PSEUDOVALUATIONS OF POLYNOMIALS

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A valuation of a ring K is a function

$$W: K \rightarrow A,$$

where A is an archimedean ordered field and W has the properties of the absolute valuation; see (**2**, chap. X). The theory was extended in 1936 by K. Mahler (**1**), who introduced the concept of pseudovaluations. Whereas for a valuation we must always have

$$W(ab) = W(a)W(b),$$

for a pseudovaluation it is sufficient that

$$W(ab) \leqslant W(a)W(b).$$

Two pseudovaluations are to be regarded as equivalent if they give the same topology. There arises the problem of determining all the independent pseudovaluations of any given ring.

This presents great difficulties in the case of the ring of polynomials in an indeterminate symbol, z, with complex coefficients. In this paper we consider only those pseudovaluations which coincide with the trivial valuation in the coefficient field. Each of these is shown to be equivalent to a sum of finitely many pseudovaluations of a simple type.

1. We begin by introducing a concept which will be prominent in our investigation.

A formal expression such as

$$\tilde{a} = p_1^{f_1} p_2^{f_2} \ldots p_t^{f_t},$$

where $p_1, p_2, \ldots, p_t$ are distinct prime polynomials and $f_1, f_2, \ldots, f_t$ are each equal to a positive integer or to zero or infinity, will be known as a pseudopolynomial.

Every pseudopolynomial can be uniquely associated with a class of equivalent pseudovaluations. Denoting the representative pseudovaluation associated with the pseudopolynomial $\tilde{a}$ by $W(\rho|\tilde{a})$, we may define:

(a) $W(\rho|1) = U(\rho) = 0$, the improper valuation.

(b) $W(\rho|0) = W_0(\rho) = \begin{cases} 0 & \text{for } \rho = 0, \\ 1 & \text{for } \rho \neq 0, \end{cases}$ the trivial valuation.

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(c) If p is a prime polynomial and f is a positive integer,

$$W(\rho|p^f) = \begin{cases} 0 & \text{for } \rho \equiv 0 \pmod{p^f}, \\ 1 & \text{for } \rho \not\equiv 0 \pmod{p^f}, \end{cases}$$

the residue class valuation.

By $W(\rho|p^\infty)$ we denote the p-adic valuation,

$$W(\rho|p^\infty) = \begin{cases} 0 & \text{for } \rho = 0, \\ e^{-f} & \text{for } \rho \neq 0,\ p^f \| \rho. \end{cases}$$

(d) Finally, if $\tilde{a} = p_1^{f_1} p_2^{f_2} \dots p_t^{f_t}$, we define:

$$W(\rho|\tilde{a}) = \sum_{\tau=1}^{t} e(f_\tau) W(\rho|p_\tau^{f_\tau}),$$

where

$$e(f) = \begin{cases} 0 & \text{for } f = 0, \\ 1 & \text{for } f \neq 0. \end{cases}$$

It is clear that $W(\rho|\tilde{a})$ is now uniquely defined, and it is easily seen that different pseudovaluations of types (a), (b), and (c) are not equivalent to each other.

2. The infinitely many pseudovaluations associated with the same finite prime polynomial, p, satisfy the relations

$$W(\rho|p) \subset W(\rho|p^2) \subset W(\rho|p^3) \subset \dots \subset W(\rho|p^\infty),$$

so that the sum of finitely many of these $W(\rho|p^f)$ is equivalent to the summand with the highest value of f.

Further, the following two statements present no difficulty:

(a) If the pseudopolynomial $\tilde{a}$ is the least common multiple of the finitely many pseudopolynomials $\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_q$ (i.e., if $\tilde{a}_l|\tilde{a}$ $(l = 1, \dots, q)$ and, if $\tilde{a}_l|\tilde{b}$ $(l = 1, \dots, q)$, then $\tilde{a}|\tilde{b}$) then

$$\sum_{l=1}^{q} W(\rho|\tilde{a}_l) \sim W(\rho|\tilde{a}).$$

(b) If $\tilde{a}|\tilde{b}$, then

$$W(\rho|\tilde{a}) \subset W(\rho|\tilde{b}).$$

We make the following two definitions:

Definition 1. If the pseudovaluations $W_1(\rho), W_2(\rho), \dots, W_r(\rho)$ are independent, then their sum,

$$W_\Sigma(\rho) = \sum_{i=1}^{r} W_i(\rho),$$

is called their direct sum.

