (3.4) we have

$$|\{x \in Q : |\Delta_R \tilde{f}(x)| \ > \lambda\}| \leq C N^{-n} \{\|\Psi^{1/(2N)} \tilde{f}\|_{H^p(\mathbb{R}^n)} / (\lambda - \varepsilon)\}^p \\ \leq C\{\|\tilde{f}\|_{H^p(\mathbb{T}^n)} / (\lambda - \varepsilon)\}^p,$$

where C is independent of ε and R. Letting $R \to \infty$ and noting ε is arbitrary, Theorem 3 (ii) is proved.

We can prove case (i) by combining the idea of the proof for case (ii) and the method of Section 2.

§5. Proofs of Theorems 4 and 5

5.1. Proof of Theorem 4

Choose a function $\Psi \in C^\infty$ such that $\Psi(x) \equiv 1$ on $[-1/2, 1/2]^n$, $0 \leq \Psi(x) \leq 1$, and $\operatorname{supp} \Psi \subseteq [-1, 1]^n$. As in the argument in Section 3, for any $\tilde{f}(x) = \sum_{k \neq 0} c_k e^{2\pi i \langle x, k \rangle}$ and any positive integer N,

$$\|G_\Phi(\tilde{f})\|_{L^p(\mathbb{T}^n)} \\ \geq \Big\{(8N)^{-n} \int_{[-4N,4N]^n} \Big\{ \int_{\mathbb{R}} |\Psi(x/N) \tilde{\Phi}_t * \tilde{f}(x)|^2 dt \Big\}^{p/2} dx \Big\}^{1/p}.$$

Thus by Lemma 1, we have that

$$\|G_\Phi(\tilde{f})\|_{L^p(\mathbb{T}^n)} \\ \geq \Big\{(8N)^{-n} \int_{[-4N,4N]^n} \Big\{ \int_{\mathbb{R}} |\Phi_t * (\Psi^{1/N} \tilde{f})(x)|^2 dt \Big\}^{p/2} dx \Big\}^{1/p} - E_N,$$

where

$$E_N = \Big\{(8N)^{-n} \int_{[-4N,4N]^n} \Big(\int_{\mathbb{R}} |J_{N,2^t}(x)|^2 dt \Big)^{p/2} dx \Big\}^{1/p}.$$

Furthermore, we have that

$$\|G_\Phi(\tilde{f})\|_{L^p(\mathbb{R}^n)} \geq (8N)^{-n/p} \|g_\Phi(\Psi^{1/N} \tilde{f})\|_{L^p(\mathbb{R}^n)} - E_N - \mathcal{E}_N \tag{5.1}$$

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with $\mathcal{E}_N = \{(8N)^{-n} \int_{|x|^*>2N} (\int_{\mathbb{R}} |\Phi_t * (\Psi^{1/N} \tilde{f})(x)|^2 dt)^{p/2} dx\}^{1/p}$, where $|x|^* = \max\{|x_1|, \ldots, |x_n|\}$. By the assumption,

$$\begin{aligned}(8N)^{-n/p} \|g_\Phi(\Psi^{1/N} \tilde{f})\|_{L^p(\mathbb{R}^n)} &\geq (8N)^{-n/p} B_1^{-1} \|\Psi^{1/N} \tilde{f}\|_{L^p(\mathbb{R}^n)} \\ &\geq C N^{-n/p} \Big\{ \int_{[-N/2, N/2]^n} |\tilde{f}(x)|^p dx \Big\}^{1/p} \\ &\geq C \|\tilde{f}\|_{L^p(\mathbb{T}^n)}.\end{aligned}$$

This shows that there is a positive constant C independent of N such that

$$\|G_\Phi(\tilde{f})\|_{L^p(\mathbb{T}^n)} \geq C \|\tilde{f}\|_{L^p(\mathbb{T}^n)} - E_N - \mathcal{E}_N.$$

