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ON GORENSTEINNESS OF HOPF MODULE ALGEBRAS

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Abstract. Let H be a Hopf algebra with a bijective antipode, A an H-simple H-module algebra finitely generated as an algebra over the ground field and module-finite over its centre. The main result states that A has finite injective dimension and is, moreover, Artin–Schelter Gorenstein under the additional assumption that each H-orbit in the space of maximal ideals of A is dense with respect to the Zariski topology. Further conclusions are derived in the cases when the maximal spectrum of A is a single H-orbit or contains an open dense H-orbit.

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1. Introduction. For some time researchers have been trying to understand the homological behaviour of infinite dimensional Hopf algebras. In [**7**], Brown and Goodearl verified that Hopf algebras in several important classes have finite injective dimension and asked whether finiteness of injective dimension is a property of all Noetherian Hopf algebras (over a field). Although this question in its full generality appears to be intractable even now, the special case of Hopf algebras which are finitely generated as algebras over the ground field and satisfy a polynomial identity has been solved by Wu and Zhang [**35**]. Moreover, such algebras are *Artin–Schelter Gorenstein* (AS-Gorenstein for short). The AS-Gorenstein rings are distinguished by an additional condition on the Ext groups which makes these rings very similar to commutative Gorenstein rings. Later Brown [**6**] posed a refined question as to whether every Noetherian Hopf algebra is AS-Gorenstein.

It is natural to consider the same problem not only for Hopf algebras themselves, but also for associative algebras on which a Hopf algebra H acts compatibly with the multiplication in the algebras. Such algebras are called H*-module algebras* (see [**20**, 4.1.1] for the precise definition). An H-module algebra A is said to be H*-simple* if A has no non-zero proper H-stable two-sided ideals. Rather vaguely, this condition corresponds to transitivity of a group action. It is quite reasonable to expect that the H-simplicity of A should imply strong ring-theoretic consequences. For example, all H-simple Artinian algebras are quasi-Frobenius rings, i.e., Artinian rings of injective dimension 0, as is shown in [**28**]. The question of Brown and Goodearl can be modified like this: *is the injective dimension of every H-simple Noetherian H-module algebra finite*?

Probably it would be too optimistic to hope that the answer is positive without some further restrictions. This paper makes an essential use of H-orbits of maximal ideals defined in terms of a certain equivalence relation for an H-module algebra A which is a finitely generated module over its centre [**27**]. In the case when H is a group algebra this notion reduces to the usual group orbits. Consider the set $\operatorname{Max} A$ of all maximal ideals of A as a topological space with respect to the Zariski topology. The main result proved in the paper is

THEOREM 1.1. *Let H be a Hopf algebra with a bijective antipode, A an H-simple H-module algebra finitely generated as an algebra over the ground field and module-finite over its centre Z. Suppose that the H-orbits of maximal ideals of A are dense in* $\operatorname{Max} A$. *Then A is AS-Gorenstein.*

When all maximal ideals of A lie in a single H-orbit, we will determine in Section 5 the multiplicities of indecomposable direct summands in the minimal injective resolution of A as a right A-module. It will be also shown, even under weaker assumptions about A, that the classical quotient ring $Q(A)$ of A is a Frobenius ring. These additions to the main result are based on the fact that, under the stated hypotheses, the localizations of A at the maximal ideals of a polynomial subring of Z are Frobenius algebras over the respective regular local rings.

If A is a Hopf algebra, then A is an A°-module algebra where A° is the dual Hopf algebra (see [**20**, Ch. 9]), and A is A°-simple when A is residually finite dimensional. However, this case in Theorem 4.5 is covered by the already mentioned result of Wu and Zhang obtained under the weaker PI assumption. What is more interesting, our results apply also to coideal subalgebras. In fact, given a right coideal subalgebra A of a Hopf algebra H such that both A and H are finitely generated as algebras and module-finite over their centres, it was proved in [**29**] that $\operatorname{Max} A$ contains a dense open H°-orbit which coincides with the whole space $\operatorname{Max} A$ precisely when A is H°-simple or, equivalently, A has no non-zero proper two-sided ideals stable under the right coaction of H.

The first two sections of the paper deal with auxiliary ring-theoretic facts. The ultimate goal is to prove that $\operatorname{Ext}^i_A(V, A) = 0$ for each simple A-module V and each integer i except for exactly one value. For this, we need to know that for each fixed i the vector space dimension of $\operatorname{Ext}^i_A(V, A)$ can be bounded by a number not depending on V. It is more convenient to work with generalized Bass numbers $\mu_i(P, A)$ by which we mean scaled dimensions of the above Ext spaces with $P \in \operatorname{Max} A$ taken to be the annihilator of V in A. In Section 2, the Bass numbers $\mu_i(P, M)$ are defined for each finitely generated A-module M, and their global boundedness is proved under suitable hypotheses. This result leads eventually to the invariance of the Bass numbers along the H-orbits in $\operatorname{Max} A$ (see Proposition 4.4).

In Section 3, it is shown that the existence of a quasi-Frobenius classical quotient ring $Q(A)$ implies that a suitable central localization of A is AS-Gorenstein. This entails the vanishing of the Bass numbers on a dense open subset of $\operatorname{Max} A$; by translating along the H-orbits the desired property is extended then to all maximal ideals. Required facts concerning $Q(A)$ have been established already in [**27**]. Heavy reliance on [**27**] makes our approach totally different from that by Wu and Zhang. In [**35**], the existence of a quasi-Frobenius quotient ring is not a prerequisite, but is derived as a consequence of the main result.

It is not clear whether the hypotheses in the main results can be weakened. For one thing, there arise difficulties in extending the results of [**27**] to the general PI case. All work in Sections 2 and 3 is also done with rings module-finite over a central subring. The assumption that A is a finitely generated algebra enables invocation of Noether's normalization lemma in the proof of Corollary 3.5. More importantly, simple modules have to be finite dimensional in the proof of Proposition 4.4. Certainly, it suffices to assume that A is finitely generated only as an algebra over its subfield of central

H-invariant elements since the ground field can be extended. The density condition on H-orbits may really hinder some applications of the results, but again it is an essential part of the technique used in the paper.

Density of H-orbits and H-simplicity of A are related concepts, although none of the two is a consequence of the other. For $P \in \operatorname{Max} A$ denote by Ω_P the H-orbit of P, and by $\mathfrak{J}_P$ the intersection of all maximal ideals lying in Ω_P. Density of Ω_P in $\operatorname{Max} A$ means that $\mathfrak{J}_P$ is contained in all maximal ideals of A, and therefore $\mathfrak{J}_P$ coincides with the intersection of all maximal ideals of A, i.e., with the Jacobson radical $\operatorname{Jrad} A$ (since A is assumed to be module-finite over its centre, the primitive ideals of A are precisely its maximal ideals). Suppose that I is an H-stable ideal of A contained in P. Then I is contained in the ideal $P_C = \{a \in A \mid Ca \subset P\}$ for each finite dimensional subcoalgebra C of H. In this case, I is contained in all ideals lying in Ω_P, so that $I \subset \mathfrak{J}_P$ as well. The condition that all H-orbits in $\operatorname{Max} A$ are dense implies therefore that $\operatorname{Jrad} A$ contains all proper H-stable ideals of A. Furthermore, by **[10]** A is a *Jacobson* (*Hilbert*) *ring*, so that each prime ideal is an intersection of primitive ideals. In particular, $\operatorname{Jrad} A$ coincides with the prime radical of A. If A is semi-prime, then $\operatorname{Jrad} A = 0$, in which case the H-simplicity of A follows from the density of H-orbits in $\operatorname{Max} A$.

In the opposite direction, if $\mathfrak{J}_P$ is stable under the action of H, then the density of the orbit Ω_P in $\operatorname{Max} A$ follows automatically from the H-simplicity of A. This happens, for example, when H is a group algebra, but not in general. The following example has been pointed out by the referee. Take $A = k[x]$ where k is a field of characteristic 0, and let H be the universal enveloping algebra of a one-dimensional Lie algebra ky. Make A into an H-module algebra so that y acts as the derivation d/dx. It is easy to see that A has no non-trivial ideals invariant under d/dx, which means that A is H-simple. However, Ω_P is just the single-element set $\{P\}$. This follows from the fact that the filter of ideals P_C, $C \in \mathcal{F}$, consists of the powers of P, but P is the only maximal ideal of A which contains a power of P. In this example the H-orbits are not dense, and so Theorem 4.5 does not apply, although the conclusion is nevertheless true. It seems reasonable to ask whether the density assumption can be removed from the hypotheses altogether.

2. Global boundedness of generalized Bass numbers. Let A be a ring, M a right A-module, and P a maximal ideal of A such that the factor ring A/P is simple Artinian. Up to isomorphism there is exactly one simple right A-module V annihilated by P. This module is a finite dimensional left vector space over the division ring $D = \operatorname{End}_A V$. For each integer i, the abelian group $\operatorname{Ext}^i_A(V, M)$ is a right vector space over D. If it has finite dimension, we define the *normalized ith Bass number* of M at P as

$$\mu_i(P, M) = \frac{\dim_D \operatorname{Ext}^i_A(V, M)}{\dim_D V}.$$

Suppose that K is a field contained in the centre of D such that $\dim_K D < \infty$. Then $\dim_K W = (\dim_D W)(\dim_K D)$ for each finite dimensional vector space W over D, whence

$$\mu_i(P, M) = \frac{\dim_K \operatorname{Ext}^i_A(V, M)}{\dim_K V}.$$

We will be concerned with the case where A is a module-finite algebra over a commutative Noetherian ring R. Under this assumption $\mathfrak{m} = P \cap R$ is a maximal ideal of R for any maximal ideal P of A, and A/P is a finite dimensional simple algebra over the field $K = R/\mathfrak{m}$. Since A is Noetherian, V has a resolution by finitely generated free A-modules. Hence, $\mathrm{Ext}^i_A(V, M)$ is a finitely generated R-module annihilated by $\mathfrak{m}$ whenever M is a finitely generated A-module. Thus $\dim_K \mathrm{Ext}^i_A(V, M) < \infty$, so that all Bass numbers are defined. When $A = R$, we have $V \cong K$, and therefore $\mu_i(\mathfrak{m}, M) = \dim_K \mathrm{Ext}^i_R(K, M)$ is the ordinary Bass number of a finitely generated module over a commutative Noetherian ring.

Denote by $\mathrm{Max}\, A$ the set of all maximal ideals of A. We say that A satisfies the *global boundedness of the Bass numbers*, GBBN for short, in degree i if the set of non-negative rational numbers

$$\{\mu_i(P, M) \mid P \in \mathrm{Max}\, A\}$$

is bounded for each finitely generated right A-module M.