Definition 2. If the pseudopolynomials $\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_q$ are mutually prime in pairs, then their least common multiple is known as their direct product.

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It is then true that:

(c) If $\tilde{a}$ is the direct product of $\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_q$, then the pseudovaluation $W(\rho|\tilde{a})$ is equivalent to the direct sum of the pseudovaluations

$$W(\rho|\tilde{a}_1),\ W(\rho|\tilde{a}_2), \ldots,\ W(\rho|\tilde{a}_q).$$

This property in fact follows quite easily from the following theorem:

THEOREM 1. *Let $p_1^{f_1}, p_2^{f_2}, \ldots, p_t^{f_t}$ be powers of finitely many different primes of P, with exponents equal to a positive integer, or to zero or infinity. Then the pseudovaluations*

$$W(\rho|p_1^{f_1}),\ W(\rho|p_2^{f_2}), \ldots,\ W(\rho|p_t^{f_t})$$

are independent of each other.

Proof. The theorem states that, given t polynomials, $\gamma_1, \gamma_2, \ldots, \gamma_t$, we can find an infinite sequence of polynomials, $\alpha_1, \alpha_2, \alpha_3, \ldots$, such that

$$\lim_{n\to\infty} W((\gamma_r - \alpha_n)|p_r^{f_r}) = 0 \qquad (r = 1, 2, \ldots, t).$$

Now, since $p_1, p_2, \ldots, p_t$ are distinct prime polynomials, we know that for any integer, f, the set of congruences

$$\alpha \equiv \gamma_r(p_r^f) \qquad (r = 1, 2, \ldots, t)$$

has the solution

$$\alpha \equiv \gamma^{(f)} \pmod{c}$$

for some polynomial $\gamma^{(f)}$, where $c = p_1^f p_2^f p_3^f \ldots p_t^f$.

We put $\alpha_n = \gamma^{(n)}$ for all n. So we have $\alpha_n \equiv \gamma_r(p_r^n)$ for all n, and the infinite sequence of α_n's thus defined clearly satisfies the conditions of the theorem.

3. Having studied the special type of pseudovaluation

$$W(\rho|\tilde{a}),$$

we now consider the general pseudovaluation

$$W(\rho)$$

and we seek to find a pseudovaluation of the above special type equivalent to it. (It will be remembered that we stipulate that $W(a) = W_0(a)$, when a is a complex number.)

For brevity we write

$$U(\rho) \text{ for } W(\rho|1),$$
$$W_0(\rho) \text{ for } W(\rho|0),$$
$$|\rho|_p \text{ for } W(\rho|p^\infty),$$

and

$$W_{p^f}(\rho) \text{ for } W(\rho|p^f) \qquad (f = 1, 2, 3, \ldots).$$

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We shall exclude the case for which $W(\rho)$ is equivalent to the improper valuation $U(\rho)$, i.e., is identically zero. We shall also exclude the uninteresting case

$$W(\rho)\begin{cases} = 0 & \text{for } \rho = 0, \\ \geqslant 1 & \text{for } \rho \neq 0, \end{cases}$$

for then $W(\rho)$ is equivalent to $W_0(\rho)$.

Hence there exists an element $\alpha \neq 0$ such that $W(\alpha) < 1$. We note that α is not a constant, for otherwise $W(\alpha) = 1$, by hypothesis.

