Ergod. Th. & Dynam. Sys. (1982), **2**, 439–463
Printed in Great Britain

Examples of conservative diffeomorphisms of the two-dimensional torus with coexistence of elliptic and stochastic behaviour

FELIKS PRZYTYCKI

Institute of Mathematics, Polish Academy of Sciences, ul. Sniadeckich 8, 00-950 *Warsaw, Poland*

(*Received* 7 *July* 1982)

Dedicated to the memory of V. M. Alexeyev

Abstract. We find very simple examples of C^∞-arcs of diffeomorphisms of the two-dimensional torus, preserving the Lebesgue measure and having the following properties: (1) the beginning of an arc is inside the set of Anosov diffeomorphisms; (2) after the bifurcation parameter every diffeomorphism has an elliptic fixed point with the first Birkhoff invariant non-zero (the KAM situation) and an invariant open area with almost everywhere non-zero Lyapunov characteristic exponents, moreover where the diffeomorphism has Bernoulli property; (3) the arc is real–analytic except on two circles (for each value of parameter) which are inside the Bernoulli property area.

Topologically after the bifurcation parameter we have hyperbolic toral automorphisms with 0 'blown up'.

1. *Introduction*

In this paper we find a simple one-parameter family of diffeomorphisms of the two-dimensional torus T^2, $H_t: T^2 \to T^2$ for $t \in [-\varepsilon, \varepsilon]$, preserving the Lebesgue measure and satisfying the properties (1)–(5) listed below.

(1) For every $t > 0$, H_t is inside the set of Anosov diffeomorphisms An (T^2). For every $t \geq 0$, H_t is topologically conjugate with the hyperbolic toral automorphism A given by the matrix $\left(\begin{smallmatrix}1 & 1\\ 1 & 2\end{smallmatrix}\right)$.

(2) The family H_t at $t = 0$ is transversal to the set Fr An (T^2) – the boundary of An (T^2). We mean by this that there exists a constant $C > 0$ such that

$$\mathrm{dist}_{C^1}(H_t, \mathrm{Fr\,An}\,(T^2)) \geq C \cdot |t|.$$

(3) For every $t < 0$ there exists an elliptic island around $0 \in T^2$. This means that the differential $DH_t(0)$ is elliptic, the eigenvalues of $DH_t(0)$ are not roots of unity of low degree and in the Birkhoff normal form the frequency of oscillations depends on the amplitude. More exactly, the first Birkhoff invariant is non-zero. Then by Kolmogorov–Arnold–Moser theory most of the neighbourhood of 0 is filled with H_t-invariant closed curves.

(4) For every $t<0$ there exists an open, non-empty H_t-invariant set $S_t \subset T^2$ on which H_t behaves stochastically. More exactly the Lyapunov characteristic exponents for $H_t|_{S_t}$ are almost everywhere non-zero and H_t restricted to S_t has the Bernoulli property.

(5) $H:[-\varepsilon, \varepsilon]\times T^2 \to T^2$ is a C^∞-function and is real–analytic except on the two families of circles $[-\varepsilon, \varepsilon]\times\{a, b\}\times S^1$.

We look for H_t in the form of a toral-linked twist mapping, see [1] and [9], i.e.

$$H_t = G_{g_t} \circ F_{f_t},$$

where

$$F_{f_t}(x, y) = (x + f_t(y), y), \quad G_{g_t}(x, y) = (x, y + g_t(x)),$$
$$f_t, g_t : \mathbb{R} \to \mathbb{R}, \quad f_t(y+1) = f_t(y) + k,$$
$$g_t(x+1) = g_t(x) + l,$$

for every $x, y \in \mathbb{R}$ and some integers k and l.

We take $f_t = \mathrm{id}$ so that

$$H_t(x, y) = (x + y, y + g_t(x + y)).$$

We take g_t satisfying the following properties:

$$\begin{cases} \text{For every } t \in [-\varepsilon, \varepsilon]\ g_t \text{ is an odd function,} \\ g_t(0) = 0, \quad g_t(1) = 1, \quad dg_0/dx(0) = d^2g_0/dx^2(0) = 0, \quad d^3g_0/dx^3(0) > 0, \\ d^2g_t/dx\,dt(x)|_{x=0,t=0} > 0 \text{ and } dg_0/dx(x) > 0 \text{ for every } x \notin Z. \end{cases} \tag{1}$$

If $t>0$, the point $0 \in T^2$ is a saddle for H_t. When t passes 0 in the negative direction two saddles p_t and q_t appear on the opposite sides of 0 on the x-axis while 0 itself becomes elliptic.

Checking the properties (1)–(3) is straightforward so we do not dwell on them. Let us only mention that H_0 corresponds to diffeomorphisms studied in [1] and [6] and that for $t<0$, $|t|$ sufficiently small, the first Birkhoff invariant at $0 \in T^2$ is non-zero since $d^3g_t/dx^3(0) \neq 0$.

Thus, the main aim of the paper is to prove property (4) for a special family g_t.

In general the stable and unstable manifolds of p_t and q_t intersect transversally (see the phase portrait in figure 1) and in such a case we do not know how to estimate the Lyapunov exponents. Moreover, in view of a recent result by R. Mañé [7] there exists a C^1-generic subset

$$\mathscr{A}_L \subset \mathrm{Diff}^1_L(T^2)\backslash \mathrm{An}_L(T^2)$$

where Lyapunov exponents are zero almost everywhere. So our H_t must be disjoint with $\mathscr{A}_L$. (The subscript L means that we consider diffeomorphisms preserving the Lebesgue measure.) In connection with the Mañé result Katok has suggested studying the Lyapunov exponents for small perturbations of H_0.

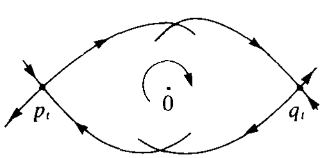

FIGURE 1

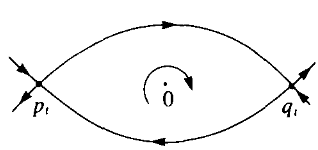

FIGURE 2

https://doi.org/10.1017/S0143385700001711 Published online by Cambridge University Press

In the case of our special H_t the saddles p_t and q_t are joined by separatrices, see figure 2. Throughout the paper we use only this property of H_t, together with properties (1). In the theorem in § 2, we consider a specific H_t only to be concrete.

Denote the domain between the separatrices by U_t. An idea which explains property (4) is that $T^2 \backslash \mathrm{cl}\, U_t$ is H_t-invariant so the behaviour along the trajectory of every point from $T^2 \backslash \mathrm{cl}\, U_t$ is hyperbolic as the trajectory keeps far away from the elliptic island around $0 \in T^2$.

In fact we 'blow up' the saddle of the Anosov diffeomorphism into the disk cl U_t. We use the Hamiltonian function $y^2 - x^2(x^2 + 2t)$. In a neighbourhood of cl U_t the saddle-like dynamics are preserved.

Section 2 is devoted to the construction of H_t. In §§ 3–5 we prove property (4) using the technique of invariant cones. In § 6 we prove that for each $t < 0$, $H_t|_{T^2 \backslash \mathrm{cl}\, U_t}$ is an almost Anosov diffeomorphism. Namely, it has continuous, uniquely integrable stable and unstable sub-bundles; it has almost everywhere non-zero Lyapunov exponents for every H_t invariant probability measure on $T^2 \backslash \mathrm{cl}\, U_t$ and it is topologically conjugate to the Anosov diffeomorphism $A|_{T^2 \backslash \{0\}}$. However proposition 3, § 6 proves that our 'blowing up' is in no sense C^1. Our study in § 6 corresponds to the Katok study for the H_0-type example [**6**] and to the Gerber and Katok study of smoothed pseudo-Anosov diffeomorphisms [**4**].

One reason why it is easy to construct our examples of coexistence is that we perturb the twist F_{id} with G_{g_t} where g_t is not periodic, i.e. the average twisting

$$\int_0^1 \frac{dg_t}{dx}(x)\, dx \neq 0.$$

The classical problem is to consider g_t to be periodic. Nevertheless the facts of local character i.e. the dynamics in the neighbourhood of cl U_t, like lemmas 2, 3, 5, concern the classical situation. See § 7 for further comments.

2. *Construction of the example*

THEOREM. *Let H_t be the one-parameter family of diffeomorphisms*

$$H_t = H_{\mathrm{id}, g_t} : T^2 \to T^2 \quad \textit{for } t \in [-\varepsilon, \varepsilon],$$

with g_t defined as follows:

$$(*) \qquad g_t(x) = 2(\sqrt{1 - t + 2x} - \sqrt{1 - t - 2x} - 2x), \quad \textit{for } |x| \le \tfrac{1}{4};$$

g_t is extended to $[-\frac{1}{4}, \frac{1}{4}] + Z$ by

$$g_t(x + n) = g_t(x) + n \quad \textit{for } x \in [-\tfrac{1}{4}, \tfrac{1}{4}],\ n \in Z$$

and extended to $]\frac{1}{4}, \frac{3}{4}[+ Z$ in anyway so that

$$\inf \{dg_t/dx\,(x) : x \in]\tfrac{1}{4}, \tfrac{3}{4}[\,\} = dg_t/dx\,(\tfrac{1}{4});$$

$$g_t - \mathrm{id} \textit{ is periodic with period } 1;$$

$$g : [-\varepsilon, \varepsilon] \times \mathbb{R} \to \mathbb{R} \textit{ is } C^\infty \textit{ and}$$

$$g|_{[-\varepsilon, \varepsilon] \times (\mathbb{R} \backslash (\frac{1}{2} \cdot Z + \frac{1}{4}))} \textit{ is real-analytic.}$$

Then the family H_t satisfies properties (1)–(5) *from the introduction.*

For example, for $x \in]\frac{1}{4}, \frac{3}{4}[$ set

$$g_t(x) = g_t(\tfrac{1}{4}) + (x - \tfrac{1}{4}) \cdot \frac{dg_t}{dx}(\tfrac{1}{4}) + \left(\int_{\frac{1}{4}}^{x} \varphi(s)\, ds\right)\left(1 - 2g_t(\tfrac{1}{4}) - \tfrac{1}{2}\frac{dg_t}{dx}(\tfrac{1}{4}) \Big/ \int_{\frac{1}{4}}^{\frac{3}{4}} \varphi(s)\, ds\right),$$

https://doi.org/10.1017/S0143385700001711 Published online by Cambridge University Press

where

$$\varphi(x)=\exp(\sin 2\pi(x+\tfrac{1}{4}))^{-1}.$$

Let us consider the following one-parameter family of Hamiltonian functions defined in the neighbourhood of $t=x=y=0$:

$$h_t(x,y)=y^2-x^2(x^2+2t).$$

For $t>0$ the Hamiltonian vector field V_t corresponding to h_t has a saddle at $0\in T^2$. For $t<0$ this saddle changes into an elliptic fixed point and V_t acquires two saddles

$$p_t=(-\sqrt{|t|},0),\quad q_t=(\sqrt{|t|},0)$$

joined by two separatrices, see figure 2 in § 1. We look for g_t such that

$$F_{\frac{1}{2}\mathrm{id}}\circ H_{\mathrm{id},g_t}\circ F_{\frac{1}{2}\mathrm{id}}^{-1}$$

has the same saddles and separatrices.

