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M-IDEALS IN $L(\ell_1, E)$

BY

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ABSTRACT. In this article it is shown that for any Banach space E, $L(\ell_1, E)$ always contains uncountably many distinct M-ideals that are closed subspaces of $K(\ell_1, E)$ and which are not complemented in $L(\ell_1, E)$. Using standard duality arguments one obtains the result that infinitely many distinct subspaces of $K(E, c_0)$ are M-ideals in $L(E, c_0)$. In particular, for the case $E = c_0$, this shows that the uniqueness conditions enjoyed by $K(\ell_p)$, $p > 1$, is not valid for $E = c_0$. The results are obtained by utilizing the identification of $L(\ell_1, E)$ with the vector-valued sequence space $\ell_\infty(E)$ and to exploit natural decompositions of $\ell_\infty(E)'$ afforded by a class of L-projections on $\ell_\infty(E)'$ induced by certain E'-valued vector measures.

1. **Introduction**. Let X be a Banach space. A continuous linear map $\tau: X \to X$ is called an L-projection if $\tau^2 = \tau$ and $\|x\| = \|\tau(x)\| + \|x - \tau(x)\|$ for all x in X [4]. A closed subspace M of a Banach space X is an M-ideal if there exists an L-projection $\tau: X' \to X'$ with $\ker \tau = M^\perp$. M-ideals were first defined and characterized in the fundamental paper of Alfsen and Effros [1].

For Banach spaces X, Y we denote by $L(X, Y)$ [respectively $K(X, Y)$] the Banach space of all continuous [respectively compact] linear maps $u: X \to Y$. Much recent attention has been devoted to the geometric problem concerning the existence and uniqueness of non-trivial M-ideals in the operator space $L(X, Y)$. Hennefeld [8] determined that $K(\ell_p)$, $1 < p < \infty$, and $K(c_0)$ are M-ideals in $L(\ell_p)$ and $L(c_0)$ respectively. That $K(\ell_2)$ is an M-ideals in $L(\ell_2)$ was first established by Dixmier [5]. Recently Flynn [7] has characterized $K(\ell_p)$ as the only non-trivial M-ideals in $L(\ell_p)$, $1 < p < \infty$. Saatkamp [10] has shown that $K(\ell_p, \ell_q)$, $1 < p \leqq q < \infty$, is an M-ideal in $L(\ell_p, \ell_q)$ while $K(\ell_1, \ell_p)$, $p \geqq 1$, and $K(\ell_p, \ell_\infty)$, $1 < p < \infty$, are not M-ideals in the corresponding spaces of linear operators. That $K(\ell_1, C[0,1])$ is not an M-ideal in $L(\ell_1, C[0,1])$ was noted by Mach and Ward [9] who also determined that $K(E, c_0)$ is always an M-ideal in $L(E, c_0)$ for any Banach space E.

In this paper we show that for any Banach space E, $L(\ell_1, E)$ always contains uncountably many distinct M-ideals that are proper closed subspaces of $K(\ell_1, E)$ and which are not complemented in $L(\ell_1, E)$. Using standard duality arguments one obtains the result that infinitely many distinct subspaces of $K(E, c_0)$ (including $K(E, c_0)$ itself) are M-ideals in $L(E, c_0)$. In particular for the case $E = c_0$ this shows that the uniqueness condition enjoyed by $K(\ell_p)$, $p > 1$, is not valid for $K(c_0)$.

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The results in this paper are completely self-contained. Our technique is to utilize the identification of $L(\ell_1, E)$ with the vector-valued sequence space $\ell_\infty(E)$ and to exploit natural decompositions of $\ell_\infty(E)'$ afforded by a class of L-projections on $\ell_\infty(E)'$ induced by certain E'-valued vector measures.

2. **Notation and Terminology**. Let E be a Banach space over R or $\mathbb{C}$. The vector-valued sequence space $\ell_\infty(E)$ is the collection of all sequences $x = (x_n)$ where $x_n \in E$ for all n and $(\|x_n\|) \in \ell_\infty$. Given the norm $\|x\| = \sup_n \|x_n\|$, $\ell_\infty(E)$ is a Banach space. The Banach spaces $c_0(E)$ and $\ell_1(E)$ are similarly defined with norms given by $\sup_n \|x_n\|$ and $\Sigma_n \|x_n\|$ respectively. $c_0(E)$ is a closed subspace of $\ell_\infty(E)$.