4. THEOREM 2. *To every pseudovaluation of the ring P (for which $W(constant) = W_0(constant)$) there corresponds a positive constant, c_1, such that, for every element $\rho \in P$,*

$$W(\rho) \leqslant c_1.$$

Proof. We have the polynomial $\alpha \neq$ a constant, such that

$$W(\alpha) < 1.$$

Let ξ be any element of P; then it can be written as a sum

$$\xi = \sum_0^f a_k \alpha^k$$

where the a_k's are polynomials such that

$$\text{degree } (a_k) < \text{degree } (\alpha) \qquad (k = 0, 1, \ldots, f).$$

Therefore

$$W(a_k) \leqslant \sum_{i=0}^{d} W(z)^i \leqslant (d+1)\max(1, W(z)^d),$$

which depends only on d, the degree of α, and on W. Hence $W(a_k)$ is bounded, say $W(a_k) \leqslant c'_1$ $(k = 0, 1, \ldots, f)$. Thus

$$W(\xi) \leqslant \sum_0^f W(a_k) W(\alpha^k)$$

$$\leqslant c'_1 \sum_0^\infty W(\alpha)^k = \frac{c'_1}{1 - W(\alpha)}, \quad \text{a constant.}$$

5. *Definition* 3. An element, $\gamma \neq 0$, of P is called a W-element if there exists a second element, $\delta \neq 0$, of P such that

$$\lim_{j\to\infty} W(\gamma^j \delta) = 0.$$

There do in fact exist W-elements, for α has the property

$$\lim_{j\to\infty} W(\alpha^j \cdot 1) = 0.$$

THEOREM 3. *If γ is a W-element and $\gamma | \gamma^*$, $\gamma^* \neq 0$, then γ^* is also a W-element.*

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Proof. Let $\gamma^* = \gamma\beta$. We have a $\delta \neq 0$ such that $\lim_{j\to\infty} W(\gamma^j\delta) = 0$. Hence

$$\lim_{j\to\infty} W(\gamma^{*j}\delta) \leqslant \lim_{j\to\infty} W(\beta^j)W(\gamma^j\delta) \leqslant c_1 \lim_{j\to\infty} W(\gamma^j\delta) = 0.$$

THEOREM 4. *If γ_1 and γ_2 are W-elements, then so is $\gamma_1 - \gamma_2$.*

Proof. We have $\delta_1 \neq 0$ such that $\lim_{j\to\infty} W(\gamma_1{}^j\delta_1) = 0$, and $\delta_2 \neq 0$ such that $\lim_{j\to\infty} W(\gamma_2{}^j\delta_2) = 0$. Now,

$$W((\gamma_1 - \gamma_2)^j\delta_1\,\delta_2) \leqslant W(\gamma_1{}^j\delta_1\,\delta_2) + W(\gamma_2{}^j\delta_1\,\delta_2) \leqslant W(\delta_2)W(\gamma_1{}^j\delta_1) + W(\delta_1)W(\gamma_2{}^j\delta_2),$$

whence $\lim_{j\to\infty} W((\gamma_1 - \gamma_2)^j\delta_1\,\delta_2) = 0$, i.e. $\gamma_1 - \gamma_2$ is also a W-element.

6. Denote by $\mathfrak{M}$ the set consisting of the null element, 0, and all W-elements of P.

(a) *If $\gamma \in \mathfrak{M}$ and $\gamma|\gamma^*$, then $\gamma^* \in \mathfrak{M}$.*

Proof. If either $\gamma = 0$ or $\gamma^* = 0$, the result is clear. Otherwise, γ, and hence also γ^*, is a W-element. Hence $\gamma^* \in \mathfrak{M}$.

(b) *If $\gamma_1 \in \mathfrak{M}$ and $\gamma_2 \in \mathfrak{M}$, then $(\gamma_1 - \gamma_2) \in \mathfrak{M}$.*

Proof. If $\gamma_1 = \gamma_2$, the result is clear; so also if $\gamma_1 = 0$ or $\gamma_2 = 0$. Otherwise, γ_1 and γ_2 are both W-elements and so, therefore, is $\gamma_1 - \gamma_2$, $\neq 0$. Hence $\gamma_1 - \gamma_2 \in \mathfrak{M}$.

From these two results, it follows that $\mathfrak{M}$ is an ideal of P, and it is not the null-ideal, since W-elements do exist. Further, since P is a principal ideal ring, we have $\mathfrak{M} = (\gamma_0)$ for some polynomial γ_0.

Definition 4. This polynomial, γ_0, defined as above, we call the principal character of $W(\rho)$.

From this definition, we have the following theorem:

THEOREM 5. *A polynomial, γ, is a W-element if and only if $\gamma \neq 0$ and $\gamma_0|\gamma$.*

The proof is self-evident.