Since A is Noetherian, GBBN always holds in degree 0. Indeed, given any finitely generated A-module M, its socle is also finitely generated, and so the socle of M is a direct sum of finitely many simple modules. Now $\mu_0(P, M) \neq 0$ if and only if $\mathrm{Hom}_A(V, M) \neq 0$ where V is the simple A-module annihilated by P, if and only if V embeds in the socle of M. It follows that $\mu_0(P, M) \neq 0$ for at most a finite number of ideals $P \in \mathrm{Max}\, A$, and the set $\{\mu_0(P, M) \mid P \in \mathrm{Max}\, A\}$ is obviously bounded. In accordance with the usual conventions of homological algebra, we also have $\mathrm{Ext}^i = 0$, and so $\mu_i(P, M) = 0$, for all $i < 0$.

In this paper, the Bass numbers will be used mainly for maximal ideals P. However, it is possible to define $\mu_i(P, M)$ for any prime ideal P of A by passing to the localization $A_\mathfrak{p} = A \otimes_R R_\mathfrak{p}$ where $R_\mathfrak{p}$ is the local ring of the prime ideal $\mathfrak{p} = P \cap R$ of the ring R. The extension $PA_\mathfrak{p}$ of P is a maximal ideal of the ring $A_\mathfrak{p}$. Put

$$\mu_i(P, M) = \mu_i(PA_\mathfrak{p}, M \otimes_A A_\mathfrak{p}).$$

This formula is consistent with the previous definition of the Bass numbers at the maximal ideals of A since on the category of finitely generated right modules over a Noetherian ring the Ext functors commute with the localization at any multiplicatively closed subset of central elements (cf. [**33**, Proposition 3.3.10]). In particular, given two right A-modules N and M where N is finitely generated, we have

$$\mathrm{Ext}^i_{A_\mathfrak{p}}(N \otimes_A A_\mathfrak{p}, M \otimes_A A_\mathfrak{p}) \cong \mathrm{Ext}^i_A(N, M) \otimes_R R_\mathfrak{p}.$$

LEMMA 2.1. *Let A be a module-finite algebra over a commutative Noetherian ring R. Let $P \in \mathrm{Max}\, A$ and $\mathfrak{m} = P \cap R$. Then for any finitely generated right A-module M there are inequalities*

$$\mu_n(\mathfrak{m}, M) \leq \sum_{q=0}^{n} \mu_{n-q}\big(P,\ \mathrm{Ext}^q_R(A, M)\big),$$

$$\mu_n(P, M) \leq \mu_n(\mathfrak{m}, M) + \mu_{n-1}(P, \widetilde{M}/M) + \sum_{r=2}^{n} \mu_{n-r}\big(P,\ \mathrm{Ext}^{r-1}_R(A, M)\big)$$

where $\widetilde{M} = \mathrm{Hom}_R(A, M)$ and the right A-module structure on each $\mathrm{Ext}^i_R(A, M)$ is obtained by functoriality from the left multiplications of A on itself. The ring A satisfies GBBN in all degrees provided so does R.

Proof. Let V be the simple right A-module annihilated by P. Put $D = \operatorname{End}_A V$ and $K = R/\mathfrak{m}$ as before. There is a convergent first quarter spectral sequence

$$\{E_r^{pq}\} \underset{p}{\Longrightarrow} \operatorname{Ext}_R^{p+q}(V, M) \quad \text{with } E_2^{pq} = \operatorname{Ext}_A^p\big(V, \operatorname{Ext}_R^q(A, M)\big)$$

(see [**33**, Exercise 5.6.3]). Since this spectral sequence is functorial in its arguments, all terms E_r^{pq} are vector spaces over D, and all differentials $d_r : E_r^{pq} \to E_r^{p+r,\, q-r+1}$ are D-linear maps. The vector space $\operatorname{Ext}_R^n(V, M)$ has an exhaustive separating filtration with factors isomorphic to E_∞^{pq} for various pairs of integers p, q such that $p + q = n$. Hence

$$\dim_K \operatorname{Ext}_R^n(V, M) = \sum_{p+q=n} \dim_K E_\infty^{pq} \le \sum_{p+q=n} \dim_K E_2^{pq}.$$

Here $\dim_K E_2^{pq} = d \cdot \mu_{n-q}\big(P, \operatorname{Ext}_R^q(A, M)\big)$ where $d = \dim_K V$. Since $\operatorname{Ext}_R^n(V, M)$ is a direct sum of d copies of $\operatorname{Ext}_R^n(K, M)$, we have $\dim_K \operatorname{Ext}_R^n(V, M) = d \cdot \mu_n(\mathfrak{m}, M)$, whence the first inequality.

Since E_{r+1}^{n0} is isomorphic to the cokernel of $d_r : E_r^{n-r,r-1} \to E_r^{n0}$, we have

$$\dim_K E_{r+1}^{n0} \ge \dim_K E_r^{n0} - \dim_K E_r^{n-r,r-1} \ge \dim_K E_r^{n0} - \dim_K E_2^{n-r,r-1}$$

for each $r \ge 2$. As $E_\infty^{n0} \cong E_{n+1}^{n0}$ embeds in $\operatorname{Ext}_R^n(V, M)$, it follows that

$$\dim_K \operatorname{Ext}_R^n(V, M) \ge \dim_K E_\infty^{n0} \ge \dim_K E_2^{n0} - \sum_{r=2}^{n} \dim_K E_2^{n-r,r-1}$$

(note that for $n = 0, 1$ the sum over r simply disappears, but the inequalities remain true). This can be rewritten as

$$\dim_K \operatorname{Ext}_A^n(V, \widetilde{M}) \le \dim_K \operatorname{Ext}_R^n(V, M) + \sum_{r=2}^{n} \dim_K \operatorname{Ext}_A^{n-r}\big(V,\ \operatorname{Ext}_R^{r-1}(A, M)\big),$$

or, dividing by $\dim_K V$,

$$\mu_n(P, \widetilde{M}) \le \mu_n(\mathfrak{m}, M) + \sum_{r=2}^{n} \mu_{n-r}\big(P,\ \operatorname{Ext}_R^{r-1}(A, M)\big).$$

The A-module $\widetilde{M}$ has the property that $\operatorname{Hom}_A(N, \widetilde{M}) \cong \operatorname{Hom}_R(N, M)$ for every right A-module N. In particular, the identity map $M \to M$ gives rise to a monomorphism of A-modules $M \hookrightarrow \widetilde{M}$. The ensuing exact sequence

$$\ldots \to \operatorname{Ext}_A^{n-1}(V, \widetilde{M}/M) \to \operatorname{Ext}_A^n(V, M) \to \operatorname{Ext}_A^n(V, \widetilde{M}) \to \ldots$$

entails $\dim_K \operatorname{Ext}_A^n(V, M) \le \dim_K \operatorname{Ext}_A^{n-1}(V, \widetilde{M}/M) + \dim_K \operatorname{Ext}_A^n(V, \widetilde{M})$, i.e.,

$$\mu_n(P, M) \le \mu_{n-1}(P, \widetilde{M}/M) + \mu_n(P, \widetilde{M}).$$

Together with the previous bound for $\mu_n(P, \widetilde{M})$ this leads to the second inequality in the statement of the lemma. Note that each $\operatorname{Ext}_R^i(A, M)$ is a finitely generated A-module. It follows from the second inequality that A satisfies GBBN in degree n whenever R

satisfies GBBN in degree n and A satisfies GBBN in all degrees less than n. Induction on n proves the final assertion of the lemma. $\square$

A commutative Noetherian ring R is *Gorenstein* if all local rings $R_\mathfrak{m}$, $\mathfrak{m} \in \operatorname{Max} R$, have finite injective dimension. There are several equivalent characterizations of this property (see Bass [**2**]). If R is Gorenstein, then its rings of fractions with respect to multiplicatively closed subsets of R are also Gorenstein and

$$\mu_i(\mathfrak{q}, R) = \begin{cases} 1 & \text{if } i = \operatorname{height} \mathfrak{q}, \\ 0 & \text{otherwise} \end{cases}$$

for each $\mathfrak{q} \in \operatorname{Spec} R$ and each integer i [**2**, p. 11]. The *Gorenstein locus* $\operatorname{Gor} R$ is the set of all prime ideals $\mathfrak{q} \in \operatorname{Spec} R$ whose local rings $R_\mathfrak{q}$ are Gorenstein.

PROPOSITION 2.2. *Let A be a module-finite algebra over a commutative Noetherian ring R. If*

$(*)$ $\operatorname{Gor} R/\mathfrak{p}$ *contains a non-empty open subset of* $\operatorname{Spec} R/\mathfrak{p}$ *for each* $\mathfrak{p} \in \operatorname{Spec} R$,

then A satisfies GBBN in all degrees. In particular, this conclusion holds when R is a homomorphic image of a Gorenstein ring.

Proof. In view of Lemma 2.1 it suffices only to show that the ring R satisfies GBBN in all degrees. So, we have to check that the set

$$X_n(R, M) = \{\mu_n(\mathfrak{m}, M) \mid \mathfrak{m} \in \operatorname{Max} R\}$$

is bounded for each integer n and each finitely generated R-module M. All factor rings of R inherit property $(*)$. Proceeding by induction on n, we may assume that the set $X_i(R', M')$ is bounded for each factor ring R' of R, each finitely generated R'-module M' and each integer $i < n$. If N is a submodule of M, then

$$\mu_n(\mathfrak{m}, M) \le \mu_n(\mathfrak{m}, N) + \mu_n(\mathfrak{m}, M/N),$$

and it follows that the set $X_n(R, M)$ is bounded whenever so are both $X_n(R, N)$ and $X_n(R, M/N)$. This observation reduces the verification to the case where M is cyclic, i.e., $M \cong R/I$ for an ideal I of R. The Noetherian induction allows us also to assume that the set $X_n(R, R/J)$ is bounded for each strictly larger ideal J of R. If I is not prime, then M contains a non-zero submodule $N \cong R/J$ for some ideal J properly containing I. Since $M/N \cong R/J'$ for another ideal J' larger than I, the boundedness of $X_n(R, M)$ is immediate. So, we may assume that $M = R/\mathfrak{p}$ where $\mathfrak{p} \in \operatorname{Spec} R$.