The union of stable and unstable manifolds for the saddles p_t and q_t in the neighbourhood of $0\in T^2$ coincides with the set of zeros of the function:

$$\begin{aligned}h_t(x,y)-h_t(p_t)&=y^2-x^2(x^2+2t)-t^2\\&=y^2-(x^2+t)^2\\&=-(x^2+t+y)(x^2+t-y).\end{aligned}$$

Consider the set of zeros of $W_t(x,y)=x^2+t+y$ and then the zeros of $W_t(x\pm\frac{1}{2}y,y)$ (broken lines in figure 3). Write these sets as graphs of the functions

$$y_t^{\pm}(x)=2(-1\mp x+\sqrt{1\pm 2x-t}).$$

Define $g_t=y_t^+-y_t^-$. We obtain the formula from the statement of the theorem.

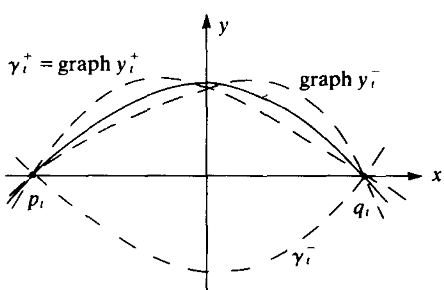

FIGURE 3

From the construction:

$$F_{\mathrm{id}}(\text{graph } y_t^+)=\text{graph } y_t^-\quad\text{and}\quad G_{g_t}(\text{graph } y_t^-)=\text{graph } y_t^+.$$

So our goal has been reached: H_{id,g_t} has the separatrices

$$\gamma_t^+=\text{graph } y_t^+\quad\text{and}\quad \gamma_t^-=-\gamma_t^+.$$

They are $F_{\frac{1}{2}\mathrm{id}}^{-1}$ images of the separatrices of V_t.

The vectors $(1\mp\sqrt{|t|},\pm 2\sqrt{|t|})$ are eigenvectors of p_t and q_t. The corresponding eigenvalues are

$$(1+\sqrt{|t|})/(1-\sqrt{|t|})\quad\text{and}\quad(1-\sqrt{|t|})/(1+\sqrt{|t|}).$$

These numbers will appear throughout the paper. Sometimes we shall use the notation $(v)_x$, $(v)_y$, $(z)_x$, $(z)_y$ to denote the x- or y-coordinate of a vector v or of a point z.

3. *Existence of invariant families of cones*

We shall describe here families of unstable and stable cones in the region $T^2\backslash\text{cl}\, U_t$, where U_t is the region between the separatrices $\gamma_t^{\pm}$.

Denote for every $a<b$, $|a-b|<1$, the strip $]a, b[\times S^1$ by $P(a, b)$.

Denote by $\mathcal{T}_t(\delta)$ the region ('triangle') bounded by the components of the stable and unstable manifolds of p_t in $\text{cl}\, P(-\sqrt{|t|}-\delta, -\sqrt{|t|})$ containing p_t and the line

$$\{x=-\sqrt{|t|}-\delta\} \quad \text{for any small } \delta>0$$

and denote $\mathcal{T}'_t(\delta)=-\mathcal{T}_t(\delta)$ (see figure 4).

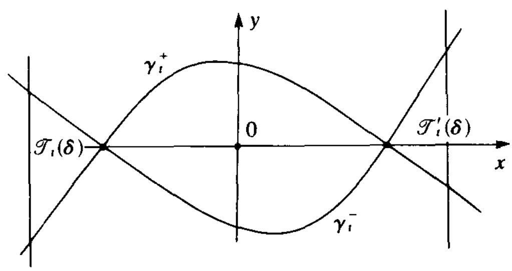

FIGURE 4

Let us start now with

LEMMA 1. *There exists a constant $C_1>0$ ($C_1 \ll 1$) such that for every $\delta: 0<\delta\le C_1$ and $t\in[-\varepsilon, 0[$, if z, $H_tz\in P(-\delta,\delta)\backslash(\text{cl}\, U_t\cup \text{Fr}\,\mathcal{T}_t(\delta)\cup\text{Fr}\,\mathcal{T}'_t(\delta))$ then there exists integers $N_1>0$, $N_2>1$ such that*

$$H_t^{-N_1}(z),\quad H_t^{N_2}(z)\in \text{cl}\, P(\delta, 1-\delta) \quad \textit{and} \quad H_t^n(z)\in P(-\delta,\delta)$$

for every $n: -N_1<n<N_2$, and one of the following possibilities occurs:

(1) *$z_n\in\mathcal{T}_t(\delta-\sqrt{|t|})$ for every $n: -N_1<n<N_2$ where by definition*

$$z_n=(x_n, y_n)=H_t^n(z);$$

(2) *$z_n\in\mathcal{T}'_t(\delta-\sqrt{|t|})$ for every $n: -N_1<n<N_2$;*

(3) *$0<y_n<2\delta+\sup\{g_t(x): x\in[-\delta,\delta]\}$ for $-N_1<n<N_2$ and the sequence (x_n), $n=-N_1,\dots,N_2$ is increasing;*

(4) *$0>y_n>-2\delta+\inf\{g_t(x): x\in[-\delta,\delta]\}$ for $-N_1<n<N_2$ and the sequence (x_n), $n=-N_1,\dots,N_2$ is decreasing.*

The proof is straightforward so it is omitted.

For $t<0$ denote by a_t the smallest positive number such that

$$dg_t/dx(a_t)=4\sqrt{|t|}/(1-\sqrt{|t|}). \tag{2}$$

Remark 1. There is no need to compute a_t exactly. Observe only that a_t exists and it is of order $\sqrt[4]{|t|}$ since

$$g_t(x)=2Q(t,x)(x^3+tx) \quad \text{where } Q(0,0)=1.$$

https://doi.org/10.1017/S0143385700001711 Published online by Cambridge University Press

This follows easily from the definition of $g_t(x)$.

For every x such that $|x|\le\sqrt{|t|}$ denote by $\mathscr{C}(x)$ the cone:

$$\mathscr{C}(x)=\{(\xi,\eta)\in\mathbb{R}^2: dy_t^+/dx(x)\le\eta/\xi\}.$$

For every $z\in T^2$ we shall identify the tangent space $T_z(T^2)$ with $\mathbb{R}^2$. Define now $\mathscr{D}(z)\subset T_z(T^2)$ for every $z=(x,y)\in T^2\backslash\operatorname{cl}U_t$ as follows:

(i) $\mathscr{D}(z)=\mathscr{C}(-\sqrt{|t|})$ if $z\in\operatorname{cl}P(a_t,1-a_t)$;

(ii) $\mathscr{D}(z)=\mathscr{C}(-\sqrt{|t|})$ if $z\in\mathscr{P}_t=P(-a_t,-\sqrt{|t|})\cup P(\sqrt{|t|},a_t)$;
and the backward trajectory $H_t^{-n}(z)$, $n=1,2,\dots$, either hits $\operatorname{cl}P(a_t,1-a_t)$ earlier than $\operatorname{cl}P(-\sqrt{|t|},\sqrt{|t|})$ or never hits $\operatorname{cl}P(-\sqrt{|t|},\sqrt{|t|})$;

(iii) $\mathscr{D}(z)=\mathscr{C}(\sqrt{|t|})$ if as in case (ii) $z\in\mathscr{P}_t$ but hits the set $\operatorname{cl}P(-\sqrt{|t|},\sqrt{|t|})$ earlier than $P(a_t,1-a_t)$;

(iv) $\mathscr{D}(z)=\mathscr{C}(\sqrt{|t|})$ if $z\in\operatorname{cl}P(-\sqrt{|t|},\sqrt{|t|})$, $H_t(z)\notin P(-\sqrt{|t|},\sqrt{|t|})$;

(v) $\mathscr{D}(z)=\mathscr{C}(x)$ if z, $H_t(z)\in P(-\sqrt{|t|},\sqrt{|t|})$ and $y>0$ ($y>0$ makes sense since, by lemma 1, $|y|$ is small);

(vi) $\mathscr{D}(z)=\mathscr{C}(-x)$ if in (v) we replace $y>0$ by $y<0$.

Now we are going to prove the invariance of this cone bundle. If z and $H_t(z)=(x_1,y_1)$ are as in cases (i) or (ii) then

$$\frac{dg_t}{dx}(x_1)\ge\frac{dg_t}{dx}(-\sqrt{|t|})$$

so

$$(DH_t)_z(\mathscr{D}(z))=(DH_t)_z(\mathscr{C}(-\sqrt{|t|}))\subset(DH_t)_{p_t}(\mathscr{C}(-\sqrt{|t|}))$$
$$=\mathscr{C}(-\sqrt{|t|})=\mathscr{D}(H_t(z)).$$

If $z=(x,y)$ as in (i) or (ii) and $H_t(z)=(x_1,y_1)$ as in (v) (or similarly (vi)), then we use the concavity of the function y_t^+. Let $z_1'=(x_1,y_1')$ be the point on the same vertical as (x_1,y_1), lying in the γ_t^+ (see figure 5). Let $H_t^{-1}(z_1')=\bar z=(\bar x,\bar y)$. Then

$$(DH_t)_z(\mathscr{D}(z))=(DH_t)_z(\mathscr{C}(-\sqrt{|t|}))\subset(DH_t)_z(\mathscr{C}(\bar x))$$
$$=(DH_t)_{\bar z}(\mathscr{C}(\bar x))=\mathscr{C}(x_1)=\mathscr{D}(H_t(z)).$$

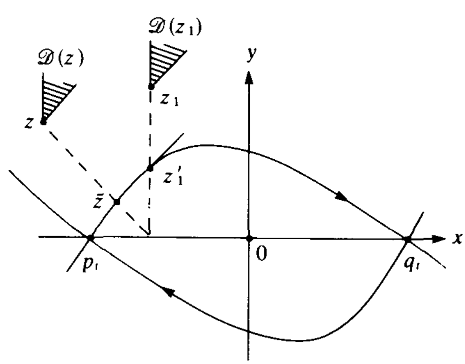

FIGURE 5

If z and $H_t(z)$ are both as in (v) (or (vi)) the argument is similar.

It is also similar for z given by (v) or (vi) and $H_t(z)$ given by (iv).