THEOREM 6. *The principal character, γ_0, of $W(\rho)$ is square-free, i.e., it is not divisible by the square of any polynomial.*

Proof. We shall show that if there exists a W-element, γ, such that $p^2|\gamma$, where p is any prime polynomial of P, then there must also exist a W-element, γ^*, such that $p|\gamma^*$ but $p^2 \nmid \gamma^*$.

Let $\gamma = p^g q$, where $g \geqslant 2$ and $(q, p) = 1$. Put $\gamma^* = pq$; then $\gamma^{*g} = \gamma\cdot q^{g-1}$. We have $\delta \neq 0$ such that $\lim_{j\to\infty} W(\gamma^j\delta) = 0$. Now

$$\gamma^{*j}\delta = \gamma^{[j/g]}\delta\cdot\gamma^{*(j-[j/g]g)}q^{(g-1)[j/g]}.$$

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Hence $W(\gamma^{*j}\delta) \leqslant W(\gamma^{[j/g]}\delta) W(\gamma^{*(j-[j/g]g)}) W(q^{(g-1)[j/g]})$. But $W(\gamma^{*(j-[j/g]g)})$ and $W(q^{(g-1)[j/g]})$ are both $\leqslant c_1$, by Theorem 2, and $W(\gamma^{[j/g]}\delta) \to 0$ as $j \to \infty$. Hence $\lim_{j\to\infty} (\gamma^{*j}\delta) = 0$, i.e., γ^* is a W-element.

7. *Definition* 5. A polynomial, $\delta \neq 0$, is called a w-element if

$$\lim_{j\to\infty} W(\gamma^j\delta) = 0$$

for every W-element γ.

It follows easily from the definition that if δ is a w-element and $\delta|\delta^*$, then δ^* is also a w-element.

Similarly, if δ_1 and δ_2 are two different w-elements, then $\delta_1 - \delta_2$ is also a w-element.

Hence the set of all w-elements, together with zero, forms an ideal, which we will call $\mathfrak{N}$. We have $\mathfrak{N} = (\delta_0)$ for some polynomial δ_0.

Definition 6. The polynomial δ_0 is called the subsidiary character of the pseudovaluation $W(\rho)$.

The following result holds:

(a) *The subsidiary character, δ_0, is different from zero.*

Proof. Every W-element can be written as $\gamma = \beta\gamma_0$, where γ_0 is the principal character; there exists a polynomial, δ_0^*, such that

$$\lim_{j\to\infty} W(\gamma_0{}^j\delta_0{}^*) = 0.$$

Hence, since $W(\gamma^j\delta_0{}^*) \leqslant W(\beta^j) W(\gamma_0{}^j\delta_0{}^*) \leqslant c_1 W(\gamma_0{}^j\delta_0{}^*)$, by Theorem 2, we have $\lim_{j\to\infty} W(\gamma^j\delta_0{}^*) = 0$. Therefore δ_0^* is a w-element and so $\delta_0|\delta_0{}^*$, and $\delta_0{}^* \neq 0$, whence $\delta_0 \neq 0$.

Furthermore:

(b) *The two characters, γ_0 and δ_0, of $W(\rho)$ are relatively prime.*

Proof. We know from the above that there is at least one w-element, δ_0^*.

Let $\delta = q\delta^*$, where $(\delta^*, \gamma_0) = 1$ and $q|\gamma_0{}^g$, for some natural integer, g.

Let γ be any W-element. Then, from Theorem 5, $\gamma_0|\gamma$; hence $\gamma_0{}^g|\gamma^g$. Therefore $q|\gamma^g$, and so $\gamma^g = q\epsilon$, for some polynomial ϵ.