Thus to complete the proof of Theorem 4, it remains to show

$$\lim_{N \to \infty} E_N = 0, \tag{5.2}$$

$$\lim_{N \to \infty} \mathcal{E}_N = 0. \tag{5.2$'$}$$

We prove (5.2′) first. Changing variables $y/N \to y$, $x/N \to x$ and $2^t N \to t$, it is easy to see that $\mathcal{E}_N$ is bounded by, up to a constant independent of N,

$$\Big\{ \int_{|x|^* \geq 2} \Big(\int_0^\infty | \int_{\mathbb{R}^n} t^{-n} \Phi(t^{-1}(x-y)) \Psi(y) \sum_{k \neq 0} c_k e^{2\pi i \langle, k, Ny \rangle} dy|^2 t^{-1} dt \Big)^{p/2} dx \Big\}^{1/p}.$$

For any $\varepsilon > 0$, there is an $M > 0$ such that $\sum_{|k| \geq M} |c_k| < \varepsilon$. Thus,

$$\mathcal{E}_N \leq \varepsilon I + \sum_{|k| < M, k \neq 0} |c_k| I_{N,k},$$

where

$$I = \Big\{ \int_{|x|^* \geq 2} \Big(\int_0^\infty \Big(\int_{\mathbb{R}^n} |t^{-n} \Phi(t^{-1}(x-y)) \Psi(y)| dy \Big)^2 t^{-1} dt \Big)^{p/2} dx \Big\}^{1/p},$$

$$I_{N,k} = \Big\{ \int_{|x|^* > 2} \Big(\int_0^\infty \Big| \int_{\mathbb{R}^n} t^{-n} \Phi\left(\frac{x-y}{t} \right) \Psi(y) e^{2\pi i \langle k, Ny \rangle} dy \Big|^2 \frac{dt}{t} \Big)^{p/2} dx \Big\}^{1/p}.$$

Clearly, to prove (5.2′), it suffices to show $I \leq C$ and $\lim_{N \to \infty} I_{N,k} = 0$ for each $k \neq 0$.

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Recall $\operatorname{supp}\Psi \subseteq [-1,1]^n$. We have that $\Phi\left(\frac{x-y}{t}\right) \le C(1+|x|/t)^{-n-\delta}$ if $y \in [-1,1]^n$ and $|x|^* \ge 2$. Thus

$$\begin{aligned}
I &\le C\Big\{\int_{|x|^*>2}\Big\{\int_0^\infty (1+|x|/t)^{-2n-2\delta}t^{-2n-1}dt\Big\}^{p/2}dx\Big\}^{1/p}\\
&\le C\Big\{\int_{|x|^*>2}\Big\{\int_0^{|x|}(t/|x|)^{2n+2\delta}t^{-2n-1}dt\Big\}^{p/2}dx\Big\}^{1/p}\\
&\qquad +C\Big\{\int_{|x|^*>2}\Big\{\int_{|x|}^\infty t^{-2n-1}dt\}^{p/2}dx\Big\}^{1/p}\\
&\le \Big\{\int_{|x|^*>2}|x|^{-np}dx\Big\}^{1/p} \le C, \text{ because } p>1.
\end{aligned}$$

Next we prove $\lim_{N\to\infty} I_{N,k} = 0$ for all $k \neq 0$. Put

$$\begin{aligned}
F_N(x,t) &= \int_{\mathbb{R}^n} t^{-n}\Phi(t^{-1}(x-y))\Psi(y)e^{2\pi i\langle k,Ny\rangle}dy,\\
G_N(x) &= \int_0^\infty |F_N(x,t)|^2 t^{-1}dt.
\end{aligned}$$

Then by the Riemann-Lebesgue theorem $F_N(x,t) \to 0$ as $N \to \infty$, since $k \neq 0$; and for $|x|^* > 2$

$$|F_N(x,t)|^2 \le C(1+|x|/t)^{-2n-2\delta}t^{-2n}$$

with a constant C independent of N. Thus by the dominated convergence theorem we see that $G_N(x) \to 0$ as $N \to \infty$ for each fixed x with $|x|^* > 2$. Furthermore, by the estimate in the previous paragraph we have

$$G_N(x) \le C|x|^{-2n}$$

with a constant C independent of N. Applying the dominated convergence theorem again, we have

$$\lim_{N\to\infty} I_{N,k} = \lim_{N\to\infty}\Big(\int_{|x|^*>2} G_N(x)^{p/2}dx\Big)^{1/p} = 0.$$

This completes the proof of (5.2′).