If z is as in (iv) or (iii) and $H_t(z)$ as in (iii), then analogously to the first-considered case:

$$(DH_t)_z(\mathscr{D}(z)) = (DH_t)_z(\mathscr{C}(\sqrt{|t|})) \subset (DH_t)_{q_t}(\mathscr{C}(\sqrt{|t|})) = \mathscr{C}(\sqrt{|t|}) = \mathscr{D}(H_t(z)).$$

Finally if z is as in (iii) or (iv) and $H_t(z) = (x_1, y_1)$ as in (i), then by (2) we have

$$dg_t/dx(x_1) \geq 4\sqrt{|t|}/(1-\sqrt{|t|}),$$

hence

$$(DH_t)_z(\mathscr{D}(z)) = (DH_t)_z(\mathscr{C}(\sqrt{|t|})) \subset \mathscr{C}(-\sqrt{|t|}) = D(H_t(z)).$$

Note that due to lemma 1 it cannot happen that z is as in case (iii) and in the same time $H_t(z)$ as in cases (iv), (v) or (vi).

So the invariance of this cone bundle has been proved. We have the cone bundle $\mathscr{D}$ over $T^2\backslash\text{cl } U_t$ and

$$DH_t(\mathscr{D}) \subset \mathscr{D}.$$

The analogous stable cone bundle $\mathscr{D}^s$ i.e. such that $DH_t^{-1}(\mathscr{D}^s) \subset \mathscr{D}^s$ can be defined by

$$\mathscr{D}^s = D(S_y \circ F_{\text{id}})(\mathscr{D})$$

where S_y is the symmetry with respect to the y-axis.

Remark 2. At this stage we can immediately deduce the existence of a set of positive Lebesgue measure with non-zero Lyapunov characteristic exponents as follows.

The set of line sub-bundles of cl ($\mathscr{D}$) over $T^2\backslash\text{cl } U_t$ is a partially ordered set, with angle order over every point. Take the bundle $L(\partial/\partial y)$ spanned by the vector field $\partial/\partial y$. For every $z \in T^2\backslash\text{cl } U_t$,

$$L(\partial/\partial y)(z) \in \text{cl } \mathscr{D}(z)$$

and the sequence $DH_t^n(L(\partial/\partial y))$ is monotonous with respect to the considered partial order. Hence the pointwise limit, a measurable line bundle, is a fixed point for DH_t (see [**2**, theorem 3.8.1] for the details). Denote this bundle by E_t. Now use the Birkhoff ergodic theorem for the function $\|DH_t|_{E_t}\|$.

Let $\lambda : T^2\backslash\text{cl } U_t \to \mathbb{R}$ be the Lyapunov characteristic exponent for the vectors from E_t. Then

$$\int_{T^2\backslash\text{cl } U_t} \lambda(z)\, dz = \int_{T^2\backslash\text{cl } U_t} \log \|DH_t|_{E_t}(z)\|\, dz$$

which is clearly positive for $|t|$ sufficiently small. So $\lambda(z)$ is positive on a set of positive Lebesgue measure. The second Lyapunov exponent, which is equal to $-\lambda(z)$, is negative on the same set. □

4. *Lyapunov characteristic exponents are non-zero almost everywhere on $T^2\backslash\text{cl } U_t$*

LEMMA 2. *Let*

$$z = (x, y) \in \text{cl } P(-\sqrt{|t|}, 0)\backslash\text{cl } U_t,$$
$$H_t(z) = (x_1, y_1) \in \text{cl } P(0, \sqrt{|t|}),$$
$$y, y_1 > 0 \quad \textit{and} \quad |x| < |x_1|.$$

https://doi.org/10.1017/S0143385700001711 Published online by Cambridge University Press

As in lemma 1 *let us put*

$$H_t^n(z) = z_n = (x_n, y_n) \quad \text{for } n \in Z.$$

Then for every $n \geq 1$

$$|x_{-n+1}| \leq |x_n| \leq |x_{-n}|. \tag{3}$$

Proof. Observe that the backward (i.e. forward under H_t^{-1}) H_t-trajectory of the point z_0 is the reflection in the y-axis of the forward trajectory under $F_{\mathrm{id}} \circ G_{g_t}$ of the point $(-x_0, y_0)$. So the latter trajectory is the sequence of points $(-x_{-n}, y_{-n})$.

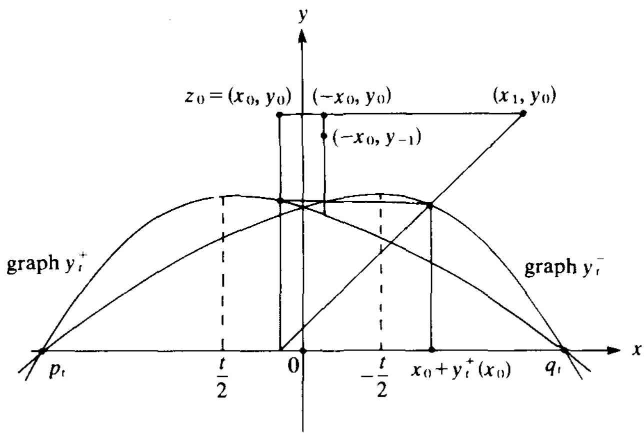

FIGURE 6

Assume that $x_0 \geq t/2$. At $t/2$ the function y_t^+ reaches its maximum (see figure 6). We have

$$\begin{aligned} y_0 - y_t^-(-x_0) &= y_0 - y_t^+(x_0) \\ &= y_0 - y_t^-(x_0 + y_t^+(x_0)) < y_0 - y_t^-(x_1), \end{aligned}$$

since $x_1 > x_0 + y_t^+(x_0)$ and the function y_t^- is decreasing to the right from $x_0 + y_t^+(x_0)$.

If $x_0 < t/2$ we have again

$$y_0 - y_t^-(-x_0) \leq y_0 - y_t^-(x_1) \tag{4}$$

since by our assumptions $-t/2 < -x_0 \leq x_1$.

In the case $-x_0 = x_1$ the lemma is trivially true so we can assume that $-x_0 < x_1$. Joint the points $(-x_0, y_0)$ and (x_1, y_0) by a curve $\alpha : [0, 1] \to \mathbb{R}^2$ which is the interval in the coordinates $(x, y - y_t^-(x))$. Due to (4), for every $s_0 \in [0, 1]$,

$$D\alpha((\partial/\partial s)(s_0)) \in (DF_{\mathrm{id}}(\mathscr{D}))(\alpha(s_0)),$$

so that for every $n \geq 0$

$$D(H_t^n \circ G_{g_t} \circ \alpha)((\partial/\partial s)(s_0)) \in \mathscr{D}(H_t^n \circ G_{g_t} \circ \alpha(s_0))$$

and the x-coordinate

$$(D(H_t^n \circ G_{g_t} \circ \alpha)((\partial/\partial s)(s_0)))_x > 0.$$

Hence $x_n > -x_{-n+1}$ for $n > 0$. This proves the left hand side inequality in (3).

To prove the right hand side inequality we observe that

$$y_0 - y_t^+(x_0) = y_0 - y_t^-(-x_0) = y_{-1} - y_t^+(-x_0).$$

We join the points (x_0, y_0) and $(-x_0, y_{-1})$ by the interval in the coordinates $(x, y - y_t^+(x))$ (unless $x_0 = 0$, which is the trivial case) and then proceed as before. □

LEMMA 3. *For every $\delta > 0$ such that $\sqrt{|t|} + \delta \le C_1$, where C_1 is the constant from lemma* 1, *if*

$$z \in \operatorname{cl} P(\sqrt{|t|} + \delta, 1 - \sqrt{|t|} - \delta)$$

and $n(z) > 0$ is the first integer such that

$$H_t^{n(z)}(z) \in \operatorname{cl} P(\sqrt{|t|} + \delta, 1 - \sqrt{|t|} - \delta),$$

if $v \in \mathscr{D}(z)$, then

$$(DH_t^{n(z)}(v))_x \ge (1 - 6\sqrt{|t|}) \cdot (DH_t(v))_x.$$

Proof. We can assume that $n(z) \ge 4$. Put

$$H_t^n(z) = z_n = (x_n, y_n),$$

assume for example that $y_n > 0$ $(n = 0, 1, \dots n(z))$, i.e. the sequence (x_n) is increasing (see lemma 1). It is possible because the case $y_n < 0$ is similar, and if $n(z) < 4$ or

$$y_n \in \mathscr{T}_t \cup \mathscr{T}_t'$$

the lemma is true for obvious reasons.

Put $DH_t^n(v) = v_n = (\xi_n, \eta_n)$ and $l_n = \xi_{n+1}/\xi_n$. Using the fact that $v_n \in \mathscr{D}$ and the description of $\mathscr{D}$ from § 3 we obtain the following estimates.

If $z_n \in P(-\sqrt{|t|} - \delta, -\sqrt{|t|})$, $\quad l_n > (1 + \sqrt{|t|})/(1 - \sqrt{|t|})$;
if $z_n \in \operatorname{cl} P(-\sqrt{|t|}, \sqrt{|t|})$, $\quad l_n \ge 1 + dy_t^+/dx\,(x_n)$;
if $z_n \in \operatorname{cl} P(\sqrt{|t|}, \sqrt{|t|} + \delta)$, $\quad l_n \ge (1 - \sqrt{|t|})/(1 + \sqrt{|t|})$.

Now look what happens to v_n under DH_t^{-1}. Equivalently, consider $DG_{g_t}^{-1}(v_{n+1})$ under DF_{id}^{-1}. For

$$z_{n+1} \in \operatorname{cl} P(-\sqrt{|t|}, \sqrt{|t|})$$

we obtain

$$l_n^{-1} \le 1 - dy_t^-/dx\,(x_{n+1}),$$

hence

$$l_n \ge (1 + dy_t^+/dx\,(-x_{n+1}))^{-1}.$$

Let $n_2 = n_2(z)$, $n_1 = n_1(z)$ and $n_3 = n_3(z)$ be respectively the smallest non-negative integers such that $x_n \ge -\sqrt{|t|}$, $x_n > 0$ and $x_n > \sqrt{|t|}$. Now make two additional assumptions:

$$n_3(z) - n_2(z) \ge 3; \tag{5}$$

$$|x_{n_1-1}| \le |x_{n_1}|. \tag{6}$$

Due to (5) and (6) the point z_{n_1-1} satisfies the assumptions of lemma 2 about z. Hence, for every k: $-1 \le k \le n_3 - 3 - n_1$

$$l_{n_1+k} \cdot l_{n_1-k-3} \ge \left(1 + \frac{dy_t^+}{dx}(-x_{n_1+k+1})\right)^{-1}\left(1 + \frac{dy_t^+}{dx}(x_{n_1-k-3})\right) \ge 1. \tag{7}$$

https://doi.org/10.1017/S0143385700001711 Published online by Cambridge University Press

We used here the fact that by lemma 2

$$|x_{n_1+k+1}| \le |x_{n_1-k-3}|$$

and that the function dy_t^+/dx is defined and decreasing between x_{n_1-k-3} and $-x_{n_1+k+1}$. It is defined because by the left hand inequality in (3)

$$|x_{n_1-k-3}| \le |x_{n_1+k+2}| \le |x_{n_3-1}| \quad \text{for } k \le n_3 - 3 - n_1.$$

We know, also by lemma 2, that $|n_2 - 1 - (n(z) - n_3)| \le 1$. So

$$\xi_{n(z)}/\xi_1 = \prod_{i=1}^{n(z)-1} l_i = \prod_{i=n_2(+1)}^{n_3-3} l_i \cdot (l_{n_2}) \cdot l_{n_3-2} \cdot l_{n_3-1} \cdot \prod_{i=1}^{n_2-1} l_i \cdot \prod_{i=n_3}^{n(z)-1} l_i$$
$$\ge \left(\frac{1-\sqrt{|t|}}{1+\sqrt{|t|}}\right)^3 > 1 - 6\sqrt{|t|}.$$

(We put the terms $+1$ and l_{n_2} into parentheses because they appear only in the case $n_3 - n_1 = n_1 - n_2 - 1$ and do not appear if $n_3 - n_1 = n_1 - n_2$.)