Let N denote the set of positive integers and $\mathcal{F}$ the power set of N. Let $A \in \mathcal{F}$, $x \in E$ and χ_A the characteristic function of A. By $\chi_A \cdot x$ we mean the sequence $(\chi_A(n)x)$. The linear span of the collection of all $\chi_A \cdot x$, $A \in \mathcal{F}$, $x \in E$ will be denoted by $S(E)$. Any such $s \in S(E)$ admits a unique representation $\Sigma_{i=1}^n \chi_{A_i} \cdot a_i$ where the a_i's are distinct elements of E and $A_1, \ldots, A_n$ is a disjoint decomposition of N. The closure of $S(E)$ in $\ell_\infty(E)$ is denoted by $k_\infty(E)$. A direct argument shows that $k_\infty(E)$ consists of those $(x_n) \in \ell_\infty(E)$ such that $\{x_n | n \in N\}$ is a relatively compact subset of E. Consequently $k_\infty(E)$ is always a proper closed subspace of $\ell_\infty(E)$ if $\dim E = \infty$.

By $bva(\mathcal{F}, E)$ we shall mean the collection of all finitely additive E-valued set functions $\mu : \mathcal{F} \to E$ such that

$$\|\mu\| = \sup\left\{\sum_{i=1}^{n} \|\mu(A_i)\| \,\middle|\, n \in N,\ N = \bigcup_{i=1}^{n} A_i \text{ (disjoint)}\right\} < \infty$$

With the above total variation norm $bva(\mathcal{F}, E)$ is a Banach space.

For $A \in \mathcal{F}$ and $x = (x_n) \in \ell_\infty(E)$ we define $\pi_A : \ell_\infty(E) \to \ell_\infty(E)$ by $\pi_A(x) = (\chi_A(n)x_n)$. If $A = \{1, 2, \ldots, m\}$ we denote π_A by π_m. The identity map on E is denoted by id_E and ${}^t u : F' \to E'$ denotes the transpose of the linear map $u : E \to F$.

3. ***L*-Projections on $\ell_\infty(E)'$**. Let E be any Banach space and let $\phi \in \ell_\infty(E)'$. A finitely additive vector measure $\mu(\phi) : \mathcal{F} \to E'$ is constructed as follows: For $A \in \mathcal{F}$ define the linear form $\mu(\phi)(A)$ on E by

$$\langle x, \mu(\phi)(A)\rangle = \phi(x_A \cdot x) \qquad (x \in E). \tag{3.1}$$

For each $x \in E$

$$|\langle x, \mu(\phi)(A)\rangle| \leqq \|\phi\| \, \|\chi_A \cdot x\|_\infty \leqq \|\phi\| \, \|x\|$$

and hence $\mu(\phi)(A) \in E'$ with $\|\mu(\phi)(A)\| \leqq \|\phi\|$. Clearly $\mu(\phi)$ is finitely additive. Further note that $\mu(\phi) = 0$ if and only if ϕ vanishes on $k_\infty(E)$. We now establish that $\mu(\phi)$ is of bounded variation.

3.2 LEMMA: *Let E be any Banach space and let $\phi \in \ell_\infty(E)'$. Then $\mu(\phi) \in bva(\mathcal{F}, E')$ and $\|\mu(\phi)\| = \|\phi | k_\infty(E)\|$.*

PROOF: Let $\phi \in \ell_\infty(E)'$; $A_1, \ldots, A_n$ a partition of N and $x_1, \ldots, x_n$ elements of E with $\|x_i\| \leqq 1$. Let $\alpha_i = sgn\, \langle x_i, \mu(\phi)(A_i)\rangle$. Then

$$\sum_{i=1}^{n} \left| \langle x_i, \mu(\phi)(A_i)\rangle \right| = \sum_{i=1}^{n} \alpha_i \langle x_i, \mu(\phi)(A_i)\rangle$$

$$= \phi\Big(\sum_{i=1}^{n} \alpha_i \chi_{A_i} \cdot x_i\Big) \leqq \|\phi\| \Big\|\sum_{i=1}^{n} \alpha_i \chi_{A_i} \cdot x_i\Big\|_\infty \leqq \|\phi\|$$

It follows that $\Sigma_{i=1}^{n} \|\mu(\phi)(A_i)\| \leqq \|\phi\|$ and so $\|\mu(\phi)\| \leqq \|\phi\|$. Consequently $\mu(\phi) \in bva(\mathcal{F}, E')$ for each $\phi \in \ell_\infty(E)'$.