Now we turn to prove (5.2). We prove

$$\lim_{N\to\infty}\int_{\mathbb{R}} |J_{N,2^t}|^2 dt = 0 \tag{5.3}$$

uniformly for $x \in \mathbb{R}^n$. The proof is similar to that of Proposition in [FS2], but here we need the idea in [Sa1]. Since

$$|J_{N,2^t}(x)| \le C \sum_{\substack{k\in\Lambda \\ k\neq 0}} |c_k| \int_{\mathbb{R}^n} |\hat{\Psi}(\xi)|\,|\hat{\Phi}(2^t k) - \hat{\Phi}(2^t k + 2^t\xi/N)|d\xi$$

and $\{c_k\}$ decays rapidly, it suffices to show

$$\Big(\int_{\mathbb{R}} \Big(\int_{\mathbb{R}^n} |\hat{\Psi}(\xi)\hat{\Phi}(2^t k)|d\xi\Big)^2 dt\Big)^{1/2} \le C, \tag{5.4}$$

$$\Big(\int_{\mathbb{R}} \Big(\int_{\mathbb{R}^n} |\hat{\Psi}(\xi)\hat{\Phi}(2^t k + 2^t\xi/N)|d\xi\Big)^2 dt\Big)^{1/2} \le C, \tag{5.5}$$

$$\lim_{N\to\infty} \Big(\int_{\mathbb{R}} \Big(\int_{\mathbb{R}^n} |\hat{\Psi}(\xi)\{\hat{\Phi}(2^t k) - \hat{\Phi}(2^t k + 2^t\xi/N)\}|d\xi\Big)^2 dt\Big)^{1/2} = 0 \tag{5.6}$$

for each $k \neq 0$, where C is independent of k and N. The proofs of (5.4) and (5.5) are the same. We prove (5.5) only. It is easy to see that Φ satisfies the assumption of Corollary 1. So, by the proof of Proposition 3 of [Sa1] we have

$$\sup_{\xi\in\mathbb{R}^n} \int_{\mathbb{R}} |\hat{\Phi}(2^t\xi)|^2 dt \le C.$$

Thus by the Minkowski inequality,

$$\begin{aligned}
&\Big(\int_{\mathbb{R}} \Big(\int_{\mathbb{R}^n} |\hat{\Psi}(\xi)\hat{\Phi}(2^t k + 2^t\xi/N)d\xi\Big)^2 dt\Big)^{1/2} \\
&\qquad \le \Big(\int_{\mathbb{R}^n} |\hat{\Psi}(\xi)|^2 \Big(\int_{\mathbb{R}} |\hat{\Phi}(2^t k + 2^t\xi/N)|^2 dt\Big)^{1/2} d\xi \\
&\qquad \le C \int_{\mathbb{R}^n} |\hat{\Psi}(\xi)|^2 d\xi \le C.
\end{aligned}$$

Finally, we prove (5.6). If we change variables $2^t \to t$, by checking the proof of (5.5) and by the dominated convergence theorem, we only need to show

$$\lim_{N\to\infty} \int_0^\infty |\hat{\Phi}(tk) + \hat{\Phi}(t(k+\xi/N))|^2 t^{-1} dt = 0, \tag{5.7}$$

for each fixed $\xi \in \mathbb{R}^n$. Put $m = k + \xi/N$. Note that

$$\begin{aligned}
&|\hat{\Phi}(2^t k) - \hat{\Phi}(tm)|^2 \\
&\quad = \iint_{\mathbb{R}^n\times\mathbb{R}^n} \Phi(x)\tilde{\Phi}(y)(e^{-2\pi i\langle x,tk\rangle} - e^{-2\pi i\langle x,tm\rangle})(e^{2\pi i\langle y,tk\rangle} - e^{2\pi i\langle y,tm\rangle})dxdy
\end{aligned}$$