In the case when (6) is not satisfied i.e. if $|x_{n_1-1}| > |x_{n_1}|$ we consider the reflection in the y-axis of the $F_{\text{id}} \circ G_{g_t}$-trajectory $(-x_{-n}, y_{-n})$ or the H_t-trajectory $z_n = (-x_{-n}, y_{-n-1})$. We can use lemma 2 for (z_n), so we obtain for every $k \ge 0$

$$|x_{n_1+k}| \le |x_{n_1-k-1}| \le |x_{n_1+k+1}|.$$

This also gives $\xi_n(z)/\xi_1 > 1 - 6\sqrt{|t|}$. The only difference in computation is that the term l_{n_1-1} has no pair, see (7). But clearly $|x_{n_1-1}| > |t|/2$, hence $l_{n_1-1} \ge 1$.

We eliminate assumption (5) in the following way.

$$\eta_{n_2-1}/\xi_{n_2-1} \ge 2\sqrt{|t|}/(1-\sqrt{|t|}),$$

i.e. it is of the order of at least $\sqrt{|t|}$. Inf $dg_t/dx \ge 3t$ for $t < 0$ and $|t|$ sufficiently small. This follows easily from the representation

$$g_t = Q(t, x) \cdot 2(x^3 + tx), \quad \text{with } Q(0,0) = 1,$$

see remark 1 in § 3. Thus

$$\eta_{n+1}/\xi_{n+1} = (\eta_n/(\xi_n + \eta_n)) + dg_t/dx\,(x_{n+1}),$$

hence if $\eta_n/\xi_n \ge K\sqrt{|t|}$, then

$$\eta_{n+1}/\xi_{n+1} \ge \min\left(\tfrac{1}{2} \cdot K, \tfrac{1}{2}\right) \cdot \sqrt{|t|} - 3|t|.$$

If we fix any integer $N > 0$ and proceed by induction starting with $k = n_2 - 1$ we can prove that for every k:

$$n_2 - 1 \le k \le n_2 + N,$$

η_k/ξ_k is of the order of $\sqrt{|t|}$ for $|t|$ sufficiently small (depending on N), hence $\eta_k/\xi_k > 0$. In particular, we can take

$$N = n_3 - n_2 < 3.$$

Then for $n: n_2 - 1 \le n \le n_3 - 1$,

$$l_n = (\xi_n + \eta_n)/\xi_n \ge 1.$$

For all other n we have trivially $l_n \ge 1$. This proves the lemma. □

Now we can estimate the Lyapunov exponents. Take the constant C_1 from lemma 1. Let $\alpha(C_1)>0$ be a constant such that

$$\alpha(C_1)<\inf\{dg_0/dx(x): x\in[C_1, 1-C_1]\}.$$

Then the similar inequality

$$\alpha(C_1)<\inf\{dg_t/dx(x): x\in[C_1, 1-C_1]\}$$

holds for every t with $|t|$ sufficiently small.

Put $Q=\operatorname{cl} P(C_1, 1-C_1)$. If $|t|$ is sufficiently small we can replace the cones $\mathcal{D}$ over Q by smaller cones

$$\mathcal{D}_Q=\{(\xi,\eta): \eta/\xi\geq -2\sqrt{|t|}/(1-\sqrt{|t|})+\alpha(C_1)\}$$

and leave the old cones over the complement of Q. Then clearly the new system of cones $\mathcal{D}'$ is also DH_t-invariant.

For $v\in\mathcal{D}'(z)$, $z\in Q$ and $n(z)>0$ the first time when $H_t^{n(z)}(z)\in Q$, we have by lemma 3

$$\begin{aligned}(DH_t^{n(z)}(v))_x/(v)_x &= ((DH_t^{n(z)}(v))_x/(DH_t(v))_x)\cdot((DH_t(v))_x/(v)_x)\\ &\geq(1-6\sqrt{t})\cdot(1-2\sqrt{t}/(1-\sqrt{|t|})+\alpha(C_1))=\lambda_t>1,\end{aligned}$$

for $|t|$ sufficiently small.

This proves that for the first return mapping $(H_t)_Q$, for almost every $z\in Q$ one of the Lyapunov exponents is not less than $\log\lambda_t$, i.e. positive and the second one is negative.

It can be easily proved by use of the Birkhoff ergodic theorem that almost every point from Q returns to Q with positive frequency, see [**1**] for example. Hence also for almost every point from the set $\bigcup_{n=-\infty}^{+\infty} H_t^n(Q)$ the Lyapunov characteristic exponents are non-zero. But the latter set by lemma 1 is equal to $T^2\backslash\operatorname{cl} U_t$. This finishes the proof that Lyapunov exponents for $H_t|_{T^2\backslash\operatorname{cl} U_t}$ are non-zero. □

5. *$H_t|_{T^2\backslash\operatorname{cl} U_t}$ has the Bernoulli property*

By the Pesin theory [**8**], for almost every $z\in T^2\backslash\operatorname{cl} U_t$ there exist local unstable and stable manifolds $W^u_{\mathrm{loc}}(z)$, $W^s_{\mathrm{loc}}(z)$. To prove the Bernoulli property, also by use of the Pesin theory, it is enough to prove that for almost every pair $z, z'\in T^2\backslash\operatorname{cl} U_t$, for every $m, n>0$, sufficiently large integers (depending on z and z')

$$H_t^n(W^u_{\mathrm{loc}}(z))\cap H^{-m}(W^s_{\mathrm{loc}}(z'))\neq\varnothing. \tag{8}$$

We consider in fact any lifts of these curves and a lift of the dynamics to $\mathbb{R}^2$ without any change of notation.

The vectors tangent to the curves $H_t^n(W^u_{\mathrm{loc}}(z))$, $H_t^{-m}(W^s_{\mathrm{loc}}(z'))$ lie in the cone bundles $\mathcal{D}$ and $\mathcal{D}^s$ respectively, hence the coordinate x is monotonic along these curves, so that we can introduce a natural orientation on those curves and denote the beginning of the curve $H^n(W^u_{\mathrm{loc}}(z))$ by $(x(n,u,b), y(n,u,b))$ and its end by $(x(n,u,e), y(n,u,e))$. Use similar notation for the ends of $H^{-m}(W^s_{\mathrm{loc}}(z))$ with u replaced by s. For almost every z, z'

$$\text{length}\, H_t^n(W^u_{\mathrm{loc}}(z)),\ \text{length}\, H_t^{-m}(W^s_{\mathrm{loc}}(z'))\xrightarrow[m,n\to\infty]{}\infty,$$

hence

$$|x(n,u,b)-x(n,u,e)|\xrightarrow[n\to\infty]{}\infty,\quad |x(m,s,b)-x(m,s,e)|\xrightarrow[m\to\infty]{}\infty.$$

From this it easily follows that

$$|y(n,u,b)-y(n,u,e)| \xrightarrow[n\to\infty]{} \infty, \quad |y(m,s,b)-y(m,s,e)| \xrightarrow[m\to\infty]{} \infty$$

and that

$$\frac{x(n,u,b)-x(n,u,e)}{y(n,u,b)-y(n,u,e)}>0, \quad \frac{x(m,s,b)-x(m,s,e)}{y(m,s,b)-y(m,s,e)}<0$$

for n, m sufficiently large.

This for geometric reasons proves (8). □

6. *Additional properties of* $H_t|_{T^2\setminus \mathrm{cl}\, U_t}$

We begin with the following lemma, where we gather standard facts about the dynamics near a saddle, which we shall need later.

LEMMA 4. *Let* $0 \in \mathbb{R}^2$ *be a saddle for a* C^2*-diffeomorphism* ϕ *of* $\mathbb{R}^2$, *with eigenvectors* $(\partial/\partial x)(0)$, $(\partial/\partial y)(0)$, *corresponding eigenvalues* $\mu > 1$, μ^{-1} *and stable and unstable manifolds coinciding respectively with the* y-th *and* x-th *axes. Let*

$$\gamma = (\gamma_1, \gamma_2) : [0,1] \to \mathbb{R}^2$$

be a C^2*-curve such that*

$$\gamma(0) = 0 \quad \text{and } d\gamma_i/ds(0) > 0 \quad \text{for } i = 1, 2.$$

Let U *be a small neighbourhood of* 0. *The curve* γ *divides the domain*

$$U^+ = U \cap \{(x,y) \in \mathbb{R}^2 : x > 0, y > 0\}$$

into U_1 *whose closure contains an interval from the* y*-axis and* U_2.

Then for every $\delta, C > 0$ *there exists an integer* $m > 0$ *such that for every*

$$z = (x_0, y_0) \in \mathbb{R}^2, \quad v \in T_z\mathbb{R}^2$$

with the properties:

$$z, \phi^N(z) \notin U, \quad \phi^n(z) \in U^+ \text{ for every } n = 1, \ldots, N-1$$

and

$$\|D\phi^N(v)\| \geq C \cdot \|v\|,$$

the following properties are true:

(*a*) $\phi^n(z) \in U_1$ *for every* n: $1 \leq n \leq (N/2) - m$;

(*b*) $\phi^n(z) \in U_2$ *for* $(N/2) + m \leq n \leq N-1$;

(*c*) $\|D\phi^{n+1}(v)\| / \|D\phi^n(v)\| > \mu - \delta$ *for* $(N/2) + m \leq n \leq N$;

(*d*) angle $(D\phi^n(v), \partial/\partial x) < \delta$ *for* $(N/2) + m \leq n \leq N$;

(*e*) *If in addition the* angle $(v, \partial/\partial y) < C^{-1}x_0$, *then for* $0 \leq n \leq (N/(2+\delta)) - m$

$$\|D\phi^{n+1}(v)\| / \|D\phi^n(v)\| < \mu^{-1} + \delta.$$

We now fix a negative t and study the individual map H_t.

PROPOSITION 1. *The measurable,* DH_t*-invariant stable and unstable sub-bundles* E^s *and* E^u, *which exist over almost whole* $T^2 \setminus \mathrm{cl}\, U_t$ *according to Pesin, are actually defined and continuous over the whole* $T^2 \setminus \mathrm{cl}\, U_t$.