By $(*)$ there exists $s \in R$ such that $s \notin \mathfrak{p}$ and the Gorenstein locus of the factor ring $R/\mathfrak{p}$ contains all prime ideals $\mathfrak{q} \in \operatorname{Spec} R/\mathfrak{p}$ with $s + \mathfrak{p} \notin \mathfrak{q}$. Let $\mathfrak{m} \in \operatorname{Max} R$ and $K = R/\mathfrak{m}$. If $\mathfrak{p} \not\subset \mathfrak{m}$, then $\mathfrak{p} + \mathfrak{m} = R$, and therefore $\operatorname{Ext}_R^n(K, M) = 0$ since this R-module is annihilated by both $\mathfrak{m}$ and $\mathfrak{p}$. In this case, $\mu_n(\mathfrak{m}, M) = 0$. Suppose that $\mathfrak{p} \subset \mathfrak{m}$. If $s \in \mathfrak{m}$, then s annihilates $\operatorname{Ext}_R^n(K, M)$. But s is a non-zerodivisor on M. The exact sequence $0 \to M \to M \to M/sM \to 0$ gives rise therefore to a surjection

$$\operatorname{Ext}_R^{n-1}(K, M/sM) \to \operatorname{Ext}_R^n(K, M)$$

which shows that $\mu_n(\mathfrak{m}, M) \le \mu_{n-1}(\mathfrak{m}, M/sM)$. Finally, suppose that $\mathfrak{p} \subset \mathfrak{m}$, but $s \notin \mathfrak{m}$. Then $\mathfrak{m}/\mathfrak{p} \in \operatorname{Gor} R/\mathfrak{p}$, i.e., the local ring $R_\mathfrak{m}/\mathfrak{p}R_\mathfrak{m}$ is Gorenstein. Since Ext commutes with the localization, we have

$$\mathrm{Ext}^n_{R/\mathfrak{p}}(K, M) \cong \mathrm{Ext}^n_{R_\mathfrak{m}/\mathfrak{p}R_\mathfrak{m}}(K, R_\mathfrak{m}/\mathfrak{p}R_\mathfrak{m}),$$

and so

$$\mu_n(\mathfrak{m}/\mathfrak{p}, M) = \dim_K \mathrm{Ext}^n_{R/\mathfrak{p}}(K, M) = \begin{cases} 1 & \text{if } n = \operatorname{height} \mathfrak{m}/\mathfrak{p}, \\ 0 & \text{otherwise.} \end{cases}$$

Now Lemma 2.1 applied with $A = R/\mathfrak{p}$ and $P = \mathfrak{m}/\mathfrak{p}$ yields

$$\mu_n(\mathfrak{m}, M) \le \sum_{i=0}^{n} \mu_{n-i}(\mathfrak{m}/\mathfrak{p}, M_i) \le 1 + \sum_{i=1}^{n} \mu_{n-i}(\mathfrak{m}/\mathfrak{p}, M_i),$$

where we put $M_i = \mathrm{Ext}^i_R(A, M)$, so that, in particular, $M_0 = \mathrm{Hom}_R(A, M) \cong M$.

By the induction hypothesis $X_{n-1}(R, M/sM)$ and all sets $X_{n-i}(R/\mathfrak{p}, M_i)$ with $i = 1, \ldots, n$ are bounded. Hence there exists an integer $a > 0$ such that

$$\begin{aligned} \mu_{n-1}(\mathfrak{m}, M/sM) &\le a \quad \text{for all } \mathfrak{m} \in \operatorname{Max} R, \\ \mu_{n-i}(\mathfrak{m}/\mathfrak{p}, M_i) &\le a \quad \text{for all } \mathfrak{m} \in \operatorname{Max} R \text{ with } \mathfrak{p} \subset \mathfrak{m} \text{ and all } i = 1, \ldots, n. \end{aligned}$$

It follows now from the previous estimates that $\mu_n(\mathfrak{m}, M) \le \max(a, 1 + na)$ for all maximal ideals $\mathfrak{m}$, and the boundedness of $X_n(R, M)$ is proved.

Finally, any homomorphic image of a commutative Gorenstein ring satisfies condition $(*)$ by [**15**, Corollary 1.6]. □

REMARK 1. Greco and Marinari [**15**, Proposition 1.7] proved that condition $(*)$ implies that $\operatorname{Gor} B$ is open in $\operatorname{Spec} B$ for each finitely generated commutative R-algebra B; when B is a domain, $\operatorname{Gor} B$ is obviously non-empty since the total quotient ring of B is a field. In particular, $(*)$ is equivalent to the condition that $\operatorname{Gor} R/\mathfrak{p}$ is open in $\operatorname{Spec} R/\mathfrak{p}$ for each $\mathfrak{p} \in \operatorname{Spec} R$. Sharp [**26**] introduced a class of commutative Noetherian rings called *acceptable* with one of the defining conditions being the openness of the Gorenstein loci for all finitely generated algebras. Thus, the conclusion of Proposition 2.2 holds whenever R is acceptable. It was also proved in [**15**] that any excellent commutative ring is acceptable.

A restriction on R in Proposition 2.2 is necessary. We will give below an example of a commutative Noetherian ring which does not satisfy GBBN in degree 1. This example employs Hochster's construction of Noetherian rings [**16**] which in turn generalizes the construction in Nagata's example of Noetherian rings of infinite Krull dimension.

Let K be an algebraically closed field, and for each integer $i \ge 1$ let R_i be a finitely generated commutative K-algebra which is a domain and which has a prime ideal $\mathfrak{p}_i$ such that $\mu_1(\mathfrak{p}_i, R_i) \ge i$. For instance, we can take R_i to be the subalgebra of the polynomial algebra $K[X]$ in one indeterminate spanned over K by 1 and by all powers X^j with $j > i$. Then $R_i = K + \mathfrak{p}_i$ where $\mathfrak{p}_i = X^{i+1}K[X]$ is a maximal ideal of R_i. The field of rational functions $Q = K(X)$ is the quotient field of R_i. Since Q is an injective R_i-module, the short exact sequence $0 \to R_i \to Q \to Q/R_i \to 0$ gives rise to an exact sequence

$$0 = \mathrm{Hom}_{R_i}(R_i/\mathfrak{p}_i, Q) \to \mathrm{Hom}_{R_i}(R_i/\mathfrak{p}_i, Q/R_i) \to \mathrm{Ext}^1_{R_i}(R_i/\mathfrak{p}_i, R_i) \to 0.$$

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It follows that $\mathrm{Ext}^1_{R_i}(R_i/\mathfrak{p}_i, R_i) \cong V/R_i$ where $V = \{a \in Q \mid \mathfrak{p}_i a \subset R_i\} = K[X]$. Hence $\mu_1(\mathfrak{p}_i, R_i) = \dim_K V/R_i = i$, and the requested conditions on R_i are fulfilled.

Now put $R' = \bigotimes_{i=1}^{\infty} R_i$ and $S = R' \setminus \bigcup_{i=1}^{\infty} \mathfrak{p}_i R'$. By [**16**, Proposition 1] the ring of fractions $R = S^{-1}R'$ is a Noetherian domain whose maximal ideals are precisely the ideals $\mathfrak{m}_i = \mathfrak{p}_i R$. Since R is flat over R_i and $R/\mathfrak{m}_i \cong R_i/\mathfrak{p}_i \otimes_{R_i} R$, we have

$$\mathrm{Ext}^1_R(R/\mathfrak{m}_i, R) \cong \mathrm{Ext}^1_{R_i}(R_i/\mathfrak{p}_i, R_i) \otimes_{R_i} R \cong \mathrm{Ext}^1_{R_i}(R_i/\mathfrak{p}_i, R_i) \otimes_{R_i/\mathfrak{p}_i} R/\mathfrak{m}_i$$

by [**5**, Section 6, Proposition 10]. Hence $\mu_1(\mathfrak{m}_i, R) = \mu_1(\mathfrak{p}_i, R_i) \geq i$ for each i. This shows that the set $\{\mu_1(\mathfrak{m}, R) \mid \mathfrak{m} \in \mathrm{Max}\, R\}$ is not bounded.

3. Gorensteinness satisfied generically. In this section, we review several notions related to Gorensteinness. For an algebra A over a commutative ring R, let us put $A^*_R = \mathrm{Hom}_R(A, R)$ and regard A^*_R as an A-bimodule in a natural way. One says that A is a *quasi-Frobenius algebra* over R if the underlying R-module of A is finitely generated projective and the following two equivalent conditions are satisfied:

(a) regarded as an A-module with respect to the left-hand structure, a direct sum of several copies of A^*_R contains A as a submodule direct summand,

(b) as an A-module with respect to the right-hand structure, A^*_R is projective.

This means that A is a left quasi-Frobenius extension of R as defined by Müller [**21**]. The class of quasi-Frobenius algebras over a commutative ring was mentioned briefly by Nakayama [**23**, p. 186] in connection with the cohomology of algebras. According to the testimony contained in [**21**], such algebras were studied by Chase and Rosenberg in an unpublished work, and Müller's paper developed the more general theory of quasi-Frobenius extensions for arbitrary rings. By [**22**, Theorem 35] the notion of quasi-Frobenius extensions is left–right symmetric when applied to algebras.

One says that A is a *Frobenius algebra* over R if the underlying R-module of A is finitely generated projective and $A^*_R \cong A$ as either right or left A-modules. This provides a generalization, due to Eilenberg and Nakayama [**11**], of the notion of Frobenius algebras over a field.

If A is a quasi-Frobenius algebra over R, then $A \otimes_R R'$ is a quasi-Frobenius algebra over R' for any homomorphism of commutative rings $R \to R'$. This is clear from the definition since $\mathrm{Hom}_{R'}(A \otimes_R R', R') \cong A^*_R \otimes_R R'$ by finiteness and projectivity of A over R. Similarly, $A \otimes_R R'$ is a Frobenius algebra over R' whenever A is a Frobenius algebra over R.

Any quasi-Frobenius algebra over a field is a *quasi-Frobenius ring*. Recall that a ring is said to be quasi-Frobenius if it is left and right Artinian, left and right self-injective. A *Frobenius ring* is any quasi-Frobenius ring whose top and socle are isomorphic as either left or right modules [**18**, Corollary 13.4.3].