Since $|\Phi(x)| \le C(1+|x|)^{-n-\delta}$, as in the proof of Proposition 3 of [Sa1] we have

$$\begin{aligned}
&\int_0^\infty |\hat{\Phi}(tk) - \hat{\Phi}(tm)|^2 t^{-1} dt \\
&\quad = (\pi/2) \iint_{\mathbb{R}^n \times \mathbb{R}^n} \Phi(x)\overline{\Phi}(y)\{-i \operatorname{sgn}\langle k, x-y\rangle + i \operatorname{sgn}(\langle x, m\rangle - \langle y, k\rangle) \\
&\qquad + i \operatorname{sgn}(\langle x, k\rangle - \langle y, m\rangle) - i \operatorname{sgn}\langle m, x-y\rangle\} dxdy \\
&\qquad + \iint_{\mathbb{R}^n \times \mathbb{R}^n} \Phi(x)\overline{\Phi}(y)\{-\log|\langle k, x-y\rangle| + \log|\langle x, m\rangle - \langle y, k\rangle| \\
&\qquad + \log|\langle x, k\rangle - \langle y, m\rangle| - \log|\langle m, x-y\rangle|\} dxdy.
\end{aligned}$$

Since $k \neq 0$, the set $\{(x,y) : \ \langle k, x-y\rangle = 0\}$ has measure 0 in $\mathbb{R}^n \times \mathbb{R}^n$. Therefore, it is easy to see that the first integral on the right hand side tends to 0 as $N \to \infty$. The same is true for the second integral. To see this, fix ξ and let

$$I_N = \iint_{\mathbb{R}^n \times \mathbb{R}^n} \Phi(x)\overline{\Phi}(y)\{\log|\langle x, k\rangle - \langle y, m\rangle| - \log|\langle m, x-y\rangle|\} dxdy.$$

Then $I_N \to 0$ as $N \to \infty$, and

$$\lim_{N\to\infty} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \Phi(x)\overline{\Phi}(y)\{\log|\langle x, m\rangle - \langle y, k\rangle| - \log|\langle k, x-y\rangle|\} dxdy = 0.$$

The proofs are similar. We prove $\lim_{N\to\infty} I_N = 0$ only. This will proves (5.7).

We assume $n \ge 2$, since the 1-dimensional case can be treated in the same way, and is easier. First, without loss of generality we may assume $k = e_1 = (1, 0, \dots, 0)$. Then we have

$$I_N = \iint_{\mathbb{R}^n \times \mathbb{R}^n} \Phi(x)\overline{\Phi}(y)\{\log|x_1 - y_1 - \langle y, \xi/N\rangle| - \log|x_1 - y_1 + \langle \xi/N, x-y\rangle|\} dxdy.$$

Put $y' = (y_2, \dots, y_n)$. By changing variables we have

$$I_N = \iint_{\mathbb{R}^n \times \mathbb{R}^n} \Phi(x)\{\overline{\Phi}(y) - \overline{\Phi}(y_1 + \langle \xi/N, x\rangle, y')\} \log|x_1 - y_1 - \langle y, \xi/N\rangle| dxdy.$$