Now let $\phi \in \ell_\infty(E)'$. By the Hahn–Banach Theorem $\exists\ \tilde{\phi} \in \ell_\infty(E)'$ such that $\tilde{\phi}|k_\infty(E) = \phi|k_\infty(E)$ and $\|\tilde{\phi}\| = \|\phi|k_\infty(E)\|$. Since $\tilde{\phi}-\phi$ vanishes on $k_\infty(E)$, $\mu(\tilde{\phi}) = \mu(\phi)$ and so $\|\mu(\phi)\| = \|\mu(\tilde{\phi})\| \leqq \|\tilde{\phi}\| = \|k_\infty(E)\|$. To establish the reverse inequality let $\epsilon > 0$ be fixed and choose a simple function s with canonical representation $s = \Sigma_{i=1}^{n} \chi_{A_i} \cdot x_i$ where $\|x_i\| \leqq 1$ for $1 \leqq i \leqq n$ and $\|\phi|k_\infty(E)\| - \epsilon \leqq |\phi(s)|$. Now $|\phi(s)| \leqq \Sigma_{i=1}^{n} |\langle x_i, \mu(\phi)(A_i)\rangle| \leqq \Sigma_{i=1}^{n} \|\mu(\phi)(A_i)\| \leqq \|\mu(\phi)\|$ and so $\|\phi|k_\infty(E)\| \leqq \|\mu(\phi)\| + \epsilon$. It follows that $\|\mu(\phi)\| = \|\phi\|k_\infty(E)\|$. □

3.3 Remark: From the above discussion the map $\mu: \ell_\infty(E)' \to bva(\mathcal{F}, E')$ is continuous and ker $\mu = k_\infty(E)^\perp$. For $\omega \in bva(\mathcal{F}, E')$ there exists, by the Hahn–Banach Theorem, $\phi_\omega \in \ell_\infty(E)'$ such that

$$\phi_\omega(s) = \sum_{i=1}^{n} \langle x_i, \omega(A_i)\rangle \qquad \Big(s = \sum_{i=1}^{n} \chi_{A_i} \cdot x_i\Big).$$

Consequently $\mu(\phi_\omega) = \omega$ and so the map μ is surjective. Thus by 3.2 the induced map $\bar{\mu}: \ell_\infty(E)'/k_\infty(E)^\perp \to bva(\mathcal{F}, E')$ is an isometric isomorphism and hence $bva(\mathcal{F}, E')$ is isometric to the dual space of $k_\infty(E)$. In particular $\ell_\infty(E)'$ is isometrically isomorphic to $bva(\mathcal{F}, E')$ if and only if dim $E < \infty$. If dim $E = \infty$ then $bva(\mathcal{F}, E')$ captures but a portion of $\ell_\infty(E)'$ (see section 4).

For $\phi \in \ell_\infty(E)'$, $\|\mu(\phi)\| \leqq \|\phi\|$ and so, in particular, $\Sigma_n \|\mu(\phi)(\{n\})\| \leqq \|\phi\|$. Thus $(\mu(\phi)(\{n\}))_n, \in \ell_1(E')$. Now for each $\phi \in \ell_\infty(E)'$ define the (clearly continuous) linear form $\tau(\phi)$ on $\ell_\infty(E)$ by

(3.4) $$\tau(\phi)(x) = \Sigma_n \langle x_n, \mu(\phi)(\{n\})\rangle \qquad (x = (x_n) \in \ell_\infty(E))$$

Since

$$\|\tau(\phi)\| = \Sigma_n \|\mu(\phi)(\{n\})\| \leqq \|\phi\| \qquad (\phi \in \ell_\infty(E)'),$$