Moreover for every $v \in E^u$, $v \neq 0$,

$$\liminf_{n\to\infty} \frac{1}{n} \log \|DH_t^n(v)\| > 0; \tag{9^u}$$

for all neighbourhoods U_1, U_2 *of* p_t *and* q_t *respectively, there exists* $\delta(U_1, U_2) > 0$ *such that*:

$$\sup\{\|DH_t^{-n}(v)\|/\|v\|: n \geq 0, v \in E^u(z), v \neq 0, \\ z \in T^2 \backslash \mathrm{cl}\,(U_t \cup U_1 \cup U_2)\} < \delta(U_1, U_2); \tag{10^u}$$

for every $v \in E^u$,

$$\lim_{n\to\infty} \|DH_t^{-n}(v)\| = 0. \tag{11^u}$$

The analogous properties hold for E^s. We denote the respective formulae by (9^s)–(11^s).

Proof. We take as E^u the line bundle E_t described in remark 2, § 3. Similarly we define E^s. For every $z \in Q = \mathrm{cl}\, P(C_1, 1-C_1)$,

$$E^u(z) \subset \mathscr{D}_Q = \mathscr{D}_Q^u$$

(see notation at the end of § 4). Clearly for every $z \in Q$,

$$E^s(z) \subset \mathscr{D}_z^s \subset \mathscr{D}_Q^s = \left\{(\xi, \eta) \in \mathbb{R}^2: \frac{2\sqrt{|t|}}{1-\sqrt{|t|}} \geq \eta/\xi \geq -1\right\}.$$

If $|t|$ is so small that

$$-2|t|/(1-\sqrt{|t|}) + \alpha(C_1) > 2\sqrt{|t|}/(1-\sqrt{|t|}),$$

then there exist two constant cones of width $\beta > 0$ which separate $\mathscr{D}_Q^u$ and $\mathscr{D}_Q^s$, hence separate $E^u|_Q$ and $E^s|_Q$ (figure 7). So there exists a number $M(\beta)$ such that if $v \in \mathscr{D}_Q^u$ is decomposed into

$$v = v_u(z) + v_s(z),$$

where

$$v_u(z) \in E^u(z), \quad v_s(z) \in E^s(z) \quad \text{and } z \in Q,$$

then

$$\|v_s(z)\|/\|v_u(z)\| < M(\beta).$$

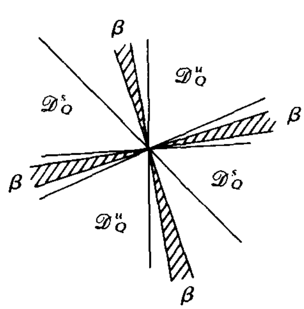

FIGURE 7

https://doi.org/10.1017/S0143385700001711 Published online by Cambridge University Press

Now assume that $z \notin W^u(p_t) \cup W^u(q_t)$ (global unstable manifolds). In this case the continuity of E^u at z can be proved similarly to the case of Anosov diffeomorphisms. Namely, let us take a constant $C_2 > C_1$ ($C_2 \approx C_1$) and denote

$$Q' = P(C_2, 1 - C_2).$$

Let $i_1 < i_2 < \cdots < i_k < \cdots$ be all consecutive non-negative integers such that $H_t^{-i_k}(z) \in Q'$. We can consider the continuity of E^u at $H_t^{-i_1}(z)$, i.e. assume that $z \in Q'$ ($i_1 = 0$).

If z' is close to z then $H_t^{-i_k}(z')$ is close to $H_t^{-i_k}(z)$ for $k = 1, \ldots, K$ with K large. Hence

$$H_t^{-i_k}(z') \in Q.$$

Let $v \in E^u(z')$ and denote $DH_t^{-i_k}(v) = v_s^k + v_u^k$ the decomposition in

$$E^s(H_t^{-i_k}(z)) \oplus E^u(H_t^{-i_k}(z)).$$

Then

$$\|v_s^k\|/\|v_u^k\| \le M_1 \cdot M(\beta) \cdot (\lambda_t - \delta)^{-2(K-k)}$$

for small $\delta > 0$. We recall from § 4 that $\lambda_t > 1$ is the constant of hyperbolicity for the differential $D((H_t)_Q)$ of the first return map $(H_t)_Q$. The coefficient M_1 appears when we pass from the x-coordinate $(\ )_x$ used as a norm on E^s and E^u in § 4 to the norm $\|\ \|$. In particular,

$$\|v_s^1(z)\|/\|v_u^1(z)\| \le M_1 \cdot M(\beta) \cdot (\lambda_t - \delta)^{-2(K-1)}$$

is small.

In the case $z \in W^u(p_t) \cup W^u(q_t)$ the continuity of E^u in z follows immediately from lemma 4(d) and the following lemma.

LEMMA 5. *For every $\delta > 0$ there exists $C(\delta) > 0$ such that if*

$$v \in E^u(z), \quad v \neq 0, \quad z \in T^2 \backslash \mathrm{cl}\, U_t$$

and for $N > 0$

$$H_t^N(z) \in \mathrm{cl}\, P(\sqrt{|t|} + \delta, 1 - \sqrt{|t|} - \delta)$$

then

$$\|DH_t^N(v)\|/\|v\| > C(\delta).$$

Proof. Let $i_1 < i_2 < \cdots$ be the sequence (finite or infinite) of all consecutive non-negative times when $H_t^{i_k}(z) \in Q$. We know that for $k = 1, 2, \ldots,$

$$(DH_t^{i_{k+1}}(v))_x/(DH_t^{i_k}(v))_x \ge \lambda_t > 1.$$

Let $n(z) \ge 0$ be the first time such that

$$H^n(z) \in \mathrm{cl}\, P(\sqrt{|t|} + \delta, 1 - \sqrt{|t|} - \delta)$$

for every $n: n(z) \le n \le i_1(z)$. Clearly the set of all possible integers $i_1(z)$–$n(z)$ is bounded from above.

It can happen that $i_1(z)$, and consequently $n(z)$, do not exist if z belongs to a component of $W^s(p_t) \backslash Q$ or $W^s(q_t) \backslash Q$ containing p_t or q_t respectively. It can also happen that $N < n(z)$ if $\delta < C_1 - \sqrt{|t|}$. However the set of all possible N in these cases is bounded from above. In the latter case this is due to lemma 1 which implies that for every $n: 0<n\le N$,

$$H_t^n(z)\in \mathrm{cl}\,P(-C_1, -\sqrt{|t|}-\delta)$$

or for every $n: 0<n\le N$,

$$H_t^n(z)\in \mathrm{cl}\,P(\sqrt{|t|}+\delta, C_1).$$

The above observations also apply to the point $H^{i_k+1}(z)$ where k is the largest integer such that $i_k<N$.

Thus, the proof of the lemma reduces to estimating

$$\|DH_t^{n(z)}(v)\|/\|v\| \quad \text{for } n(z) \text{ large}.$$

Let $z=(x, y)$ and for example $y>0$. Consider the case when

$$z, H_t(z)\in \mathrm{cl}\,P(0, \sqrt{|t|}).$$

Put

$$z=z_0=(x_0, y_0), \quad z_1=F_{\mathrm{id}}(z)=(x_1, y_0)$$

and consider the points $z_i'=(x_i', y_i')$ lying on the same vertical as z_i belonging to γ_t^+, for $i=0$ and to graph y_t^- for $i=1$, see figure 8.

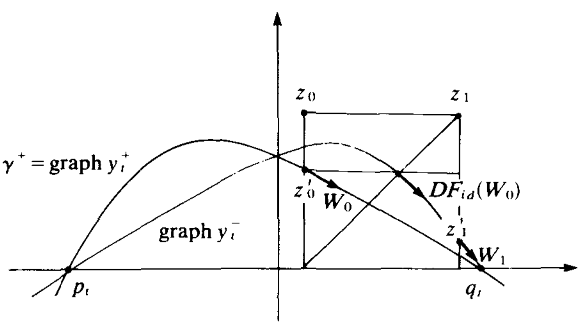

FIGURE 8

Denote by W_0 the vector tangent to γ_t^+ at z_0' such that $(W_0)_x=1$ and by W_1 the vector tangent to graph y_t^- at z_1' such that

$$(W_1)_x=(DF_{\mathrm{id}}(W_0))_x.$$

Denote the vectors W_i at z_i instead of at z_i' by $W_i(z_i)$, for $i=0, 1$. Instead of v it is enough to consider $W_0(z_0)$.

Put $u=DF_{\mathrm{id}}(W_0(z_0))-W_1(z_1))$. Since

$$DF_{\mathrm{id}}(W_0(z_0))=DF_{\mathrm{id}}(W_0)$$

if we identify the respective tangent spaces, we have

$$\begin{aligned}(u)_y&=\left(\frac{dy_t^-}{dx}((F_{\mathrm{id}}(z_0'))_x)-\frac{dy_t^-}{dx}(x_1)\right)\cdot(W_1)_x\\&\ge\left(\sup\left\{\frac{d^2y_t^-}{dx^2}(x): x\in[-\sqrt{|t|}, \sqrt{|t|}]\right\}\right)\cdot(y_0'-y_0)\cdot(W_1)_x\\&>\tfrac{3}{2}\cdot(y_0-y_0')\cdot(1-\sqrt{|t|})/(1+\sqrt{|t|})>y_0-y_1'.\end{aligned}$$

Of course $(u)_x=0$.

https://doi.org/10.1017/S0143385700001711 Published online by Cambridge University Press

Let $I:[0,1]\to T^2$ be the interval, joining z_1' with z_1. Using the convexity of the function g_t in the domain $[0,\frac{1}{4}]$ one can prove by induction that for every $s\in[0,1]$ and $n: 0\le n\le n(z)-1$:

$$(D(H_t^n\circ G_{g_t})(u))_x\ge (D(H_t^n\circ G_{g_t}\circ I)((\partial/\partial s)(s)))_x.$$

Since

$$D(H_t^n\circ G_{g_t})(W_1(z_1)),\, D(H_t^n\circ G_{g_t})(u)\in\mathcal{D}\quad \text{for } n\ge 1,$$

we have

$$(D(H_t^n\circ G_{g_t})(W_1(z_1)))_x,\, (D(H_t^n\circ G_{g_t})(u))_x>0.$$

So

$$(DH_t^{n(z)}(W_0(z_0)))_x\ge\int_0^1 (D(H_t^{n(z)-1}\circ G_{g_t}\circ I)((\partial/\partial s)(s)))_x\,ds$$
$$+(D(H_t^{n(z)-1}\circ G_{g_t})(W_1(z_1)))_x>\delta,$$

since $(H_t^{n(z)-1}G_{g_t})(z_1')$ stays in γ_t^+, hence in $P(-\sqrt{|t|},\sqrt{|t|})$ and

$$(H_t^{n(z)-1}G_{g_t})(z_1)\in P(\sqrt{|t|}+\delta,\,1-\sqrt{|t|}-\delta).$$

We have considered the case $z, H_t(z)\in \operatorname{cl} P(0,\sqrt{|t|})$.