We have thus

$$\begin{aligned} 0 &\leqslant \lim_{j\to\infty} W(\gamma^j\delta^*) = \lim_{j\to\infty} W(\gamma^j\cdot\gamma^g\delta^*) \\ &= \lim_{j\to\infty} W(\gamma^j\cdot q\epsilon\delta^*) = \lim_{j\to\infty} W(\gamma^j\delta_0{}^*\cdot\epsilon) \\ &\leqslant W(\epsilon)\lim_{j\to\infty} W(\gamma^j\delta_0{}^*) = 0, \end{aligned}$$

since γ is a W-element and δ_0^* a w-element. Hence δ^* is a w-element, and so $\delta_0|\delta^*$. But $(\delta^*, \gamma_0) = 1$. Hence, a fortiori, $(\delta_0, \gamma_0) = 1$.

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8. Now let

$$\Omega^*(\rho) = \begin{cases} U(\rho) & \text{for } \gamma_0 = 1, \\ \sum_{p|\gamma_0} |\rho|_p & \text{for } \gamma_0 \neq 1, \end{cases}$$

and

$$\Omega^{**}(\rho) = \begin{cases} U(\rho) & \text{for } \delta_0 = 1, \\ \sum_{p^f \| \delta_0} W_{p^f}(\rho) & \text{for } \delta_0 \neq 1. \end{cases}$$

We proved in the last section that γ_0 and δ_0 are relatively prime; hence $\tilde{a} = \gamma_0^\infty \delta_0 \neq 0$ is a direct product.

Thus

$$W(\rho|\tilde{a}) \sim \Omega^*(\rho) + \Omega^{**}(\rho)$$

is a direct sum. We shall now attempt to show that $W(\rho)$ and $W(\rho|\tilde{a})$ are equivalent.

9. Let $\alpha_1, \alpha_2, \alpha_3, \ldots$ be an infinite sequence of elements of P such that $\lim_{j\to\infty} W(\alpha_j) = 0$. We wish to show that then both (a) $\lim_{j\to\infty} \Omega^*(\alpha_j) = 0$ and (b) $\lim_{j\to\infty} \Omega^{**}(\alpha_j) = 0$.

Put $\theta_j = (\alpha_j, \gamma_0{}^j\delta_0)$, and so $\theta_j = \phi_j \alpha_j + \psi_j \gamma_0{}^j \delta_0$ for some two elements, ϕ_j and ψ_j, of P. By Theorem 2 we have $W(\phi_j) \leqslant c_1$ and $W(\psi_j) \leqslant c_1$. Hence $W(\theta_j) \leqslant c_1\{W(\alpha_j) + W(\gamma_0{}^j\delta_0)\}$. But $W(\alpha_j)$ and $W(\gamma_0{}^j\delta_0)$ both tend to zero as $j \to \infty$; hence

$$\lim_{j\to\infty} W(\theta_j) = 0.$$

Thus the new sequence, $\theta_1, \theta_2, \theta_3, \ldots$, has also the limit zero with respect to W.

We have

$$\theta_j = (\alpha_j, \gamma_0{}^j\delta_0) = (\alpha_j, \gamma_0{}^j)(\alpha_j, \delta_0) \qquad (j = 1, 2, 3, \ldots)$$

and so the two statements: (a) $\lim_{j\to\infty} \Omega^*(\alpha_j) = 0$ and (b) $\lim_{j\to\infty} \Omega^{**}(\alpha_j) = 0$ are equivalent to the following:

(a′) All elements θ_j with sufficiently large index j, are divisible by an arbitrarily large power of γ_0;

(b′) All elements θ_j with sufficiently large index j, are divisible by δ_0.

We shall prove these two statements by indirect means.

10. First suppose (a′) to be false, and therefore that there exists an infinite subsequence, $\theta_{j_1}, \theta_{j_2}, \theta_{j_3}, \ldots$, such that $\gamma_0{}^a \| \theta_{j_\nu}$ $(\nu = 1, 2, 3, \ldots)$. This is only possible if $\gamma_0 \neq 1$, and so $\gamma_0 = p_1 p_2 \ldots p_f$, where $p_1, p_2, \ldots, p_f$ are finitely many different prime elements.

Because of our assumption, we can write

$$\theta_{j_\nu} = (\alpha_{j_\nu}, \delta_0) p_1{}^{e_{1\nu}} p_2{}^{e_{2\nu}} \ldots p_f{}^{e_{f\nu}} \qquad (\nu = 1, 2, 3, \ldots),$$

where the exponents, $e_{1\nu}, e_{2\nu}, \ldots, e_{f\nu}$, are non-negative rational integers, which are not all unbounded as $\nu \to \infty$.