$\tau: \ell_\infty(E)' \to \ell_\infty(E)'$ is continuous. Now $\tau(\phi)(x) = \lim_{m\to\infty} \phi(\pi_m(x))$ for each $x \in \ell_\infty(E)$ (and consequently $\lim_{m\to\infty} \phi((id\text{-}\pi_m)(x))$ exists for each $x \in \ell_\infty(E)$). It follows that for each $\phi \in \ell_\infty(E)'$, $\tau(\phi)$ agrees with ϕ on $c_0(E)$ and so ker $\tau = c_0(E)^\perp$. Since $\mu(\tau(\phi))(A) = \mu(\phi)(A)$ for finite sets $A \in \mathcal{F}$ it is evident that $\tau^2 = \tau$. We now establish that τ is an L-projection on $\ell_\infty(E)'$.

3.5 LEMMA: *For each* $\phi \in \ell_\infty(E)'$

$$\|\phi\| = \|\tau(\phi)\| + \|\phi - \tau(\phi)\|$$

PROOF: Since $\Sigma_n\|\mu(\phi)(\{n\})\| \leqq \|\phi\|$ it follows that $\sigma_m = {}'\pi_m$ converges to τ in the norm topology of $\ell_\infty(E)'$. Moreover π_m is an M-projection on $\ell_\infty(E)$ and hence ${}'\pi_m$ is an L-projection on $\ell_\infty(E)'$. Consequently τ is an L-projection. [see [2], [3]]. □

The τ-map can now be used to generate a commuting family $\{\tau_A|A \in \mathcal{F}\}$ of L-projections on $\ell_\infty(E)'$. For $A \in \mathcal{F}$ let $\sigma_A = {}'\pi_A : \ell_\infty(E)' \to \ell_\infty(E)'$ and define $\tau_A = \sigma_A \circ \tau$. That is,

$$\tau_A(\phi)(x) = \sum_{n \in A} \langle x_n, \mu(\phi)(\{n\})\rangle \qquad (x \in \ell_\infty(E),\ \phi \in \ell_\infty(E)').$$

Since $\mu(\tau_A(\phi)(\{n\})) = \chi_A(n)\mu(\phi)(\{n\})$ we have

$$\begin{aligned}\tau_A(\tau_A(\phi)(x)) &= \sum_{n \in A} \langle x_n, \mu(\tau_A(\phi))(\{n\})\rangle \\ &= \sum_{n \in A} \langle x_n, \mu(\phi)(\{n\})\rangle \\ &= \tau_A(\phi)\end{aligned}$$

and thus $\tau_A^2 = \tau_A$. Furthermore, one easily checks for $A, B \in \mathcal{F}$

[a] $$\tau_A \circ \tau_B = \tau_{A\cap B}$$

[b] $$\tau_{A\cup B} = \tau_A + \tau_B - \tau_{A\cap B}$$

σ_A is an L-projection on $\ell_\infty(E)'$ since π_A is an M-projection on $\ell_\infty(E)$ [1]. Thus

$$\begin{aligned}\|\tau_A(\phi)\| + \|\phi - \tau_A(\phi)\| &= \|\sigma_A(\tau(\phi))\| + \|\phi - \sigma_A(\tau(\phi))\| \\ &\leqq \|\sigma_A(\tau(\phi))\| + \|\tau(\phi) - \sigma_A(\tau(\phi))\| + \|\phi - \tau(\phi)\| \\ &= \|\tau(\phi)\| + \|\phi - \tau(\phi)\| \\ &= \|\phi\| \qquad \text{(by 3.5)}\end{aligned}$$

Hence $\{\tau_A|A \in \mathcal{F}|$ is a family of L-projections on $\ell_\infty(E)'$.

For $A \in \mathcal{F}$ define

$$c_A(E) = \{x = (x_n) \in c_0(E)|x_n = 0 \text{ for } n \notin A\}.$$

Note $c_N(E) = c_0(E)$.