The case $z\in P(-C_1,0)$ reduces to the previous one since

$$(DH_t^{n+1}(v))_x/(DH_t^n(v))_x\ge 1\quad \text{for every } n=0,1,\dots,n_1(z)-2,$$

where $n=n_1(z)$ is the first time when

$$H_t^n(z)\in\operatorname{cl} P(0,\sqrt{|t|}).$$

Then also

$$H_t^{n_1(z)+1}(z)\in\operatorname{cl} P(0,\sqrt{|t|})$$

due to the assumption that $n(z)$ is large.

Also for $z\in\mathcal{T}_t(C_1-\sqrt{|t|})\cup\mathcal{T}_t'(C_1-\sqrt{|t|})$ we have

$$(DH_t^{n+1}(v))_x/(DH_t^n(v))_x>1\quad\text{for every } n=0,1,\dots,n(z)-1.$$

The less trivial case is when $z\in P(\sqrt{|t|},\sqrt{|t|}+\delta)$. We still assume that $y>0$:

Let $n=n'(z)>0$ be the first time that

$$H_t^{-n}(z)\in\operatorname{cl} P(0,\sqrt{|t|}-\delta).$$

Notice that it is enough to prove the lemma only for $\delta\ll\sqrt{|t|}$. Now we shall use lemma 4 for the saddle q_t, its neighbourhood: the square with the sides $x=\sqrt{|t|}\pm\delta$ and $y=\pm\delta$ and for the curve $\{x=\sqrt{|t|}, y\ge 0\}$. For that we need to change coordinates. Its assumptions, for the vector $DH_t^{-n'(z)}(v)$ tangent of $H_t^{-n'(z)}(z)$ are satisfied due to the proved case of lemma 5.

So, by lemma 4(*c*)

$$\|DH_t^{n+1}(v)\|'/\|DH_t^n(v)\|'\ge 1 \tag{12}$$

for every n such that

$$(n(z)+n'(z))/2+m-n'(z)\le n\le n(z).$$

Here we use the Euclidean norm $\|\ \|'$ connected with the coordinates of lemma 4.
By lemma 4(*a*), since $z \in P(\sqrt{|t|}, \sqrt{|t|}+\delta)$

$$n'(z) \geq \frac{n(z)+n'(z)}{2} - m.$$

Hence (12) holds for every $n = 2m, \ldots, n(z)-1$. So

$$\|DH_t^{n(z)}(v)\|/\|v\| \geq CL^{-2m},$$

where L is the Lipschitz constant for H_t^{-1} and C is a coefficient connected with the change of the norms. This ends the proof of lemma 5. □

We still need to prove $(9^{u(s)})$–$(11^{u(s)})$ in proposition 1. Let us start with (9^u). This is obvious for

$$z \in W^s(p_t) \cup W^s(q_t).$$

To prove the other case it is enough to find $\mu_t > 1$ such that for every $z \in Q$, if the first positive integer $n(z)$ such that $H_t^{n(z)} \in Q$ is larger than a constant integer N,

$$(DH_t^n(v))_x/(v)_x \geq \mu_t^n \quad \text{for } n = 0, 1, \ldots, n(z).$$

Then we would obtain in (9^u) the estimate by

$$\min(N^{-1} \log \lambda_t, \log \mu_t).$$

Let $z = (x, y) \in Q$, $n(z)$ be as above with $y > 0$. Let $n = n_1(z)$ be the first positive integer such that

$$H_t^n(z) \in P(0, C_1).$$

We extend the notation from the proof of lemma 3, § 4: For every $n = 0, 1, \ldots, n(z)$ put

$$R(n) = (DH_t^n(v))_x/(v)_x = \prod_{k=0}^{n-1} l_k$$

where $l_n = (DH_t^{n+1}(v)_x/(DH_t^n(v))_x$. Put as usual $H_t^n(z) = (x_n, y_n)$ and, furthermore:

$r_n = 1 - 2\sqrt{|t|}/(1-\sqrt{|t|}) + \alpha(C_1)$ for $n = 0$;

$r_n = (1+\sqrt{|t|})/(1-\sqrt{|t|})$ for $n \geq 0$ and such that $x_n < -\sqrt{|t|}$;

$r_n = 1 + dy_t^+/dx(x_n)$ if $-\sqrt{|t|} \leq x_n$ and $n \leq n_1(z)-2$ and also for $n = n_1(z)-1$ we assume that $|x_{n_1(z)-1}| > |x_{n_1(z)}|$;

$r_n = (1 - dy_t^-/dx(x_{n+1}))^{-1}$ if $x_{n+1} \leq \sqrt{|t|}$ and $n \geq n_1(z)$ and also for $n = n_1(z)-1$ we assume that $|x_{n_1(z)-1}| \leq |x_{n_1(z)}|$;

$r_n = (1-\sqrt{|t|})/(1+\sqrt{|t|})$ if $x_{n+1} > \sqrt{|t|}$ and $n < n(z)$.

Recall that $l_n \geq r_n$. Due to lemma 5 we can use lemma 4(*b*) and (*c*), so there exists $m > 0$ such that for every n satisfying:

$$n_4(z) = n_1(z) + (n(z) - n_1(z))/2 + m \leq n \leq n(z)$$

we have

$$x_n > \sqrt{|t|} \quad \text{and} \quad l_n > ((1+\sqrt{|t|})/(1-\sqrt{|t|}))^{\frac{1}{2}}. \tag{13}$$

So

$$R_{n_4(z)}=\prod_{i=0}^{n_4(z)-1} l_i \geq \left(\prod_{i=0}^{n(z)-1} r_i\right)\cdot\left(\prod_{i=n_4(z)}^{n(z)-1} r_i\right)^{-1}$$
$$\geq \lambda_t\cdot\left(\frac{1-\sqrt{|t|}}{1+\sqrt{|t|}}\right)^{-n(z)/5} \geq \left(\left(\frac{1+\sqrt{|t|}}{1-\sqrt{|t|}}\right)^{\frac{1}{5}}\right)^{n_4(z)}. \tag{14}$$

We have used the fact that for large $n(z)$, $n_4(z)<\frac{4}{5}n(z)$. This is true due to the definition of $n_4(z)$ and due to lemma 2, § 4 which gives $|n_1(z)-n(z)/2|\leq 1$.

For $n\geq n_4(z)$, we have due to (13):

$$R(n)=R(n_4(z))\cdot\left(\prod_{i=n_4(z)}^{n-1} l_i\right)$$
$$\geq\left(\left(\frac{1+\sqrt{|t|}}{1-\sqrt{|t|}}\right)^{\frac{1}{5}}\right)^{n_4(z)}\left(\left(\frac{1+\sqrt{|t|}}{1-\sqrt{|t|}}\right)^{\frac{1}{2}}\right)^{n-n_4(z)}>\left(\left(\frac{1+\sqrt{|t|}}{1-\sqrt{|t|}}\right)^{\frac{1}{5}}\right)^{n}.$$

For $n\leq n_4(z)$ similar estimates follow from

$$R(n)\geq\prod_{i=0}^{n-1} r_i, \quad \prod_{i=0}^{n_4(z)-1} r_i\geq\left(\left(\frac{1+\sqrt{|t|}}{1-\sqrt{|t|}}\right)^{\frac{1}{5}}\right)^{n_4(z)}$$

and from the fact that the sequence r_i, $i=0,\ldots,n_4(z)$ is decreasing.

Concluding, we can take

$$\mu_t=((1+\sqrt{|t|})/(1-\sqrt{|t|}))^{\frac{1}{5}}.$$

A more careful estimate in (14) would show that we could take

$$\mu_t=((1+\sqrt{|t|})/(1-\sqrt{|t|}))^{\frac{1}{3}-\delta}$$

for arbitrarily small $\delta>0$.

Now let us prove (10^u). Let U_1, U_2 contain respectively some balls $B(p_t,\delta)$, $B(q_t,\delta)$. For

$$z\in\operatorname{cl} P(\sqrt{|t|}+\delta/2,\, 1-\sqrt{|t|}-\delta/2)$$

(10^u) follows from lemma 5. If

$$z\in\operatorname{cl} P(-\sqrt{|t|}-\delta/2,\, \sqrt{|t|}+\delta/2),$$

then z is within the distance of at least $\delta/2$ from the components W^p and W^q of

$$W^u(p_t)\cap\operatorname{cl} P(-\sqrt{|t|}-\delta/2,\, -\sqrt{|t|})$$

or

$$W^u(q_t)\cap\operatorname{cl} P(\sqrt{|t|},\, \sqrt{|t|}+\delta/2)$$

containing p_t or q_t, respectively, since W^p and W^q are almost horizontal if $|t|$ is sufficiently small. So after bounded time $n>0$ and some time $m\geqslant 0$ during which DH_t^{-1} contracts on E^u,

$$H_t^{-n}(z)\in\operatorname{cl} P(\sqrt{|t|}+\delta/2,\, 1-\sqrt{|t|}-\delta/2)$$

and we have the previous case.

(11^u) is obvious in the case $z \in W^u(p_t) \cup W^u(q_t)$. In the other case it follows from (10^u) and from the fact that

$$(DH_t^{-i_{k+1}}(v))_x/(DH_t^{-i_k}(v))_x \le \lambda_t^{-1}$$

for every two consecutive times i_k, i_{k+1} when the backward trajectory $H_t^{-n}(z)$ of z hits Q. The proof of proposition 1 is finished. □

COROLLARY 1. *For every H_t-invariant probability measure on $T^2\backslash \mathrm{cl}\ U_t$, the Lyapunov characteristic exponents are almost everywhere non-zero and of opposite signs.*

Proof. This corollary follows from (9^u) and (9^s).

COROLLARY 2. *For every $\delta > 0$ there exists $C_0(\delta) > 0$ such that for every $\gamma^u : [0, 1] \to T^2\backslash \mathrm{cl}\ U_t$ an integral curve for E^u, if*

$$\mathrm{dist}\,(\gamma^u, \{p_t\} \cup \{q_t\}) \ge \delta$$

then

$$\sup_{n \ge 0}\,(\mathrm{length}\ H_t^{-n}(\gamma^u)/\mathrm{length}\ \gamma^u) \le C_0(\delta) \tag{15}$$

and

$$\lim_{n\to\infty} \mathrm{length}\ H_t^{-n}(\gamma^u) = 0.$$

The analogous facts hold for integral curves for E^s.