From now on, we assume that *A is a ring module-finite over a Noetherian central subring R*. Then A is Noetherian too. The *upper grade* of a maximal ideal P of A is defined as

$$\text{upper grade}\, P = \sup\{i \mid \mathrm{Ext}^i_A(V, A) \neq 0\}$$

where V is any non-zero right A/P-module. The ring A is said to be (right) *injectively homogeneous* over R if the upper grade is finite for each $P \in \mathrm{Max}\, A$ and if each pair of maximal ideals $P,\ P' \in \mathrm{Max}\, A$ with $P \cap R = P' \cap R$ have equal upper grades. This notion was studied by Brown and Hajarnavis more generally for Noetherian rings integral over central subrings [**8**]. The special case where R is local was considered earlier by Vasconcelos [**32**]. One difference with the definition given in [**8**] is that we do not require A to have finite (right) injective dimension. By [**8**, Lemma 3.1]

$$\operatorname{injdim} A = \sup\{\text{upper grade } P \mid P \in \operatorname{Max} A\}.$$

Applying this formula to the localization $A_{\mathfrak{m}} = A \otimes_R R_{\mathfrak{m}}$ where $\mathfrak{m} \in \operatorname{Max} R$, we get

$$\operatorname{injdim} A_{\mathfrak{m}} = \sup\{\text{upper grade } P \mid P \in \operatorname{Max} A,\ P \cap R = \mathfrak{m}\}.$$

Since A has finitely many maximal ideals lying above $\mathfrak{m}$, it follows that, whenever A is right injectively homogeneous over R, the ring $A_{\mathfrak{m}}$ has finite injective dimension, and $A_{\mathfrak{m}}$ is right injectively homogeneous over $R_{\mathfrak{m}}$. The prefix "right" may henceforth be omitted since this notion is left–right symmetric [**8**, Corollary 4.4]. Furthermore, under the same assumption that A is injectively homogeneous over R,

$$\text{upper grade } P = \operatorname{height} \mathfrak{m} \quad \text{for any } P \in \operatorname{Max} A \text{ with } P \cap R = \mathfrak{m}$$

by [**8**, Corollary 3.5] since this equality can be checked locally (note that the height of a prime ideal was called the rank in [**8**]). Therefore, $\operatorname{injdim} A < \infty$ if and only if $\operatorname{Kdim} R < \infty$ where Kdim stands for the (classical) Krull dimension. Since the ring extension $R \subset A$ satisfies Going-Up and Incomparability by [**3**], we have

$$\operatorname{Kdim} A = \operatorname{Kdim} R = \operatorname{injdim} A.$$

A sequence of elements $x_1, \ldots, x_n \in R$ is said to be A-*regular* if $\sum_{j=1}^{n} Ax_j \neq A$ and x_i is a non-zerodivisor on $A/\sum_{j=1}^{i-1} Ax_j$ for each i. For a prime ideal P of A the length of any A-regular sequence contained in the prime ideal $P \cap R$ of R does not exceed the height of P. The ring A is called *centrally Macaulay* over R or just R-*Macaulay* if $P \cap R$ contains an A-regular sequence of length equal to the height of P for each $P \in \operatorname{Max} A$. This implies, in particular, that P and $P \cap R$ have equal heights. Brown, Hajarnavis and MacEacharn [**9**] extended many classical facts of the commutative theory to centrally Macaulay rings. In [**8**, Theorem 3.4], it is proved that A is R-Macaulay whenever A is injectively homogeneous over R (by resorting to the localizations $A_{\mathfrak{m}}$ this result extends immediately to cover the case when the injective dimension of A is infinite). Since we consider only the case where A is a finitely generated R-module, the ring A is centrally Macaulay over R if and only if A is Cohen–Macaulay as an R-module.

Concerning results of [**8**], we note that Theorems 5.3 and 5.5 in that paper are not completely correct. The multiplicities of indecomposable summands in the minimal injective resolution of A as a right A-module are not given in general by the uniform dimensions of the prime factor rings of A as can be seen already in the case of injective dimension 0. Indeed, the property that A is isomorphic to the direct sum of the injective hulls of the right A-modules A/P, $P \in \operatorname{Max} A$, distinguishes Frobenius rings in the class of all quasi-Frobenius rings. We will return to this question later in Section 5.

Lemma 3.1. *Suppose that R is a commutative Gorenstein ring contained in the centre of A. If A is a quasi-Frobenius algebra over R, then A is injectively homogeneous over R. Furthermore,* $\operatorname{injdim} A_{\mathfrak{m}} = \operatorname{injdim} R_{\mathfrak{m}}$ *for each* $\mathfrak{m} \in \operatorname{Max} R$.

Proof. To prove that A is injectively homogeneous over R it suffices to look at the localizations at the maximal ideals of R. So we may assume R to be local with a maximal ideal $\mathfrak{m}$. Let $x_1, \ldots, x_n \in \mathfrak{m}$ be a sequence of maximal length which is regular on R. Put $I = \sum_{i=1}^{n} Rx_i$. Since R is Gorenstein, R is a Cohen–Macaulay ring and its ideal I is irreducible [**2**, Theorem 4.1], so that R/I is an artinian commutative ring with a simple socle. In particular, the ring R/I is self-injective [**12**, Proposition 21.5], i.e., quasi-Frobenius (moreover, it is a Frobenius ring). The R/I-algebra A/IA is obtained from A by base change $R \to R/I$. Hence A/IA is a quasi-Frobenius extension of R/I, and therefore A/IA is a quasi-Frobenius ring by [**21**, Satz 3]. Since any quasi-Frobenius ring contains all simple modules in its socle [**18**, Theorem 13.4.2], A/IA is injectively homogeneous of injective dimension 0. Since R is local, any projective R-module is free. In particular, so is A. This makes it clear that the sequence $x_1, \ldots, x_n$ remains regular on A. Now it follows from [**8**, Theorem 4.3] that A is injectively homogeneous over R with injdim $A = n$. Similarly, injdim $R = n$. □

Given $s \in R$, we will denote by $R[s^{-1}]$ the ring of fractions of R with respect to the multiplicatively closed set of powers of s. Put $A[s^{-1}] = A \otimes_R R[s^{-1}]$.

LEMMA 3.2. *Suppose that R is a Noetherian domain with the quotient field $Q(R)$. Put $Q(A) = A \otimes_R Q(R)$. If M is a finitely generated (right) A-module such that the $Q(A)$-module $M \otimes_R Q(R)$ is projective, then there exists an element $0 \neq s \in R$ such that $M \otimes_R R[s^{-1}]$ is a projective $A[s^{-1}]$-module.*

Proof. Take any epimorphism of A-modules $\varphi : F \to M$ where F is a free A-module of finite rank. Then $\varphi' = \varphi \otimes \mathrm{Id} : F \otimes_R Q(R) \to M \otimes_R Q(R)$ is a split epimorphism of $Q(A)$-modules by the projectivity hypothesis. There exists a $Q(A)$-linear map $\psi' : M \otimes_R Q(R) \to F \otimes_R Q(R)$ such that $\varphi'\psi' = \mathrm{Id}$. Since A is a Noetherian ring, the finitely generated A-module M is finitely presented. Hence

$$\mathrm{Hom}_{Q(A)}\big(M \otimes_R Q(R), N \otimes_R Q(R)\big) \cong \mathrm{Hom}_A(M, N) \otimes_R Q(R)$$

for each (right) A-module N by [**33**, Lemma 3.3.8]. In particular, $\psi' = \psi \otimes u^{-1}$ for some $\psi \in \mathrm{Hom}_A(M, F)$ and $0 \neq u \in R$. Then $\varphi\psi \otimes 1 = \mathrm{Id} \otimes u$ in $(\mathrm{End}_A M) \otimes_R Q(R)$, and it follows that there exists $0 \neq v \in R$ such that $v\varphi\psi = vu\,\mathrm{Id}$. Taking $s = vu$, we see that

$$\varphi \otimes \mathrm{Id} : F \otimes_R R[s^{-1}] \to M \otimes_R R[s^{-1}]$$

is a split epimorphism of $A[s^{-1}]$-modules. Hence, $M \otimes_R R[s^{-1}]$ is a direct summand of a free $A[s^{-1}]$-module. □

PROPOSITION 3.3. *Let A be a module-finite algebra over a commutative Noetherian domain R whose Gorenstein locus contains a non-empty open subset of* Spec R. *If $Q(A) = A \otimes_R Q(R)$ is a quasi-Frobenius algebra over the quotient field $Q(R)$ of R, then there exists $0 \neq s \in R$ such that $A[s^{-1}]$ is injectively homogeneous over $R[s^{-1}]$.*

Proof. By the hypothesis there exists $0 \neq t \in R$ such that $R[t^{-1}]$ is a Gorenstein ring. Moreover, we can find such an element t with the property that $A[t^{-1}]$ is a free $R[t^{-1}]$-module. Since $R[t^{-1}]$ has the same quotient field as R and the pair $R[t^{-1}]$, $A[t^{-1}]$ satisfies the same assumptions as R and A, we may assume from the very beginning that R is Gorenstein and A is free as an R-module. Now

$$A_R^* \otimes_R Q(R) \cong \mathrm{Hom}_{Q(R)}\big(Q(A), Q(R)\big)$$

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is right $Q(A)$-projective since $Q(A)$ is a quasi-Frobenius algebra. Lemma 3.2 applied to $M = A_R^*$ shows that there exists $0 \neq s \in R$ such that

$$A_R^* \otimes_R R[s^{-1}] \cong \operatorname{Hom}_{R[s^{-1}]}\big(A[s^{-1}], R[s^{-1}]\big)$$

is right $A[s^{-1}]$-projective. This means that $A[s^{-1}]$ is a quasi-Frobenius algebra over the commutative Gorenstein ring $R[s^{-1}]$. By Lemma 3.1 $A[s^{-1}]$ is injectively homogeneous over $R[s^{-1}]$. $\square$

Following Wu and Zhang [**34**], we say that A is (right) *AS-Gorenstein* if A has finite right injective dimension, say d, and for each simple right A-module V the left A-module $\operatorname{Ext}_A^d(V, A)$ is also simple, while $\operatorname{Ext}_A^i(V, A) = 0$ whenever $i \neq d$ (cf. [**7**, 1.14]). In this case, A is injectively homogeneous over R with $\operatorname{injdim} A_{\mathfrak{m}} = d$ for all $\mathfrak{m} \in \operatorname{Max} R$. Since A satisfies a polynomial identity by module-finiteness over R, it follows from [**35**, Proposition 3.2] that A is also left AS-Gorenstein. Therefore, the prefix may be omitted.

LEMMA 3.4. *If* height $\mathfrak{m} = n$ *for all* $\mathfrak{m} \in \operatorname{Max} R$ *and* A *is injectively homogeneous over* R*, then* A *is AS-Gorenstein of injective dimension* n.