3.6 PROPOSITION. *Let* $A \in \mathcal{F}$

[a] $c_A(E)$ *is an* M*-ideal in* $\ell_\infty(E)$

[b] *If* A *is infinite then* $c_A(E)$ *is not complemented in* $\ell_\infty(E)$

[c] $c_A(E)''$ *is isometrically isomorphic to* $\ell_\infty(E'')$. *In particular, if* A *is infinite,* $c_A(E)$ *is not a dual space.*

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PROOF: [a] It suffices to note ker $\tau_A = c_A(E)^\perp$. Let $x \in c_0(E)$, $\phi \in \ell_\infty(E)'$. Then

$$\phi(x) = \tau(\phi)(x) = \Sigma_n \langle x_n, \mu(\phi)(\{n\})\rangle$$
$$= \sum_{n \in A} \langle x_n, \mu(\phi)(\{n\})\rangle + \sum_{n \in N-A} \langle x_n, \mu(\phi)(\{n\})\rangle.$$

If $\tau_A(\phi) = 0$ then $\mu(\phi)(\{n\}) = 0$ for $n \in A$ and hence $\phi(x) = 0$ if $x \in c_A(E)$. Similarly if $\phi \in c_A(E)^\perp$ then $\mu(\phi)(\{n\}) = 0$ for $n \in A$ and so $\tau_A(\phi) = 0$. Thus $\ell_\infty(E)' = c_A(E)^\perp \oplus_{\ell_1} \text{im}\, \tau_A$.

[b] Let $A \in \mathcal{F}$ be infinite and suppose there exists a continuous projection ξ on $\ell_\infty(E)$ with im $\xi = c_A(E)$. Let $x_0 \in E$, $x_0' \in E'$ be such that $\langle x_0, x_0'\rangle = 1$ and let $\iota: N \to A$ be the natural order preserving bijection. Denote by $\hat{\iota}$ the induced embedding of ℓ_∞ into $\ell_\infty(E)$. That is, for $\beta \in \ell_\infty$,

$$\hat{\iota}(\beta)(n) = \beta_{\iota^{-1}(n)} x_0 \text{ if } n \in A, \qquad \hat{\iota}(\beta)(n) = 0 \text{ if } n \notin A$$

Finally let $\bar{p}$ be the mapping from $c_A(E)$ into c_0 given by

$$\bar{p}(x) = (\langle p_{\iota(n)}(x), x_0'\rangle)_n \qquad (x \in c_A(E))$$

where $p_n: \ell_\infty(E) \to E$ is the n-th coordinate projection map on $\ell_\infty(E)$. Note that $\hat{\iota}(\beta) \in c_A(E)$ if $\beta \in c_0$. Consequently, for $\beta \in c_0$, $\bar{p} \circ \xi \circ \hat{\iota}(\beta) = \bar{p}(\xi(\hat{\iota}(\beta))) = \bar{p}(\hat{\iota}(\beta)) = (\langle p_{\iota(n)}(\hat{\iota}(\beta)), x_0'\rangle)_n = (\langle \beta_n x_0, x_0'\rangle)_n = \beta$. Thus $\bar{p} \circ \xi \circ \hat{\iota}$ gives a continuous projection of ℓ_∞ onto c_0 which is impossible.

[c] if A is infinite then $c_A(E)$ is isometrically isomorphic to $c_0(E)$. The result follows by standard duality arguments. □

For any Banach space $E, L(\ell_1, E)$ is isometrically isomorphic to $\ell_\infty(E)$ under the isometry ρ_E given by $\rho_E(u) = (u(e_n))_n$. One directly verifies that $\rho_E(K(\ell_1, E)) = k_\infty(E)$. For each $A \in \mathcal{F}$ we define $K_A(\ell_1, E) = \overset{-1}{p_E}(c_A(E))$. Note that $K_A(\ell_1, E)$ is always a proper closed subspace of $K(\ell_1, E)$. From 3.6 we have

3.7 COROLLARY: *Let E be any Banach space and let $A \in \mathcal{F}$*
[a] *$K_A(\ell_1, E)$ is an M-ideal in $L(\ell_1, E)$*
[b] *$K_A(\ell_1, E)$ is not complemented in $L(\ell_1, E)$ if A is infinite*
[c] *$K_A(\ell_1, E)''$ is isometrically isomorphic to $L(\ell_1, E'')$ if A is infinite.*

Now using the transpose map $t: L(E, c_0) \to L(\ell_1, E')$ and corollary 3.7 we obtain a family of M-ideals in $L(E, c_0)$ by pulling back the M-ideals $K_A(\ell_1, E')$. We begin with an elementary lemma.