Proof.

$$\lim_{n\to\infty} \mathrm{length}\ H_t^{-n}(\gamma^u) = \lim_{n\to\infty} \int_0^1 \|D(H_t^{-n} \circ \gamma^u)(\partial/\partial s)(s)\|\, ds$$
$$= \int_0^1 (\lim_{n\to\infty} \|D(H_t^{-n} \circ \gamma^u)(\partial/\partial s)(s))\|\, ds = 0.$$

We used the fact that the integrands are uniformly bounded by (10^u) and converge to 0 pointwise by (11^u). (10^u) gives (15) with

$$C_0(\delta) = C(B(p_t, \delta), B(q_t, \delta)).$$
□

Let $z = (x_0, y_0) \in T^2\backslash \mathrm{cl}\ U_t$. Consider the rectangle

$$S = \{(x, y) : x_0 - \delta \le x \le x_0 + \delta,\ y_0 - K\delta \le y \le y_0 + K\delta\},$$

where $K = 1 + \sup dg_t/dx$ and δ such that $S \cap \mathrm{cl}\ U_t = \varnothing$ and

$$(K+1) \cdot \delta \cdot C_0(\mathrm{dist}\,(S, \{p_t\} \cup \{q_t\})) \ll 1.$$

Let $\gamma^u \ni z$ be a maximal integral curve for E^u in S. By the definition of K it joins the left and right hand sides of S. Then γ^u, the candidate for a local unstable manifold has the following characterization:

$$\gamma^u = \{z' \in S : \mathrm{dist}\,(H_t^{-n}(z'), H_t^{-n}(z)) \le \mathrm{dist}\,(z', z) \cdot (K+1)$$
$$\times C_0(\mathrm{dist}\,(S, \{p_t\} \cup \{q_t\})) \text{ for every } n \ge 0\}.$$

The inclusion '⊂' follows from (15) in corollary 2. To prove '⊃' take

$$u=(x_1, y_1)\in S\backslash\gamma^u$$

and put $u'=(x_1, y_1')$ the point on the same vertical as u, in γ^u. Take the interval I joining u with u'. For every $n\geq 2$ the vectors tangent to $H_t^{-n}(I)$ belong to the stable cones $\mathcal{D}^s$. Hence

$$\begin{aligned}\sup_{n\geq 0}\operatorname{dist}(H_t^{-n}(u), H_t^{-n}(z)) &\geq \sup_{n\geq 0}\operatorname{dist}(H_t^{-n}(u), H_t^{-n}(u'))\\ &\quad -\sup_{n\geq 0}\operatorname{dist}(H_t^{-n}(u'), H_t^{-n}(z))\\ &\geq \tfrac{1}{2}L^{-1}-(K+1)\cdot\delta\cdot C_0(\operatorname{dist}(S,\{p_t\}\cup\{q_t\}))\geq \text{const}>0.\end{aligned}$$

L is the Lipschitz constant for H_t^{-1}.

The above characterization of γ^u and an analogous characterization of γ^s prove:

COROLLARY 3. *The line bundles E^u and E^s are uniquely integrable.*

Remark. The bundles E^u and E^s extend to the continuous bundles $\bar{E}^u$ and $\bar{E}^s$ over $T^2\backslash(U_t\cup\{p_t\}\cup\{q_t\})$ which are tangent to $\gamma_t^{\pm}$ over $\gamma_t^{\pm}$. It is easy to see that $(10^{u(s)})$, $(11^{u(s)})$, corollary 2 and corollary 3 hold if $E^{u(s)}$ is replaced by $\bar{E}^{u(s)}$.

PROPOSITION 2. *There exists a continuous semiconjugacy $\varphi: T^2 \xrightarrow{\text{onto}} T^2$ from H_t to the Anosov automorphism A* (i.e. $\varphi\circ H_t=A\circ\varphi$) *such that $\varphi^{-1}(0)=\operatorname{cl}U_t$ and $\varphi|_{T^2\backslash\operatorname{cl}U_t}$ is* 1–1. *This means that $H_t|_{T^2\backslash\operatorname{cl}U_t}$ is topologically conjugated with $A|_{T^2\backslash\{0\}}$.*

Compare this proposition with property (1) of H_t, $t\geq 0$, from the introduction.

Proof. The existence of a semiconjugacy follows from [**3**, proposition 2.1].

Denote by $\tilde{\varphi}$, $\tilde{H}$, $\tilde{A}$ lifts of φ, H_t, A to $\mathbb{R}^2$ keeping $0\in\mathbb{R}^2$ invariant, such that

$$\tilde{\varphi}\circ\tilde{H}=\tilde{A}\circ\tilde{\varphi}.$$

$\tilde{\varphi}-\text{id}$ is a bounded function and $\tilde{A}$ is expansive in the following sense:

$$\sup_{n\in Z}\operatorname{dist}(\tilde{A}^n(z), \tilde{A}^n(z'))=\infty$$

for every $z, z'\in\mathbb{R}^2$, $z\neq z'$.

Hence $\tilde{\varphi}(z)\neq\tilde{\varphi}(z')$ is equivalent to

$$\sup_{n\in Z}\operatorname{dist}(\tilde{H}^n(z), \tilde{H}^n(z'))=\infty. \tag{16}$$

Denote by Π the projection $\Pi:\mathbb{R}^2\to\mathbb{R}^2/Z^2=T^2$. Then

$$\varphi(\Pi(z))\neq\varphi(\Pi(z'))$$

is equivalent to (16) for every pair $z+w$, z' where $w\in Z^2$.

Due to this criterion we immediately have $\varphi(\operatorname{cl}U_t)=0$. To finish the proof it is enough to check that for every pair $z\in\Pi^{-1}(T^2\backslash\operatorname{cl}U_t)$, $z'\in\Pi^{-1}(T^2\backslash U_t)$, we have

$$\sup_{n\in Z}\operatorname{dist}(\tilde{H}^n(z), \tilde{H}^n(z'))=\infty.$$

We shall only check the case when $z=z_0=(x_0, y_0)$ and $z'=(x', y')$ are close to 0 and $y_0, y'>0$ and leave the other cases to the reader.

https://doi.org/10.1017/S0143385700001711 Published online by Cambridge University Press

Consider the new coordinates $x, \beta(x, y) = y - y_t^+(x, y)$ in a neighbourhood W of 0. Put

$$V_1 = \{z = (x, y) \in W: (x - x_0) \cdot (\beta(z) - \beta(z_0)) \geq 0\}$$

and

$$V_2 = W \setminus V_1.$$

See figure 9.

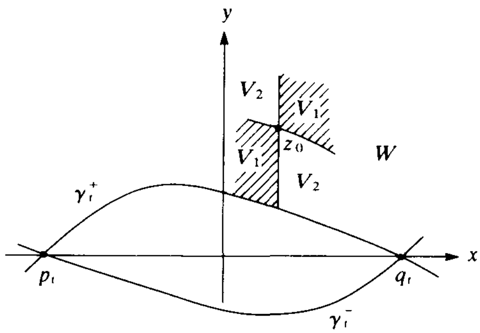

FIGURE 9

Join $z = z_0$ with z' by the interval $I: [0, 1] \to \mathbb{R}^2$ in the coordinates (x, β). We lift our DH_t (DH_t^{-1})-invariant cones and bundles $E^{u(s)}$ to $T(\mathbb{R}^2)$ and use the same notation for them as in $T(T^2)$. Now if $z' \in V_1$

$$D(\tilde{H}^n \circ I)(\partial/\partial s)(s) \in \mathscr{D} \tag{17}$$

for every $n = 1, 2, \ldots$ and $s \in [0, 1]$ except maybe $s = s_0$ such that $I(s_0) \in \gamma_t^+$ where $\mathscr{D}$ has not been defined. There exists at most one such s_0 since z and z' do not both belong to γ_t^+. Hence

$$D(\tilde{H}^n \circ I)(\partial/\partial s)(s) \notin \bar{E}^s.$$

This is true for $s \neq s_0$ since

$$\mathscr{D} \cap \operatorname{int} \mathscr{D}^s = \emptyset$$

and in fact $E^s \subset \operatorname{int} \mathscr{D}^s$.

For every $s \neq s_0$ decompose

$$D(\tilde{H} \circ I)(\partial/\partial s)(s) = v_1(s) + v_2(s)$$

according to the decomposition $E^s \oplus E^u$. We have the function $\|v_1(s)\|$ bounded from above on $[0, 1]\setminus J$ where J is a neighbourhood of s_0 and also $\|v_2(s)\| > 0$ for every $s \in [0, 1]\setminus J$.

Then by (9^u)–(11^u)

$$\text{length}\,(\tilde{H}^n \circ I|_{[0,1]\setminus J}) \xrightarrow[n\to\infty]{} \infty$$

so length $(\tilde{H}^n \circ I) \to_{n\to\infty} \infty$.

Since by (17) the functions $(D\tilde{H}^n(\partial/\partial s)(s))_x$ have constant signs and

$$(D\tilde{H}^n(\partial/\partial s)(s))_y/(D\tilde{H}^n(\partial/\partial s)(s))_x \leq 1 + \sup dg_t/dx$$

is uniformly bounded,

$$\text{dist}\,(\tilde{H}^n(z), \tilde{H}^n(z')) \xrightarrow[n\to\infty]{} \infty.$$

The proof for $z' \in V_2$ is similar. In that case expansiveness occurs under backward iterates. The proof of proposition 2 is finished. □

It occurs that $\varphi|_{T^2 \setminus \mathrm{cl}\, U_t}$ cannot be C^1. Moreover, we shall prove the following proposition.

PROPOSITION 3. *There exist no C^1-diffeomorphisms*

$$B : T^2 \to T^2 \quad \text{and} \quad \varphi : (T^2 \setminus \mathrm{cl}\, U_t) \to T^2 \setminus \{0\}$$

such that $\varphi \circ H_t = B \circ \varphi$.

Proof. We use the method used by Gerber and Katok [**4**] to prove the analogous fact for pseudo-Anosov homeomorphisms. Due to proposition 2 we can find a Markov partition for $H_t|_{T^2 \setminus \mathrm{cl}\, U_t}$ containing the cells M_i, $i = 1, 2$ being closures of a neighbourhood of cl U_t intersected with $\mathcal{T}_t$ and $\{y > 0\} \setminus (\mathcal{T}_t \cup \mathcal{T}'_t)$ respectively. So there exist sequences of H_t-periodic points z_n, w_n with periods $\alpha_n, \beta_n \to \infty$ such that

$$z_n \in \mathrm{int}\, M_1, \quad w_n \in \mathrm{int}\, M_2,\, z_n,\, w_n \xrightarrow[n \to \infty]{} D,$$

which is a fundamental domain in $W^s(p_t)$, and there exists a constant integer $N > 0$ such that for every $i, n : 0 \leq i \leq \alpha_n - N$,

$$H^i(z_n) \in M_1$$

and for every $i, n : 0 \leq i \leq \beta_n - N$,

$$H^i(w_n) \in M_2$$

(see figure 10).

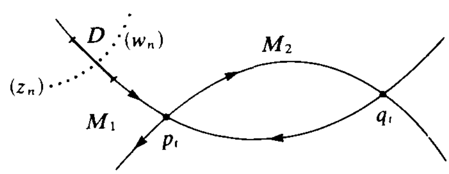

FIGURE 10

Clearly the Lyapunov exponents $\lambda_\pm(z_n)$ converge to the logarithms of the eigenvalues at p_t, i.e. to

$$\lambda_\pm^{(z)} = \pm \log (1 + \sqrt{|t|})/(1 - \sqrt{|t|}).$$

For $v \in E^u(w_n)$ we have clearly

$$\frac{1}{n_1(w_n)} \log \|DH_t^{n_1(w_n)}(v_n)\| \approx \log \left(\frac{1 + \sqrt{|t|}}{1 - \sqrt{|t|}} \right)$$

for n large, where $n_1(w_n)$ is the first time when

$$(H_t^{n_1(w_n)}(w_n))_x > 0.$$

The bundle $\bar{E}^u$ is Lipschitz continuous at $\gamma^\pm$. This is a property of dynamics around the saddle p_t, compare with [**5**, theorem 6.3.b].