Proof. The hypothesis entails $\operatorname{injdim} A = \operatorname{Kdim} R = n$. Since A is centrally Macaulay over R [**8**, Theorem 3.4], each maximal ideal of R contains an A-regular sequence of length n. Suppose that $I = Ax_1 + \cdots + Ax_n$ where $x_1, \ldots, x_n \in R$ is an A-regular sequence. Each prime ideal P of A containing I has to be a maximal ideal since

$$\operatorname{height} P \geq n = \operatorname{Kdim} A.$$

Hence $\operatorname{Kdim} A/I = 0$, and so A/I is a module-finite algebra over a commutative Noetherian ring R' of Krull dimension 0. It follows that R' and A/I are Artinian. Next, repeated application of the Rees Theorem [**25**, Theorem 9.37] yields

$$\operatorname{Ext}_A^i(V, A) \cong \operatorname{Ext}_{A/I}^{i-n}(V, A/I)$$

for each right A/I-module V in all degrees i. It follows that $\operatorname{Ext}_A^i(V, A) = 0$ for all $i < n$ since $\operatorname{Ext}_{A/I}^j = 0$ for $j < 0$ and $\operatorname{Ext}_{A/I}^j(V, A/I) = 0$ for all $j > 0$ since $\operatorname{Ext}_A^i(?, A) = 0$ for $i > n$. The latter means that A/I is a right self-injective ring. It is therefore quasi-Frobenius. By a basic property of quasi-Frobenius rings the functor $\operatorname{Hom}_{A/I}(?, A/I)$ gives a duality between the categories of right and left modules over A/I [**1**, Theorem 30.7]. Hence

$$\operatorname{Ext}_A^n(V, A) \cong \operatorname{Hom}_{A/I}(V, A/I)$$

is a simple left A/I-module whenever V is a simple right one. Now any simple right A-module is annihilated by some maximal ideal of R and therefore by an A-regular sequence of length n. Hence, all the above conclusions hold with V taken to be such a module. $\square$

REMARK 2. The fact that $\operatorname{Ext}_A^n(?, A)$ takes simple A-modules to simple ones can be deduced from the Ischebeck spectral sequence [**17**] which shows that

$$V \cong \operatorname{Ext}_A^n\big(\operatorname{Ext}_A^n(V, A), A\big)$$

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when it is known already that $\mathrm{Ext}^i_A(V, A) = 0$ for all $i \neq n$. However, the arguments essentially taken from **[8]** allow us to avoid the use of this spectral sequence.

A Noetherian PI ring B of finite injective dimension d is called *right injectively smooth* if $\mathrm{Ext}^d_B(V, B) \neq 0$ for all simple right B-modules V **[31]**. Lemma 3.4 can be reformulated by saying that A is AS-Gorenstein provided that A is injectively smooth (cf. [**31**, Theorem 3.8]).

We will call a commutative Noetherian ring R *equidimensional* if $\mathrm{Kdim}\, R/\mathfrak{p} = \mathrm{Kdim}\, R$ for each minimal prime ideal $\mathfrak{p}$ of R. Recall that an *embedded prime* of R is any associated prime ideal of R which is not a minimal prime ideal. If R has no embedded primes, then the set of zerodivisors of R is the union of its minimal primes. In this case, the total quotient ring $Q(R)$ of R is a Noetherian ring of Krull dimension 0, i.e., an Artinian ring.

COROLLARY 3.5. *Let R be an equidimensional finitely generated commutative algebra over a field k with the total quotient ring $Q(R)$. Suppose that R has no embedded primes. If A is a module-finite algebra over R such that $Q(A) = A \otimes_R Q(R)$ is a quasi-Frobenius ring, then there exists a non-zerodivisor s of R such that the ring $A[s^{-1}]$ is AS-Gorenstein.*

Proof. By Noether's Normalization Lemma R is integral, therefore module-finite, over a subalgebra R' isomorphic to a polynomial algebra, say in n indeterminates, over k. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ be all the minimal primes of R, and let $\mathfrak{p}'_i = \mathfrak{p}_i \cap R'$ for each i. Then $\bigcap \mathfrak{p}'_i$ is a nilpotent ideal of R' since $\bigcap \mathfrak{p}_i$ is the nil radical of R. Since R' is a domain, we must have $\mathfrak{p}'_i = 0$ for at least one i. On the other hand, $\mathrm{Kdim}\, R'/\mathfrak{p}'_i$ does not depend on i since R is equidimensional and $\mathrm{Kdim}\, R'/\mathfrak{p}'_i = \mathrm{Kdim}\, R/\mathfrak{p}_i$ by Going-Up and Incomparability. It follows that there are no inclusions between the ideals $\mathfrak{p}'_1, \dots, \mathfrak{p}'_r$. Therefore $\mathfrak{p}'_i = 0$ for all i. Then each non-zero element of R' is a non-zerodivisor of R since it is contained in none of the associated primes of R.

Let $Q(R')$ be the quotient field of R'. Now $R \otimes_{R'} Q(R')$ is a partial quotient ring of R. Since this ring is a finite dimensional algebra over $Q(R')$, all its non-zerodivisors are invertible elements. It follows that $Q(R) \cong R \otimes_{R'} Q(R')$, and therefore $Q(A) \cong A \otimes_{R'} Q(R')$ is a finite dimensional algebra over the field $Q(R')$. This means that the quasi-Frobenius ring $Q(A)$ is a quasi-Frobenius algebra over $Q(R')$.

The ring R' is a Noetherian domain of finite global dimension n. In particular, $\mathrm{injdim}\, R' < \infty$ as well, which means that R' is Gorenstein. So the hypotheses of Proposition 3.3 are satisfied for the R'-algebra A. There exists $0 \neq s \in R'$ such that $A[s^{-1}]$ is injectively homogeneous over $R'[s^{-1}]$. Since $R'[s^{-1}]$ is a finitely generated domain over the field k with the quotient field isomorphic to $Q(R')$, we have

$$\mathrm{height}\, \mathfrak{m} = \mathrm{Kdim}\, R'[s^{-1}] = \text{transcendence degree } Q(R')/k = n$$

for each $\mathfrak{m} \in \mathrm{Max}\, R'[s^{-1}]$ (see, e.g., [**12**, 8.2.1, Theorem A]). By Lemma 3.4 $A[s^{-1}]$ is AS-Gorenstein. □

4. The main result. Let H be a Hopf algebra over a field k with the comultiplication $\Delta : H \to H \otimes H$, the co-unit $\varepsilon : H \to k$, and the antipode $S : H \to H$. Let A be a left H-module algebra. One of the tools we need is twistings of A-modules introduced in **[30]**. For a right A-module V and a right H-comodule U consider the

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vector spaces

$$U \otimes V = U \otimes_k V \quad \text{and} \quad [U, V] = \mathrm{Hom}_k(U, V)$$

as right A-modules with respect to the actions

$$(u \otimes v)a = \sum_{(u)} u_{(0)} \otimes v\big((Su_{(1)})a\big), \qquad (fa)(u) = \sum_{(u)} f(u_{(0)})(u_{(1)}a)$$

where $a \in A$, $u \in U$, $v \in V$ and $f \in \mathrm{Hom}_k(U, V)$. Here the comodule structure map $U \to U \otimes H$ is written symbolically as $u \mapsto \sum_{(u)} u_{(0)} \otimes u_{(1)}$.

LEMMA 4.1. *Given two right A-modules V and W, there are k-linear bijections*

$$\mathrm{Ext}^i_A(U \otimes V, W) \cong \mathrm{Ext}^i_A\big(V, [U, W]\big).$$

Proof. For $i = 0$, the isomorphism $\mathrm{Hom}_A(U \otimes V, W) \cong \mathrm{Hom}_A\big(V, [U, W]\big)$ natural in V and W was verified in [**30**, Lemma 1.1]. It shows that the functors $U \otimes ?$ and $[U, ?]$ defined on the category of right A-modules form an adjoint pair. Both functors are exact, so it follows that $U \otimes ?$ preserves projectivity of A-modules. If $F_\bullet$ is a projective resolution of V then $U \otimes F_\bullet$ is a projective resolution of $U \otimes V$. It remains to exploit the isomorphism of complexes $\mathrm{Hom}_A(U \otimes F_\bullet, W) \cong \mathrm{Hom}_A\big(F_\bullet, [U, W]\big)$. □

LEMMA 4.2. *Suppose that the antipode S is bijective. If U is a right H-comodule with* $\dim_k U = d < \infty$ *and M is a free right A-module of rank* 1, *then* $[U, M]$ *is a free right A-module of rank d.*

Proof. There is a left H-module structure on M which makes M into an equivariant right A-module in the sense that the following compatibility condition is satisfied:

$$h(va) = \sum_{(h)} (h_{(1)}v)(h_{(2)}a) \quad \text{for all } h \in H,\ v \in M,\ a \in A.$$

Indeed, since $M \cong A$ as right A-modules, we can transfer the given H-module structure on A to M. By [**30**, Lemma 1.2] there is an isomorphism of right A-modules

$$[U, M] \cong [U_{\mathrm{triv}}, M]$$

where U_{triv} is the vector space U equipped with the trivial right H-comodule structure $u \mapsto u \otimes 1$ for $u \in U$. Since the action of A on $[U_{\mathrm{triv}}, M]$ is given by the rule

$$(fa)(u) = f(u)a, \quad f \in \mathrm{Hom}_k(U, V),\ a \in A,\ u \in U,$$

taking values of linear maps $U \to M$ at basis elements of U gives an isomorphism of A-modules $[U_{\mathrm{triv}}, M] \cong M^d$. Thus $[U, M] \cong M^d$, and the conclusion is clear. □

Another important tool is a certain equivalence relation $\sim_H$ on the subset $\mathrm{Spec}_f A$ of the prime spectrum $\mathrm{Spec}\, A$ consisting of those prime ideals P of A for which there exists no infinite strictly ascending chain $P_0 \subset P_1 \subset \cdots$ in $\mathrm{Spec}\, A$ starting at $P_0 = P$. The existence of this relation was proved in [**27**, Theorem 1.1] under the assumption that A is module-finite over its centre Z. From now on, we will assume that A satisfies this condition.

Denote by $\mathcal{F}$ the family of all finite dimensional subcoalgebras of H, and for each $C \in \mathcal{F}$ and $P \in \operatorname{Spec} A$ define an ideal

$$P_C = \{a \in A \mid Ca \subset P\}.$$

Given two prime ideals $P,\ P' \in \operatorname{Spec}_f A$, one has $P \sim_H P'$ if and only if P' is a prime minimal over P_C for some $C \in \mathcal{F}$. The $\sim_H$-equivalence classes are called the *H-orbits*.

In view of [**27**, Proposition 5.10] the H-orbit of any maximal ideal of A consists of maximal ideals. Thus, the set $\operatorname{Max} A$ of all maximal ideals is a union of H-orbits. For $P,\ P' \in \operatorname{Max} A$ one has $P \sim_H P'$ if and only if $P_C \subset P'$ for some $C \in \mathcal{F}$.

Note that P_C coincides with the kernel of the linear map $\tau : A \to \operatorname{Hom}_k(C, A/P)$ defined by the rule $\tau(a)(c) = ca + P$ for $a \in A$ and $c \in C$. Hence $\dim_k A/P_C < \infty$ for any $C \in \mathcal{F}$ whenever $\dim_k A/P < \infty$. In this case, $\dim_k A/P' < \infty$ for each P' in the H-orbit of P.

The maximal ideals of A are in a bijective correspondence with the isomorphism classes of simple right A-modules. It turns out that the H-orbit relation on $\operatorname{Max} A$ can be described in terms of the twisting operations with modules. In the next lemma, we will consider only maximal ideals of finite codimension in A since this is the only case needed in the final result.