Put $\eta = e_1 + \xi/N$, then $|\eta| \cong 1$ if N is sufficiently large. Take a rotation A_N such that $A_N^{-1}\eta = |\eta|e_1$. Then, changing variable, we see that

$$\begin{aligned} I_N &= \iint \Phi(x)\{{}_N\overline{\Phi}(y) - {}_N\overline{\Phi}(y + \langle \xi/N, x\rangle A_N^{-l} e_1)\} \log |x_1 - |\eta| y_1| dx dy \\ &= \iint \Phi(x)\{\overline{{}^N\Phi}(y_1 + x_1, |\eta| y') - \overline{{}^N\Phi}((y_1 + x_1, |\eta| y') + |\eta|\langle \xi/N, x\rangle A_N^{-1} e_1)\} \\ &\qquad\qquad \log |y_1| dx dy, \end{aligned}$$

where ${}_N\Phi(y) = \Phi(A_N y)$, ${}^N\Phi(y) = |\eta|^{-1}\Phi(|\eta|^{-1}A_N y)$. Let

$$II_N(x) = \int_{\mathbb{R}^n} |\overline{{}^N\Phi}(y_1 + x_1, |\eta| y') - \overline{{}^N\Phi}((y_1 + x_1, |\eta| y') + |\eta|\langle \xi/N, x\rangle A_N^{-1} e_1)|\, |\log |y_1||\, dy.$$

We prove

$$II_N(x) \le C_\gamma(1 + |x|^\gamma), \tag{5.8}$$

where $0 < \gamma < \min\{1, \delta\}$ and C_γ is independent of x and N (N is sufficiently large). Since

$$|\log |t|| \le C\chi_{(0,1]}(t)t^{-\gamma} + C\chi_{(0,\infty)}(t)t^\gamma, \tag{5.9}$$

(5.8) is an immediate consequence of the following estimates:

$$\int_{\mathbb{R}^n} |{}^N\Phi(y_1 + x_1, |\eta| y')|\, |y_1|^\gamma dy \le C(1 + |x|^\gamma), \tag{5.10}$$

$$\int_{\mathbb{R}^n} |{}^N\Phi((y_1 + x_1, |\eta| y') + |\eta|\langle \xi/N, x\rangle A_N^{-1} e_1)|\, |y_1|^\gamma dy \le C(1 + |x|^\gamma), \tag{5.11}$$

$$\int_{\mathbb{R}^n} |{}^N\Phi(y_1 + x_1, |\eta| y')|\chi_{(0,1]}(|y_1|)|y_1|^{-\gamma} dy \le C, \tag{5.12}$$

$$\int_{\mathbb{R}^n} |{}^N\Phi((y_1 + x_1, |\eta| y') + |\eta|\langle \xi/N, x\rangle A_N^{-1} e_1)|\chi_{(0,1]}(|y_1|)|y_1|^{-\gamma} dy \le C. \tag{5.13}$$

Put $P(x) = (1 + |x|)^{-n-\delta}$. Then $|{}^N\Phi(x)| \le CP(x)$. So, the proofs of (5.10) and (5.11) are easy. The proofs of (5.12) and (5.13) are similar. We prove (5.13) only. Put $A_N^{-1}e_1 = (a_1, a')$ and $\tilde{P}(x') = (1 + |x'|)^{-n-\delta}$. Since $P(x) \le \tilde{P}(x')$, the integral in (5.13) is bounded by

$$\begin{aligned} &C \int_{\mathbb{R}^n} \tilde{P}(|\eta| y' + |\eta|\langle \xi/N, x\rangle a')\chi_{(0,1]}(|y_1|)|y_1|^{-\gamma} dy \\ &\qquad = C|\eta|^{-(n-1)} \int_{\mathbb{R}^n} \tilde{P}(y')\chi_{(0,1]}(|y_1|)|y_1|^{-\gamma} dy. \end{aligned}$$

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This proves (5.13).