Hence we can use lemma 4(e) for a neighbourhood of q_t and conclude by use of lemma 4(c) that

$$\log (\|DH_t^{\beta_n-N}(v_n)\|/\|DH_t^{n_1(w_n)}(v_n)\| \approx 0$$

for n large.

By lemma 2 $|n_1(w_n)-\beta_n/2|$ is uniformly bounded for all n. So, the Lyapunov exponents $\lambda_\pm(w_n)$ converge to

$$\lambda_\pm^{(w)} = \pm\tfrac{1}{2}\log((1+\sqrt{|t|})/(1-\sqrt{|t|})).$$

If φ and B existed, the Lyapunov exponents over $\varphi(z_n)$ and $\varphi(w_n)$ would also converge to $\lambda_\pm^{(z)}$, $\lambda_\pm^{(w)}$ respectively.

Meanwhile, if one of the eigenvalues of $DB(0)$ were 0, then

$$\lim_{n\to\infty}\frac{1}{n}\log\|DB^{\alpha_n}_{\varphi(z_n)}\|=0 \quad \text{or} \quad \lim_{n\to\infty}\frac{1}{n}\log\|DB^{-\alpha_n}_{\varphi(z_n)}\|=0,$$

so $\log\lambda_+^{(z)}$ and $\log\lambda_-^{(z)}$ could not have different signs. This is a contradiction.

If 0 were a saddle for B then $\varphi(z_n), \varphi(w_n) \to_{n\to\infty} \varphi(D)$ – a fundamental domain in a local stable manifold for B at 0. But the set of limit spaces of the sequences $D\varphi(E^u(z_n))$ and $D(E^u(w_n))$ is disjoint with the bundle tangent to $\varphi(D)$. Hence one of the Lyapunov exponents at $\varphi(z_n)$ and at $\varphi(w_n)$ converge to the same number, to the logarithm of an eigenvalue of $DB(0)$. So $\lambda_+^{(z)}=\lambda_+^{(w)}$. This is a contradiction. □

7. *Final remarks*

Remark 1. We do not know whether there exists a family g_t satisfying property (1) § 1, with separatrices joining p_t with q_t for the corresponding H_t, such that g_t on $\mathbb{R}$, and hence, H_t on T^2, is real-analytic.

The problem is to solve the system of functional equations:

$$y_t^-(x+y_t^+(x)) = y_t^+(x) \quad g_t = y_t^+ - y_t^-$$

close to $t=0$, $x=0$, in real-analytic functions satisfying property (1) (its part, at $t=x=0$), so that $g_t - \mathrm{id}$ is *periodic* with period 1.

The periodicity condition does not hold for g_t defined by (*) in the theorem in § 2. There, the functions g_t have real poles.

We can attempt to solve the problem by starting with the family of the Hamiltonian functions:

$$h_t(x,y) = (\tfrac{4}{5}x^2-(y+\sqrt{C}-t)^2+C)\cdot(\tfrac{4}{5}x^2-(y-\sqrt{C}+t)^2+C),$$

for a constant $C>0$.

Then we obtain $g_t - \mathrm{id}$ bounded (not periodic unfortunately).

We have chosen the above h_t so that the set of their zeros consists of branches of hyperboles. The choice is motivated by the fact that if we want g_t to be real-analytic, then graph y_t^+ must coincide with the unstable manifold of q_t for H_t (when $x\to+\infty$). So, when $x\to+\infty$, graph y_t^+ must be within the finite distance from the unstable manifold of 0 for the Anosov diffeomorphism A, which is the straight line $(2x/\sqrt{5})-y=0$. This is so because of the existence of a semiconjugacy from H_t to A, see proposition 3, § 6.

Remark 2. We could consider directly the time-one diffeomorphism $\bar{H}_{t,1}$ for the Hamiltonian vector field corresponding to the function

$$h_t(x, y) = y^2 - x^2(x^2 + 2t),$$

see § 2. The trouble then is with a simple extension of this diffeomorphism from a neighbourhood of cl U_t to the whole T^2. Such $\bar{H}_{t,1}$ on U_t would be integrable (i.e. $U_t \backslash \{0\}$ would consist of closed invariant curves).

Our H_t's, close to 0, are perturbations of such $\bar{H}_{t,1}$. The intuition to treat H_t as a time-one solution for a differential equation has been basic to the existence of invariant cones (such a cone cannot pass to the other side of the trajectory of the flow, figure 11) and in lemma 2.

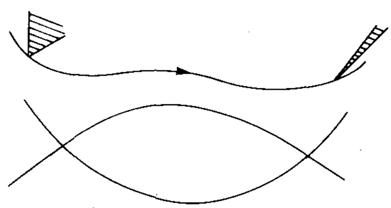

FIGURE 11

Remark 3. In the proof of proposition 3 § 6 we used the fact that in the construction of H_t, $t < 0$ only two out of four sectors between stable and unstable manifolds of a saddle of an Anosov diffeomorphism were 'blown up'.

We can however use the Hamiltonian function:

$$y^2(y^2 + 2t) - x^2(x^2 + 2t) = (y^2 - x^2)(x^2 + y^2 + 2t).$$

For $t < 0$, the separatrices γ_i, $i = 1, \ldots, 4$, joining the saddles $p_i = (\pm\sqrt{|t|}, \pm\sqrt{|t|})$, $i = 1, \ldots, 4$, form a circle S_t, see figure 12.

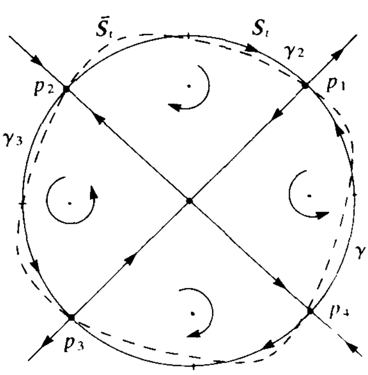

FIGURE 12

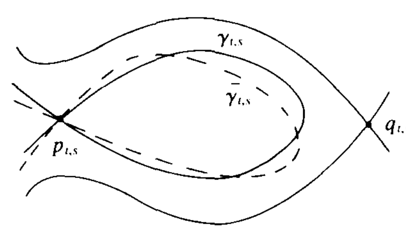

FIGURE 13

Now we should either somehow extend the time-one diffeomorphism for the resulting Hamiltonian vector field or find functions $f = f_t$, $g = g_t$, $\mathbb{R} \to \mathbb{R}$ with property (1) § 1 such that the toral linked twist mapping $H_{f,g}$ still preserves the saddles p_i and a closed curve $\bar{S}_t$, built from separatrices $\bar{\gamma}_i$, $i = 1, \ldots, 4$ (close to S_t).

https://doi.org/10.1017/S0143385700001711 Published online by Cambridge University Press

For each individual t it is easy to find such f, g of class C^∞ as follows: Define any reasonable $f = g$ in a small neighbourhood of $\pm\sqrt{2|t|}$, then extend four small arcs $F_{-\frac{1}{2}f}(\gamma_{2(4)})$, $G_{\frac{1}{2}g}(\gamma_{1(3)})$ to a curve $\bar{S}_t'$ (invariant under rotation by $\pi/2$) and, using also $\bar{S}_t'$ symmetric to $\bar{S}_t$ with respect to the x- or y-axis, find f and g.

Is it possible to find such f_t, g_t real-analytic, at least in a neighbourhood of $t = x = y = 0$?

The whole theory from this paper holds for the resulting $H_t = H_{f,g}$ except for proposition 3. Can the resulting H_t on $T^2\backslash\text{cl } U_t$ (U_t is the domain bounded by $\bar{S}_t$) be C^1-conjugate with $A|_{T^2\backslash\{0\}}$? The obstruction used in the proof of proposition 3 disappears in this case.

Remark 4. We can consider a secondary bifurcation $H_{t,s}$ of H_t. Let us start with the Hamiltonian function:

$$h_{t,s}(x, y) = y^2 - x^2(x^2 + sx + 2t).$$

See the phase portrait of figure 13.

Now as in remark 3 we can look for functions $f_{t,s}$, $g_{t,s}$, $t < 0$ such that H_{f_t,g_t} preserves the saddles $p_{t,s}$, $q_{t,s}$ and a separatrix $\bar{\gamma}_{t,s}$ close to $\gamma_{t,s}$ from $p_{t,s}$ (or $q_{t,s}$) to itself.

Observe that we dropped the assumption from property (1) § 1 that $g_{t,s}$ is an odd function, since for $s \neq 0$ $h_{t,s}$ is not an even function with respect to x.

As in remark 3 it is easy to find $f_{t,s}$, $g_{t,s}$ C^∞ for each individual t, s.

Is it possible to find $f_{t,s}$, $g_{t,s}$ real-analytic at least in a neighbourhood of $t = s = x = y = 0$?

Are the Lyapunov exponents outside the separatrix $\gamma_{t,s}$ different from zero for $H_{t,s}$, $s \neq 0$, $t < 0$?

The author gratefully acknowledges the financial support of the Stiflung Volkswagenwerk for a visit to the Institut des Hautes Études Scientifique during which this paper was partially written.

REFERENCES

[1] R. Burton & R. Easton. Ergodicity of linked twist mappings. *Global Theory of Dynamical Systems: Lectures Notes in Math.* No. 819. Springer: Berlin, 1980, pp. 35–49.
[2] R. E. Edwards. *Functional Analysis.* Holt, Rinehart and Winston: New York, 1965.
[3] J. Franks. Anosov diffeomorphisms. *Global Analysis. Proc. Symp. Pure Math.* **14** (1970), 61–93.
[4] M. Gerber & A. Katok. Smooth models of Thurston's pseudo-Anosov maps. *Ann. Sci. de l'Ecole Normale Superieure,* 4^e serie, **15** (1982), 173–204.
[5] M. Hirsch & Ch. Pugh. Stable manifolds and hyperbolic sets. *Global Analysis, Proc. Symp. Pure Math.* **14** (1970), 133–163.
[6] A Katok. Bernoulli diffeomorphisms on surfaces. *Ann. Math.* **110** (1979), 529–547.
[7] R. Mañé. The Lyapunov exponents of generic area preserving diffeomorphisms. Preprint.
[8] Ya. B. Pesin. Lyapunov characteristic exponents and smooth ergodic theory. *Uspekhi Math. Nauk.* **32** No. 4 (1977), 55–112. English translation: *Russ. Math. Surveys* **32** No. 4 (1977), 55–114.
[9] F. Przytycki. Linked twist mappings: Ergodicity. Preprint *I.H.E.S.*, February 1981. (To appear in part in *Ann. Scientifiques de l'Ecole Norm. Sup.*)