LEMMA 4.3. *Assume S to be surjective. Let $V,\ V'$ be two simple finite dimensional right A-modules, and let $P,\ P'$ be their annihilators in A. In order that P and P' lie in the same H-orbit, it is necessary and sufficient that V' be isomorphic to a composition factor of $U \otimes V$ for some finite dimensional right H-comodule U.*

Proof. If $\rho : U \to U \otimes H$ is a right H-comodule structure on a finite dimensional vector space U, then $\rho(U) \subset U \otimes C$ for some $C \in \mathcal{F}$. Now $S(C) \in \mathcal{F}$ as well, and from the explicit formula for the twisted action of A it is clear that the ideal

$$P_{S(C)} = \{a \in A \mid S(C)a \subset P\}$$

annihilates $U \otimes V$. Hence $P_{S(C)} \subset P'$, and therefore $P \sim_H P'$, whenever V' occurs as a composition factor of $U \otimes V$.

Conversely, suppose that $P \sim_H P'$. Then $P_D \subset P'$ for some $D \in \mathcal{F}$. Since S is surjective, there exists $C \in \mathcal{F}$ such that $D \subset S(C)$. Take $U = C$ with the comodule structure given by the comultiplication $\Delta : C \to C \otimes C$. If $a \in A$ annihilates $C \otimes V$, then

$$v\big(S(c)a\big) = (\varepsilon \otimes \mathrm{Id})\Big(\sum_{(c)} c_{(1)} \otimes v\big(S(c_{(2)})a\big)\Big) = (\varepsilon \otimes \mathrm{Id})\big((c \otimes v) \cdot a\big) = 0$$

for all $c \in C$ and $v \in V$, i.e., $S(C)a$ is contained in the annihilator P of V. This shows that the annihilator of $C \otimes V$ in A coincides with the ideal $P_{S(C)}$. On the other hand, since $\dim_k(C \otimes V) < \infty$, the right A-module $C \otimes V$ has finite length. If $V_1, \ldots, V_n$ are the simple factors in its composition series, then the product of their annihilators $P_1, \ldots, P_n$ taken in a suitable order annihilates $C \otimes V$, whence

$$\prod P_i \subset P_{S(C)} \subset P_D \subset P'.$$

It follows that $P_i \subset P'$ for at least one i by the primeness of P'. Hence $P' = P_i$ since $P_i \in \operatorname{Max} A$. This entails $V_i \cong V'$. $\square$

Now we are ready to prove the invariance of the Bass numbers along the H-orbit of a maximal ideal of finite codimension in A. I do not know whether the next result can be valid under less-restrictive assumptions about P.

PROPOSITION 4.4. *Let H be a Hopf algebra with a bijective antipode, A an H-module algebra module-finite over a Noetherian central subring R, and P a maximal ideal of finite codimension in A. Put*

$$\Omega = \{P' \in \operatorname{Max} A \mid P \sim_H P'\}.$$

Suppose that either Ω is finite or R has the property that the Gorenstein locus of $R/\mathfrak{p}$ contains a non-empty open subset of $\operatorname{Spec} R/\mathfrak{p}$ *for each* $\mathfrak{p} \in \operatorname{Spec} R$. *Then*

$$\mu_i(P', A) = \mu_i(P, A) \quad \text{for all } P' \in \Omega \text{ and all } i \geq 0.$$

Proof. We may fix i. In both cases, the set $X = \{\mu_i(P', A) \mid P' \in \Omega\}$ is finite. This conclusion is obvious when Ω is finite. Under the other assumption about R Proposition 2.2 ensures that A satisfies GBBN in degree i, whence X is bounded. However, each rational number in the set X is written as an integer fraction whose denominator is bounded by the number r of elements generating A as an R-module. Indeed, if V' is the simple right A-module annihilated by $P' \in \operatorname{Max} A$, then

$$\mu_i(P', A) = \frac{\dim_{R/\mathfrak{m}'} \operatorname{Ext}^i_A(V', A)}{\dim_{R/\mathfrak{m}'} V'}$$

with $\dim_{R/\mathfrak{m}'} V' \leq \dim_{R/\mathfrak{m}'} A/P' \leq r$ where $\mathfrak{m}' = R \cap P'$. Hence there are finitely many possibilities both for the denominator and the numerator, and the finiteness of X is clear.

In particular, X has a largest element. Changing P if necessary, we may assume that $\mu_i(P, A)$ is this largest element, so that $\mu_i(P', A) \leq \mu_i(P, A)$ for all $P' \in \Omega$. Suppose that W is any right A-module of finite length whose composition factors have annihilators lying in Ω. Then

$$\frac{\dim_k \operatorname{Ext}^i_A(W, A)}{\dim_k W} \leq \mu_i(P, A). \tag{$\star$}$$

If W is a simple right A-module annihilated by some $P' \in \Omega$, then the left-hand side coincides with $\mu_i(P', A)$, and the inequality above holds by the choice of P. In general ($\star$) is proved by induction on the length of W. Considering any short exact sequence of right A-modules $0 \to W' \to W \to W'' \to 0$ where W' and W'' have smaller length than W, we get

$$\begin{aligned}\dim_k \operatorname{Ext}^i_A(W, A) &\leq \dim_k \operatorname{Ext}^i_A(W', A) + \dim_k \operatorname{Ext}^i_A(W'' A) \\ &\leq (\dim_k W' + \dim_k W'')\, \mu_i(P, A) = (\dim_k W)\, \mu_i(P, A).\end{aligned}$$

A similar induction shows that, whenever equality is attained in ($\star$), each composition factor of W is annihilated by an ideal $P' \in \Omega$ such that $\mu_i(P', A) = \mu_i(P, A)$.

Let V be the simple right A-module annihilated by P. If U is a right H-comodule of finite dimension d, then

$$\operatorname{Ext}^i_A(U \otimes V,\ A) \cong \operatorname{Ext}^i_A\big(V,\ [U, A]\big) \cong \operatorname{Ext}^i_A(V, A^d)$$

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by Lemmas 4.1 and 4.2. Hence

$$\frac{\dim_k \operatorname{Ext}^i_A(U \otimes V, A)}{\dim_k(U \otimes V)} = \frac{d \cdot \dim_k \operatorname{Ext}^i_A(V, A)}{d \cdot \dim_k V} = \mu_i(P, A).$$

By Lemma 4.3 each composition factor of $U \otimes V$ is annihilated by a maximal ideal of A lying in Ω. Hence the previous argument applies with $W = U \otimes V$, and equality is attained in ($\star$) for this choice of W. It follows that $\mu_i(P', A) = \mu_i(P, A)$ for any $P' \in \Omega$ which annihilates some composition factor of $U \otimes V$.

Finally, given any $P' \in \Omega$, by Lemma 4.3 there exists a finite dimensional right H-comodule U such that the simple right A-module annihilated by P' occurs as a composition factor of $U \otimes V$. This entails the required equality. $\square$

Recall that A is assumed to be an H-module algebra module-finite over its centre Z. The algebra A is called *H-semiprime* if A contains no non-zero nilpotent H-stable ideals, and A is called *H-prime* if $IJ \neq 0$ for any two non-zero H-stable ideals I and J of A. Denote by $Q(Z)$ the total ring of fractions of Z. Assuming that A is Noetherian and H-semiprime, it was proved in [**27**, Theorem 1.3] that A has a quasi-Frobenius classical quotient ring $Q(A)$ isomorphic with $A \otimes_Z Q(Z)$. This implies, in particular, that all non-zerodivisors of Z remain non-zerodivisors in A and that $Q(Z)$ is itself an Artinian ring, so that Z cannot have embedded primes. If A is Noetherian and H-prime, then Z is equidimensional by [**27**, Proposition 5.15].

THEOREM 4.5. *Let H be a Hopf algebra with a bijective antipode, A an H-simple H-module algebra finitely generated as an algebra over the ground field and module-finite over its centre Z. Suppose that the H-orbits of maximal ideals of A are dense in* Max A. *Then A is AS-Gorenstein.*

Proof. By the Artin–Tate Lemma [**19**, 13.9.10], Z is a finitely generated algebra over the ground field k. In particular, Z is Noetherian, and so too is A. By the Hilbert Nullstellensatz all maximal ideals of A have finite codimension. Since A has no non-trivial H-stable ideals, A is H-prime. By the previously mentioned results from [**27**] this implies that Z is equidimensional, Z has no embedded primes, and $A \otimes_Z Q(Z)$ is a quasi-Frobenius ring. By Corollary 3.5, there exists a non-zerodivisor s of Z such that the ring $A[s^{-1}]$ is AS-Gorenstein. Consider the open subset

$$D_s = \{P \in \operatorname{Max} A \mid s \notin P\}$$

of the maximal spectrum. Since Z is a Jacobson ring, it has a maximal ideal $\mathfrak{m}$ such that $s \notin \mathfrak{m}$, and any maximal ideal of A lying over $\mathfrak{m}$ is contained in D_s. This shows that D_s is non-empty.

Put $d = \operatorname{injdim} A[s^{-1}]$. If V is a simple right A-module with annihilator $P \in D_s$, then V and $\operatorname{Ext}^i_A(V, A)$ are $A[s^{-1}]$-modules since s acts invertibly on V. Therefore

$$\operatorname{Ext}^i_A(V, A) \cong \operatorname{Ext}^i_A(V, A) \otimes_Z Z[s^{-1}] \cong \operatorname{Ext}^i_{A[s^{-1}]}(V, A[s^{-1}]) = 0$$

for all $i \neq d$. This means that $\mu_i(P, A) = 0$ for all $i \neq d$.

If Ω is any H-orbit in Max A, then $\Omega \cap D_s \neq \varnothing$ since Ω is dense in Max A by the hypothesis. Thus, given any $P \in \operatorname{Max} A$, there exists $P' \in D_s$ lying in the H-orbit of P, so that $P \sim_H P'$. Proposition 4.4 entails $\mu_i(P, A) = \mu_i(P', A) = 0$ for all $i \neq d$. In other words, each maximal ideal of A has upper grade equal to d. Therefore A is injectively homogeneous with height $\mathfrak{m} = \operatorname{injdim} A_{\mathfrak{m}} = d$ for each $\mathfrak{m} \in \operatorname{Max} Z$. By Lemma 3.4 A is AS-Gorenstein. □

REMARK 3. In Theorem 4.5 it suffices to assume that the antipode S of H is only surjective rather than bijective, and even this condition can be dropped altogether if the orbits with respect to the Hopf subalgebra $S^2(H)$ of H are used instead of the H-orbits. The point is that without any conditions on S the conclusion of Lemma 4.2 still holds for the A-module $[U^*, M]$ in place of $[U, M]$, and in Lemma 4.3 the maximal ideals P, P' lie in the same $S^2(H)$-orbit if and only if V' is isomorphic to a composition factor of $U^* \otimes V$ for some finite dimensional right H-comodule U. Proposition 4.4 will still be valid if Ω is taken to be an $S^2(H)$-orbit.