We may assume, without loss of generality, that the exponents $e_{1\nu}, e_{2\nu}, \ldots, e_{g\nu}$ all tend to infinity with ν and that the other exponents, $e_{g+1,\nu}, e_{g+2,\nu}, \ldots, e_{f\nu}$, are therefore bounded for all ν. Here we have $g \leqslant f - 1$ (and possibly $g = 0$).

Hence the expression

$$(\alpha_{j_\nu}, \delta_0) p_{g+1}^{e_{g+1,\nu}} p_{g+2}^{e_{g+2,\nu}} \ldots p_f^{e_{f\nu}}$$

can take at most finitely many values.

Since θ_{j_ν} is an infinite subsequence, we may, by taking a further subsequence, assume that

$$(\alpha_{j_\nu}, \delta_0) p_{g+1}^{e_{g+1,\nu}} p_{g+2}^{e_{g+2,\nu}} \ldots p_f^{e_{f\nu}} = t$$

where t is a fixed element of P.

So we have

$$\theta_{j_\nu} = t p_1^{e_{1\nu}} p_2^{e_{2\nu}} \ldots p_g^{e_{g\nu}}$$

where the exponents, $e_{i\nu}, e_{2\nu}, \ldots, e_{g\nu}$, are non-negative rational integers which tend to infinity with ν.

Hence, for all sufficiently large ν, θ_{j_ν} is divisible by an arbitrarily large power of $\gamma_0^* = p_1 p_2 \ldots p_g$.

Further, $\lim_{\nu \to \infty} W(\theta_{j_\nu}) = 0$.

Denote by $\gamma \neq 0$ an element of P such that

$$\gamma_0^* | \gamma, \qquad (\gamma, p_{g+1} p_{g+2} \ldots p_f) = 1.$$

Let $j_{\nu(i)}$, for all sufficiently large natural integers i, be the greatest j_ν for which $\theta_{j_\nu} | \gamma^i t$. Since $(\gamma_0^*)^i | \gamma^i t$, it is clear that $j_{\nu(i)} \to \infty$ with i. But, from the definition of $j_{\nu(i)}$, we have $\gamma^i t = \lambda_i \theta_{j_{\nu(i)}}$ for some element λ_i; and, from Theorem 2,

$$W(\lambda_i) \leqslant c_1.$$

Hence

$$W(\gamma^i t) \leqslant c_1 W(\theta_{j_{\nu(i)}}),$$

and so

$$0 \leqslant \lim_{i \to \infty} W(\gamma^i t) \leqslant c_1 \lim_{i \to \infty} W(\theta_{j_{\nu(i)}}) = 0.$$

Therefore γ^i must be a W-element. But this contradicts Theorem 5, for, by its construction, γ is not divisible by γ_0.

11. From now on, therefore, we may assume that (a′), and hence (a), is true.

Now let us suppose that (b′) is false; there exists, therefore, an infinite subsequence, $\theta_{j_1}, \theta_{j_2}, \theta_{j_3}, \ldots$, of the sequence θ_j, all elements of which are not divisible by δ_0.

Since δ_0 possesses only finitely many divisors, we may assume, by replacing θ_{j_ν} by an infinite subsequence of itself, that $(\theta_{j_\nu}, \delta_0) = (\alpha_{j_\nu}, \delta_0) = \delta_0^*$ for all ν, where δ_0^* is a proper divisor of δ_0 (i.e. if $\delta_0 = \delta_0^* \delta_0^{**}$, then $\delta_0^{**} \neq 1$). Hence we have

$$\theta_{j_\nu} = \delta_0^* (\alpha_{j_\nu}, \gamma_0^{j_\nu}) \qquad (\nu = 1, 2, 3, \ldots).$$

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It is evident that, for all sufficiently large ν, θ_{j_ν} is divisible by an arbitrarily large power of γ_0.

For every sufficiently large natural integer, i, let $j_{\nu(i)}$ be the greatest j_ν for which $\theta_{j_\nu}|\gamma^i\delta$, where γ is an arbitrary W-element and δ is a polynomial such that $\delta_0^*|\delta$ and $(\delta, \delta_0^{**}) = 1$.