Now we have

$$I_N \le \int_{|x|<M} |\Phi(x)| II_N(x)dx + \int_{|x|\ge M} |\Phi(x)| II_N(x)dx =: I_{N,M} + J_{N,M}.$$

Given $\varepsilon > 0$, by (5.8) we can find $M > 0$ such that $J_{N,M} \le \varepsilon$ uniformly in N. Therefore, to prove $\lim_{N\to\infty} I_N = 0$, it sufficies to show that $\lim_{N\to\infty} I_{N,M} = 0$ for each fixed $M > 0$. Put

$$III_N(x,y) = |{}^N\Phi(y_1 + x_1, |\eta|y') - {}^N\Phi((y_1 + x_1, |\eta|y') + |\eta|\langle \xi/N, x\rangle A_N^{-1}e_1)|.$$

We prove

$$\lim_{L\to\infty} \int_{|y|\ge L} III_N(x,y)|\log|y_1||dy = 0 \tag{5.14}$$

uniformly in N and x satisfying $|x| \le M$. By (5.9), it suffices to show the following:

$$\lim_{L\to\infty} \int_{|y|\ge L} |{}^N\Phi(y_1 + x_1, |\eta|y')|y_1|^\gamma dy = 0, \tag{5.15}$$

$$\lim_{L\to\infty} \int_{|y|\ge L} |{}^N\Phi((y_1 + x_1, |\eta|y' + |\eta|\langle \xi/N, x\rangle A_N^{-1}e_1)|y_1|^\gamma dy = 0. \tag{5.16}$$

$$\lim_{L\to\infty} \int_{|y|\ge L} |{}^N\Phi(y_1 + x_1, |\eta|y')|\chi_{(0,1]}(|y_1|)|y_1|^{-\gamma} dy = 0, \tag{5.17}$$

$$\lim_{L\to\infty} \int_{|y|\ge L} |{}^N\Phi((y_1 + x_1, |\eta|y') + |\eta|\langle \xi/N, x\rangle A_N^{-1}e_1| \times \chi_{(0,1]}(|y_1|)|y_1|^{-\gamma} dy = 0, \tag{5.18}$$

where each convergence is uniform in N and x with $|x| \le M$.

The proofs of (5.15) and (5.17) are similar to those of (5.16) and (5.18), respectively. We prove (5.16) and (5.18) only. Now, the integral in (5.16) is bounded by

$$C \int_{|y|\ge L} P((y_1 + x_1, |\eta|y') + |\eta|\langle \xi/N, x\rangle A_N^{-1}e_1)|y_1|^\gamma dy.$$

By changing variables we see that this is bounded by

$$C \int_{|y|>L/2} P(y)(|y_1|^\gamma + |x|^\gamma)dy,$$

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if $|x| \le M$ and L is sufficiently large. This proves (5.16). To prove (5.18) we use the same notation as that in the proof of (5.13). Then we easily see that the integral in (5.18) is bounded by

$$C \int_{|y|\ge L} \tilde{P}(|\eta|y' + |\eta|\langle \xi/N, x\rangle a')\chi_{(0,1]}(|y_1|)|y_1|^{-\gamma}dy$$
$$\le C|\eta|^{-(n-1)} \int_{|y'|\ge L/2} \tilde{P}(y')dy' \int_{-1}^{1} |y_1|^{-\gamma}dy_1,$$

if $|x| \le M$ and L is sufficiently large. This proves (5.18).

Finally we prove

$$\lim_{N\to\infty} \int_{|y|<L} III_N(x,y)|\log|y_1||dy = 0 \tag{5.19}$$

for each x with $|x| \le M$, since

$$II_N(x) = \int_{|y|<L} III_N(x,y)|\log|y_1||dy + \int_{|y|\ge L} III_N(x,y)|\log|y_1||dy.$$

By (5.14) and (5.19) along with the dominated convergence theorem, it follows that $\lim_{N\to\infty} I_{N,M} = 0$, which will complete the proof of $\lim_{N\to\infty} I_N = 0$.