REMARK 4. The crucial step in the approach of Wu and Zhang was to prove that a Noetherian PI algebra A is AS-Gorenstein whenever all functors $\operatorname{Ext}^i_A(?, A)$ are exact on the category of left A-modules of finite length and there exists a subset $\Phi \subset \mathbb{Z}$ such that for each simple left A-module V one has $\operatorname{Ext}^i_A(V, A) \neq 0$ if and only if $i \in \Phi$ [**35**, Proposition 3.2]. If A is finitely generated, then all A-modules of finite length have finite dimension, and if A is also a Hopf algebra, then the required condition follows at once from the facts that for each finite dimensional left A-module V the left A-module $\operatorname{Hom}_k(V, A) \cong A \otimes V^*$ is free and

$$\operatorname{Ext}^i_A(V, A) \cong \operatorname{Ext}^i_A(k \otimes V, A) \cong \operatorname{Ext}^i_A\big(k, \operatorname{Hom}_k(V, A)\big).$$

If A is an H-module algebra, Lemma 4.1 alone does not yield such a conclusion. This explains the necessity of some other methods.

5. The minimal injective resolution. Let A be a ring module-finite over a central subring R. Assume that R and A are Noetherian. Then any injective A-module is a direct sum of indecomposable submodules [**18**, Theorem 6.6.4]. Furthermore, each indecomposable injective I has a single associated prime ideal of A called the *assassinator* of I, and for each prime ideal P there exists a unique, up to isomorphism, indecomposable injective right A-module whose assassinator is P [**13**, p. 423]. This result was extended later to fully bounded Noetherian rings (see [**14**, Theorem 9.15]).

The multiplicities with which indecomposable injectives occur in minimal injective resolutions can be expressed readily in terms of the Bass numbers:

LEMMA 5.1. *For each $P \in \operatorname{Spec} A$ denote by I_P the indecomposable injective right A-module with assassinator P and denote by* udim A/P *the uniform dimension (Goldie rank) of the prime factor ring A/P. Let $0 \to M \to E_0 \to E_1 \to \dots$ be the minimal injective resolution of a right A-module M. Writing*

$$E_i = \bigoplus_{P \in \operatorname{Spec} A} I_P^{a_i(P,M)},$$

we have $a_i(P, M) = \mu_i(P, M)\,(\operatorname{udim} A/P)$ for each i.

Proof. Suppose first that $P \in \operatorname{Max} A$. Let V be the simple right A-module annihilated by P, and let $D = \operatorname{End}_A V$. Then I_P is the injective hull of V. Hence V is isomorphic with the socle of I_P and V does not embed in any $I_{P'}$ for $P' \neq P$. This means that $\operatorname{Hom}_A(V, I_P) \cong D$ and $\operatorname{Hom}_A(V, I_{P'}) = 0$ for all primes $P' \neq P$. Since E_i is the injective hull of the image of E_{i-1}, the socle of E_i is contained in that image, and therefore the map $E_i \to E_{i+1}$ vanishes on the socle of E_i. It follows that the complex $\mathrm{Hom}_A(V, E_\bullet)$ has zero differentials. Hence

$$\mathrm{Ext}^i_A(V, M) \cong \mathrm{Hom}_A(V, E_i)$$

as vector spaces over D. Note that $\dim_D \mathrm{Hom}_A(V, E_i) = a_i(P, M)$, while

$$\dim_D \mathrm{Ext}^i_A(V, M) = \mu_i(P, M)n \quad \text{where } n = \dim_D V.$$

By the Wedderburn Theorem, the simple artinian ring A/P is isomorphic to the full matrix ring $\mathrm{Mat}_n(D)$, so that $\mathrm{udim}\, A/P = n$ as well, for udim A/P is equal to the length of A/P in this case. Comparison of dimensions yields the required equality.

The general case is proved by localization. Let $\mathfrak{p} \in \mathrm{Spec}\, R$. By [**8**, Lemma 5.4] $E_\bullet \otimes_A A_\mathfrak{p}$ is the minimal injective resolution of the right $A_\mathfrak{p}$-module $M \otimes_A A_\mathfrak{p}$. Now $I_P \otimes_A A_\mathfrak{p}$ is an indecomposable injective right $A_\mathfrak{p}$-module with assassinator $PA_\mathfrak{p}$ when $P \cap R \subset \mathfrak{p}$ and $I_P \otimes_A A_\mathfrak{p} = 0$ otherwise. If $P \cap R = \mathfrak{p}$, then $PA_\mathfrak{p} \in \mathrm{Max}\, A_\mathfrak{p}$. Since $a_i(P, M) = a_i(PA_\mathfrak{p}, M \otimes_A A_\mathfrak{p})$ and the ring A/P has the same uniform dimension as its classical quotient ring $Q(A/P) \cong A_\mathfrak{p}/PA_\mathfrak{p}$, the formula for $a_i(P, M)$ follows from the previously considered case of maximal ideals. □

LEMMA 5.2. *Suppose that R is a Gorenstein ring and $A_\mathfrak{m}$ is a Frobenius algebra over $R_\mathfrak{m}$ for each $\mathfrak{m} \in \mathrm{Max}\, R$. Then A is injectively homogeneous over R. Moreover,*

$$\mu_i(P, A) = \begin{cases} 1 & \text{if } i = \text{height } P, \\ 0 & \text{otherwise} \end{cases}$$

for all $P \in \mathrm{Spec}\, A$. Denote by $E(A/P)$ the injective hull of A/P. The terms of the minimal injective resolution of A as a right A-module are given by

$$E_i = \bigoplus_{\{P \in \mathrm{Spec}\, A \,|\, \text{height } P = i\}} E(A/P).$$

Proof. By Lemma 3.1 $A_\mathfrak{m}$ is injectively homogeneous over $R_\mathfrak{m}$ for each $\mathfrak{m} \in \mathrm{Max}\, R$. This implies that A is injectively homogeneous over R. The Bass numbers can be determined by passing to the localizations $A_\mathfrak{p}$ for $\mathfrak{p} \in \mathrm{Spec}\, R$. If $\mathfrak{m}$ is any maximal ideal of R such that $\mathfrak{p} \subset \mathfrak{m}$, then $A_\mathfrak{p} \cong A_\mathfrak{m} \otimes_{R_\mathfrak{m}} R_\mathfrak{p}$. It follows that $A_\mathfrak{p}$ is a Frobenius algebra over $R_\mathfrak{p}$ for each $\mathfrak{p} \in \mathrm{Spec}\, R$.

So we may assume R to be local with a maximal ideal $\mathfrak{m}$. As in the proof of Lemma 3.1, there exists an A-regular sequence $x_1, \ldots, x_h \in \mathfrak{m}$ of length h equal to the height of $\mathfrak{m}$. Put

$$R' = R/I \quad \text{and} \quad A' = A/IA \quad \text{where } I = Rx_1 + \cdots + Rx_h.$$

Then R' is a local Artinian ring with a nilpotent maximal ideal $\mathfrak{m}' = \mathfrak{m}/I$ and with a simple socle S. This means that R' is a Frobenius ring. Next, $A' \cong A \otimes_R R'$ is a Frobenius algebra over R'. Since A' is a free R'-module and S coincides with the annihilator of $\mathfrak{m}'$ in R, the annihilator of $\mathfrak{m}'$ in A' is equal to SA'. Note that $SA' \cong A' \otimes_{R'} S \cong A'/\mathfrak{m}'A'$ as $S \cong R'/\mathfrak{m}'$. Since the right socle of A' is annihilated by $\mathfrak{m}'$, it coincides with the socle of SA' as a right A'-module, and therefore is isomorphic with the right socle of $A'/\mathfrak{m}'A'$. But $A'/\mathfrak{m}'A'$ is a Frobenius algebra over the field $R'/\mathfrak{m}'$.

Hence

$$\operatorname{soc} A' \cong \operatorname{soc} A'/\mathfrak{m}'A' \cong A'/J'$$

where J' is the Jacobson radical of A'. In other words, A' is itself a Frobenius ring. (This is a special case of a general result [**24**, Satz 6] concerning transitivity of Frobenius ring extensions.) If V is any right A'-module, then

$$\operatorname{Ext}^i_A(V, A) \cong \operatorname{Ext}^{i-h}_{A'}(V, A')$$

by the Rees Theorem. Since A' is self-injective, it follows that $\operatorname{Ext}^i_A(V, A) = 0$ for all $i \neq h$. Taking V to be the simple right A-module with annihilator $P \in \operatorname{Max} A$, we get $\mu_i(P, A) = 0$ for $i \neq h$. On the other hand,

$$\operatorname{Ext}^h_A(V, A) \cong \operatorname{Hom}_A(V, A') \cong \operatorname{Hom}_A(V, \operatorname{soc} A') \cong \operatorname{Hom}_A(V, A'/J').$$

If $n = \dim_D V$ where $D = \operatorname{End}_A V$, then V occurs in A'/J' as a direct summand with multiplicity n. Hence $\dim_D \operatorname{Ext}^h_A(V, A) = n$, and so $\mu_h(P, A) = 1$. Furthermore, $h = \operatorname{height} P$ since $P \cap R = \mathfrak{m}$ and A is R-Macaulay.

The formula for $\mu_i(P, A)$ is thus proved. The final conclusion follows from Lemma 5.1 applied to $M = A$. Indeed, $E(A/P) \cong I_P^n$ where $n = \operatorname{udim} A/P$ since the ring A/P is an essential extension of a direct sum of n uniform right ideals, while I_P is isomorphic with the A-module injective hull of any uniform right ideal of A/P. □

LEMMA 5.3. *Suppose that A is finitely generated projective as an R-module. Let $\mathfrak{m}$ be a maximal ideal of R. If $A/\mathfrak{m}A$ is a Frobenius algebra over $R/\mathfrak{m}$, then $A_\mathfrak{m}$ is a Frobenius algebra over $R_\mathfrak{m}$.*

Proof. We may assume the ring R to be local. Since $A/\mathfrak{m}A$ is a Frobenius algebra over the field $R/\mathfrak{m}$, there exists an $R/\mathfrak{m}$-linear map $f_\mathfrak{m} : A/\mathfrak{m}A \to R/\mathfrak{m}$ whose kernel contains no non-zero one-sided ideals of $A/\mathfrak{m}A$. This map can be lifted to an R-linear map $f : A \to R$. Define a right A-linear map $\varphi : A \to A^*_R$ by the formula $\varphi(a) = f \cdot a$ for $a \in A$. By the choice of $f_\mathfrak{m}$ the reduction of φ modulo $\mathfrak{m}$ is an isomorphism of vector spaces $A/\mathfrak{m}A \cong (A/\mathfrak{m}A)^*$ over $R/\mathfrak{m}$. Since both A and A^*_R are finitely generated free R-modules, φ has to be bijective. Thus $A \cong A^*_R$ as right A-modules, which means that A is a Frobenius algebra over R. □

LEMMA 5.4. *Suppose that R is a regular ring and A is injectively homogeneous over R. Suppose also that*

$$\mu_{\operatorname{height} P}(P, A) = 1 \quad \text{for all } P \in \operatorname{Max} A.$$

Then $A_\mathfrak{m}$ is a Frobenius algebra over $R_\mathfrak{m}$ for each $\mathfrak{m} \in \operatorname{Max} R$.