Since $\gamma_0^i|\gamma^i\delta$, it is clear that $j_{\nu(i)} \to \infty$ as $i \to \infty$. But, from the definition of $j_{\nu(i)}$, we have

$$\gamma^i\delta = \mu_i \theta_{j_\nu(i)}$$

for some polynomial μ_i; and, from Theorem 2, $W(\mu_i) \leqslant c_1$. Hence

$$W(\gamma^i\delta) \leqslant c_1 W(\theta_{j_\nu(i)})$$

and, since

$$\lim_{i\to\infty} W(\theta_{j_\nu(i)}) = 0,$$

we have

$$\lim_{i\to\infty} W(\gamma^i\delta) = 0.$$

Therefore δ is a w-element, and this cannot be, since δ_0 does not divide δ.

12. We have now also proved (b′) and so (b).

Clearly the limit equations (a) and (b) can be more briefly expressed in the form

$$\Omega^* \subset W \quad \text{and} \quad \Omega^{**} \subset W.$$

So we have

$$\Omega^* + \Omega^{**} \subset W.$$

To conclude this paper, we shall show that

$$W \subset \Omega^* + \Omega^{**}$$

and therefore

$$W \sim \Omega^* + \Omega^{**}.$$

To this end, let $\alpha_1, \alpha_2, \alpha_3, \ldots$ be an infinite sequence of elements of P such that both

$$\lim_{j\to\infty} \Omega^*(\alpha_j) = 0$$

and

$$\lim_{j\to\infty} \Omega^{**}(\alpha_j) = 0.$$

It will be proved that it is then true that

$$\lim_{j\to\infty} W(\alpha_j) = 0.$$

Clearly we may assume without loss of generality that every member of the series α_j is different from zero.

We know that, for all sufficiently large natural integers, j, α_j is divisible by an arbitrarily large power of γ_0 and by δ_0.

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Let $i(j)$ be the greatest natural integer for which

$$\gamma_0^{i(j)} | \alpha_j.$$

It is clear that $i(j) \to \infty$ as $j \to \infty$.

Put $\alpha_j = \gamma_0^{i(j)} \rho_j$ where ρ_j is an element of P different from zero.

Since γ_0 is prime to δ_0, it is evident that, for sufficiently large j, ρ_j is divisible by δ_0. So, for large values of j, we may put $\rho_j = \delta_0 \sigma_j$, where, by Theorem 2, $W(\sigma_j) \leqslant c_1$. We have therefore

$$0 \leqslant \lim_{j\to\infty} W(\alpha_j) \leqslant c_1 \lim_{j\to\infty} (\gamma_0^{i(j)} \delta_0) = 0.$$

So, the proof of the equivalence

$$W \sim \Omega^* + \Omega^{**}$$

is now completed. By §8, this can be put in the form

$$W(\rho) \sim W(\rho | \tilde{a})$$

where $\tilde{a}$ denotes the pseudopolynomial $\gamma_0^{\infty} \delta_0$, which is uniquely determined by the pseudovaluation $W(\rho)$.

In the proof we have assumed that $W(\rho)$ is equivalent neither to the trivial valuation nor to the improper valuation. But, in these two cases, we have, in the notation of the first section, $W(\rho) \sim W(\rho|0)$ and $W(\rho) \sim W(\rho|1)$ respectively, so that we have a result of the same form.

To conclude our discussion, we can therefore formulate the following theorem:

To every pseudovaluation, $W(\rho)$, of P (which is equivalent to the trivial valuation over the constant field) there corresponds a pseudopolynomial, $\tilde{a}$, such that

$$W(\rho) \sim W(\rho | \tilde{a})$$

where $W(\rho|\tilde{a})$ denotes the special pseudovaluation corresponding to $\tilde{a}$.

With the exception of the two special cases, $\tilde{a} = 0$ and $\tilde{a} = 1$, we have

$$\tilde{a} = \gamma_0^{\infty} \delta_0,$$

where γ_0 is the principal character and δ_0 the subsidiary character of $W(\rho)$.

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