To prove (5.19), we split the domain of integration:

$$\int_{|y|<L} III_N(x,y)|\log|y_1||dy$$
$$= \left\{ \int_{|y|<L,|y_1|<\rho} + \int_{|y|<L,|y_1|\ge\rho} \right\} III_N(x,y)|\log|y_1||dy.$$

Since III_N is bounded and $\log|y_1|$ is locally integrable, given $\varepsilon > 0$ there exists $\rho > 0$ such that the first integral on the right hand side is less than ε. Thus it suffices to show that

$$\lim_{N\to\infty} \int_{|y|<L,|y_1|\ge\rho} III_N(x,y)|\log|y_1||dy = 0$$

for each fixed $\rho > 0$. Since $\log|y_1|$ is bounded on $\{y : |y| < L, |y_1| \ge \rho\}$, we have

$$\int_{|y|<L,|y_1|\ge\rho} III_N(x,y)|\log|y_1||dy \le C \int_{\mathbb{R}^n} III_N(x,y)dy$$
$$= C \int_{\mathbb{R}^n} |\Phi(y) - \Phi(y + \langle \xi/N, x\rangle e_1)|dy.$$

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The last integral tends to 0 as $N \to \infty$, since $\Phi \in L^1$. This shows (5.19), which completes the proof of (5.7).

5.2. Proof of Theorem 5

Choose the function Ψ as in 5.1. Following the proof of (5.1) and the assumption we obtain that

$$\begin{aligned} \|G_\Phi(\tilde{f})\|^p_{L^p(\mathbb{T}^n)} &\geq CN^{-n}\|g_\Phi(\Psi^{1/N}\tilde{f})\|^p_{L^p(\mathbb{R}^n)} - E_N^p - \mathcal{E}_N^p \\ &\geq CN^{-n}\|\Psi^{1/N}\tilde{f}\|^p_{H^p(\mathbb{R}^n)} - E_N^p - \mathcal{E}_N^p. \end{aligned} \tag{5.20}$$

Using exactly the same proof as in the proof of Theorem 4, we have that E_N and $\mathcal{E}_N$ tend to zero as N goes to infinity. Thus to prove the theorem, it suffices to prove

$$\liminf_{N\to\infty} N^{-n}\|\Psi^{1/N}\tilde{f}\|^p_{H^p(\mathbb{R}^n)} \geq C\|\tilde{f}\|^p_{H^p(\mathbb{T}^n)}.$$

Take a $\phi \in \mathcal{S}(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n}\phi(x)dx \neq 0$. Then by the definition of the Hardy space,

$$\begin{aligned} N^{-n}\|\Psi^{1/N}\tilde{f}\|^p_{H^p(\mathbb{R}^n)} &= N^{-n}\int_{\mathbb{R}^n}\sup_{t>0}\left|\int_{\mathbb{R}^n}\phi_t(x-y)\Psi(y/N)\tilde{f}(y)dy\right|^p dx \\ \geq N^{-n}&\int_{[-N/2,N/2]^n}\int_{\mathbb{R}^n}\sup_{0<t\leq R}\left|\int_{\mathbb{R}^n}\phi_t(x-y)\Psi(y/N)\tilde{f}(y)dy\right|^p dx. \end{aligned}$$

So by Lemma 1, for each fixed $R > 0$,

$$\begin{aligned} &N^{-n}\|\Psi^{1/N}\tilde{f}\|^p_{H^p(\mathbb{R}^n)} \\ &= N^{-n}\int_{[-N/2,N/2]^n}|\Psi(x/N)|^p\sup_{0<t\leq R}|\tilde{\phi}_t * \tilde{f}(x)|^p dx - o(1) \\ &= N^{-n}\int_{[-N/2,N/2]^n}\sup_{0<t\leq R}|\tilde{\phi}_t * \tilde{f}(x)|^p dx - o(1) \\ &= \int_Q \sup_{0<t\leq R}|\tilde{\phi}_t * \tilde{f}(x)|^p dx - o(1), \qquad \text{as} \quad N \to \infty. \end{aligned}$$

This shows

$$\liminf_{N\to\infty} N^{-n}\|\Psi^{1/N}\tilde{f}\|^p_{H^p(\mathbb{R}^n)} \geq C\int_Q \sup_{0<t\leq R}|\tilde{\phi}_t * \tilde{f}(x)|^p dx.$$

Letting $R \to \infty$, we obtain (5.21). Now the theorem is proved.

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