Proof. We may assume again that R is local with a maximal ideal $\mathfrak{m}$. Then $\mathfrak{m}$ is generated by a regular sequence $x_1, \ldots, x_h$ where $h = \operatorname{height} \mathfrak{m} = \operatorname{injdim} A$. Since A is module-finite and centrally Macaulay over R, it is a Cohen–Macaulay R-module. But a finitely generated module over a regular local commutative ring is free if and only if it is Cohen–Macaulay of maximal Krull dimension [**4**, p. 53, Corollary 2]. It follows that A is a free R-module. Then $x_1, \ldots, x_h$ is an A-regular sequence, and so for each right A'-module V where $A' = A/\mathfrak{m}A$ we have

$$\operatorname{Ext}^i_A(V, A) \cong \operatorname{Ext}^{i-h}_{A'}(V, A')$$

by the Rees Theorem. In particular, $\mathrm{Ext}^1_{A'}(V, A') = 0$ for each V, which implies that the Artinian ring A' is right self-injective. Take V to be a simple module with annihilator $P \in \mathrm{Max}\, A$, and put $n = \dim_D V$ where $D = \mathrm{End}_A V$. Since $\mu_h(P, A) = 1$, we have $\dim_D \mathrm{Ext}^h_A(V, A) = n$. The isomorphism $\mathrm{Ext}^h_A(V, A) \cong \mathrm{Hom}_{A'}(V, A')$ now shows that V occurs in the socle of A' with multiplicity n. Since this holds for each simple V, we get $\mathrm{soc}\, A' \cong A'/J'$ as right A'-modules where J' stands for the Jacobson radical of A'. Thus A' is a Frobenius ring. Since A' is a finite dimensional algebra over the field $R/\mathfrak{m}$, it is a Frobenius algebra [**18**, Theorem 13.5.7]. By Lemma 5.3 A is a Frobenius algebra over R. □

LEMMA 5.5. *Let* $\mathfrak{m} \in \mathrm{Max}\, R$ *and* $h = \mathrm{height}\, \mathfrak{m}$. *If A is injectively homogeneous over R, then* $\prod_{P \in \mathrm{Max}_{\mathfrak{m}} A} \mu_h(P, A) = 1$ *where* $\mathrm{Max}_{\mathfrak{m}} A = \{P \in \mathrm{Max}\, A \mid P \cap R = \mathfrak{m}\}$.

Proof. Passing to $R_{\mathfrak{m}}$ and $A_{\mathfrak{m}}$, we may assume R to be local. Then $\mathrm{injdim}\, A = h$ and A is AS-Gorenstein by Lemma 3.4. Hence the assignment

$$V \mapsto \mathrm{Ext}^h_A(V, A)$$

gives a bijection between the isomorphism classes of simple right A-modules and simple left ones. Put $K = R/\mathfrak{m}$. The set $\mathrm{Max}_{\mathfrak{m}} A = \mathrm{Max}\, A$ is finite since its elements are in a bijective correspondence with the maximal ideals of the finite dimensional algebra $A/\mathfrak{m}A$ over the field K. Let $\mathrm{Max}\, A = \{P_1, \dots, P_n\}$, and for each i denote by V_i the simple right A-module annihilated by P_i. The simple left A-module annihilated by P_i is then $V_i^* = \mathrm{Hom}_K(V_i, K)$. Hence, there exists a permutation π of indices $1, \dots, n$ such that $\mathrm{Ext}^h_A(V_i, A) \cong V^*_{\pi(i)}$ for each i. Since $\dim_K V_i^* = \dim_K V_i$, we get

$$\prod_{i=1}^n \mu_h(P_i, A) = \prod_{i=1}^n \frac{\dim_K \mathrm{Ext}^h_A(V_i, A)}{\dim_K V_i} = \prod_{i=1}^n \frac{\dim_K V_i^*}{\dim_K V_i} = 1.$$

□

THEOREM 5.6. *Let H be a Hopf algebra with a bijective antipode, A an H-simple H-module algebra finitely generated as an algebra over the ground field and module-finite over its centre Z. Let* $0 \to A \to E_0 \to E_1 \to \dots$ *be the minimal injective resolution of A as a right A-module. If the set* $\mathrm{Max}\, A$ *is a single H-orbit, then*

$$E_i = \bigoplus_{\{P \in \mathrm{Spec}\, A \,\mid\, \mathrm{height}\, P = i\}} E(A/P)$$

for each i where $E(A/P)$ stands for the injective hull of A/P.

Proof. By Theorem 4.5 A is AS-Gorenstein. Let $d = \mathrm{injdim}\, A$. Then all maximal ideals of A have upper grade equal to d. In particular, A is injectively homogeneous over a central subring R whenever A is module-finite over R. By Noether's Normalization Lemma, we may take R to be a polynomial algebra over the ground field. Then R is a regular commutative ring. By Proposition 4.4 the function $P \mapsto \mu_d(P, A)$ is constant on each H-orbit in $\mathrm{Max}\, A$. Since $\mathrm{Max}\, A$ is a single H-orbit, this function is constant on the whole set $\mathrm{Max}\, A$. Let c be its value, so that $\mu_d(P, A) = c$ for all $P \in \mathrm{Max}\, A$. By Lemma 5.5 $c^n = 1$ where n is the cardinality of $\mathrm{Max}_{\mathfrak{m}} A$ for some $\mathfrak{m} \in \mathrm{Max}\, R$. Since c is a positive rational number, we get $c = 1$. Thus $\mu_d(P, A) = 1$ for all $P \in \mathrm{Max}\, A$. Note that $\mathrm{height}\, \mathfrak{m} = \mathrm{injdim}\, A_{\mathfrak{m}} = d$ for all $\mathfrak{m} \in \mathrm{Max}\, R$, and also $\mathrm{height}\, P = \mathrm{height}(P \cap R) = d$

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for all $P \in \operatorname{Max} A$ since A is R-Macaulay. An application of Lemmas 5.4 and 5.2 completes the proof. $\square$

In the last result of this paper, the dense open H-orbit Ω does not necessarily coincide with the whole set $\operatorname{Max} A$, and then the algebra A may not be H-simple. As mentioned in the introduction, such a situation occurs when dealing with coideal subalgebras of Hopf algebras.

THEOREM 5.7. *Let H be a Hopf algebra with a bijective antipode, A an H-prime H-module algebra finitely generated as an algebra over the ground field and module-finite over its centre Z. If* $\operatorname{Max} A$ *contains a dense open H-orbit Ω, then the classical quotient ring $Q(A)$ is a Frobenius ring.*

Proof. As recalled in Section 4, the conditions imposed on A imply that Z is an equidimensional finitely generated algebra over the ground field, Z has no embedded primes, and A has a quasi-Frobenius classical quotient ring $Q(A) \cong A \otimes_Z Q(Z)$. Let R be a subalgebra of Z such that Z is module-finite over R and R is isomorphic to a polynomial algebra, say in n indeterminates, over the ground field. As in the proof of Corollary 3.5 we conclude that $Q(A) \cong A \otimes_R Q(R)$ where $Q(R)$ stands for the quotient field of R and there exists $0 \neq s \in R$ such that $A[s^{-1}]$ is AS-Gorenstein of injective dimension n. Then $\mu_i(P, A) = 0$ for all $i \neq n$ and all $P \in D_s$ where

$$D_s = \{P \in \operatorname{Max} A \mid s \notin P\}.$$

Since D_s is a non-empty open subset of $\operatorname{Max} A$ and Ω is dense in $\operatorname{Max} A$, we get $\Omega \cap D_s \neq \varnothing$. Proposition 4.4 now entails $\mu_i(P, A) = 0$ for all $i \neq n$ and all $P \in \Omega$.

Denote by I the intersection of all ideals $P \in \operatorname{Max} A \setminus \Omega$. Since Ω is open in $\operatorname{Max} A$, we have

$$\Omega = \{P \in \operatorname{Max} A \mid I \not\subset P\}.$$

By [**10**, Theorem 4.3] A is a Jacobson ring, i.e., each prime ideal of A is an intersection of maximal ideals. This implies that $\operatorname{Max} A$ is a dense subset of $\operatorname{Spec} A$. Hence Ω is dense in $\operatorname{Spec} A$ too, but then Ω intersects each irreducible component of the Noetherian space $\operatorname{Spec} A$. It follows that I is contained in none of the minimal prime ideals of A. Then $IQ(A) \cong I \otimes_R Q(R)$ is a two-sided ideal of $Q(A)$ contained in none of the maximal ideals. Hence $1 \in IQ(A)$, which shows that $I \cap R \neq 0$.

Now pick any maximal ideal $\mathfrak{m}$ of R such that $I \cap R \not\subset \mathfrak{m}$. We have $\operatorname{Max}_{\mathfrak{m}} A \subset \Omega$ since $I \not\subset P$ for each $P \in \operatorname{Max}_{\mathfrak{m}} A$. It follows that $\mu_i(P, A) = 0$ for all $i \neq n$ and all $P \in \operatorname{Max}_{\mathfrak{m}} A$. Hence $A_{\mathfrak{m}}$ is injectively homogeneous over $R_{\mathfrak{m}}$ with $\operatorname{injdim} A_{\mathfrak{m}} = n$. By Proposition 4.4 the function $P \mapsto \mu_n(P, A)$ has a constant value, say c, on the H-orbit Ω and therefore on its subset $\operatorname{Max}_{\mathfrak{m}} A$. Applying Lemma 5.5, we deduce that $c = 1$. Thus $\mu_n(P, A) = 1$ for all $P \in \operatorname{Max}_{\mathfrak{m}} A$. Since $R_{\mathfrak{m}}$ is a regular ring, Lemma 5.4 applied to the ring $A_{\mathfrak{m}}$ shows that $A_{\mathfrak{m}}$ is a Frobenius algebra over $R_{\mathfrak{m}}$. But the quotient field of the domain $R_{\mathfrak{m}}$ may be identified with $Q(R)$. Hence

$$Q(A) \cong A_{\mathfrak{m}} \otimes_{R_{\mathfrak{m}}} Q(R)$$

is a Frobenius algebra over the field $Q(R)$. Then $Q(A)$ is Frobenius as a ring. $\square$

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