3.8 LEMMA: *Let E be any Banach space and let $t: L(E, c_0) \to L(\ell_1, E')$ denote the transpose map. Then $t(K(E, c_0) = K_N(\ell_1, E')$.*

PROOF: Let $u \in K_N(\ell_1, E')$ and define $v: E \to c_0$ by $v(x) = (\langle x, u(e_n)\rangle)$. Clearly v is continuous and for $x \in E$, $\langle x, {}^t v(e_n)\rangle = \langle v(x), e_n\rangle = \langle x, u(e_n)\rangle$. Thus ${}^t v = u$. Moreover $v \in K(E, c_0)$ since u is compact.

Now for $x' \in E'$, $\beta \in c_0$ let $w: E \to c_0$ denote the rank-1 operator $w(x) = \langle x, x' \rangle \beta$. For each $x \in E$, $n \in N$ $\langle x, {}^t w(e_n) \rangle = \langle w(x), e_n \rangle = \beta_n \langle x, x' \rangle$. Thus $\|{}^t w(e_n)\| \leqq \|x'\| \, |\beta_n|$ and consequently ${}^t w \in K_N(\ell_1, E')$. Since $K(E, c_0)$ is the closed linear span of rank-1 maps it follows that ${}^t u \in K_N(\ell_1, E')$ for each $u \in K(E, c_0)$. □

Let Z be a Banach space, M, Y closed subspaces such that $M \subset Y \subset Z$ and suppose M is an M-ideal in Z. Let τ denote the L-projection on Z' with kernel $M^\perp$. For $\phi \in Y'$ let $\bar{\phi}$ be any continuous linear extension of ϕ to all of Z and define $\bar{\tau}(\phi) = \tau(\bar{\phi})|Y$. It is easy to see that $\bar{\tau}: Y' \to Y'$ is an L-projection on Y' with ker $\bar{\tau} = M^\perp$ (in Y').

3.9 Corollary: *For each $A \in \mathcal{F}$ let $M_A(E, c_0) = t^{-1}(K_A(\ell_1, E'))$. Then $M_A(E, c_0)$ is an M-ideal in $L(E, c_0)$ and if A is infinite, $M_A(E, c_0)''$ is isometrically isomorphic to $L(\ell_1, E''')$.*

Proof: Let $t: L(E, c_0) \to L(\ell_1, E)$ be the transpose map. By 3.8 we have $t(M_A(E, c_0)) = K_A(\ell_1, E')$ and $K_A(\ell_1, E') \subset t(L(E, c_0)) \subset L(\ell_1, E')$. By 3.7 $K_A(\ell_1, E')$ is an M-ideal in $L(\ell_1, E')$ and so by the above discussion $K_A(\ell_1, E')$ is an M-ideal in $t(L(E, c_0))$. Since t is an isometry onto its image it follows that $M_A(E, c_0) = t^{-1}(K_A(\ell_1, E'))$ is an M-ideal in $L(E, c_0)$. □

3.10 Remark: It is of interest to unravel the identifications and exhibit explicitly the form of the L-projections on $L(E, c_0)'$ with kernel $M_A(E, c_0)^\perp$.

We have $t: L(E, c_0) \to L(\ell_1, E')$ (isometry into) and $\rho_{E'}: L(\ell_1, E') \to \ell_\infty(E')$ (isometry onto). We have the following where $\gamma = \rho_{E'} \circ t$

$$\begin{array}{ccc} M_A(E, c_0) & \hookrightarrow & L(E, c_0) \\ \gamma \downarrow & & \gamma \downarrow \\ c_A(E') & \hookrightarrow & \operatorname{im} \gamma \subset \ell_\infty(E') \end{array}$$

Let τ_A be the L-projection on $\ell_\infty(E')'$ with ker $\tau_A = c_A(E')^\perp$ and $\bar{\tau}_A$ the induced L-projection on $(\operatorname{im} \gamma)'$ with ker $\bar{\tau}_A = c_A(E')^\perp$ (in $(\operatorname{im} \gamma)'$). The L-projection on $L(E, c_0)'$ (with kernel $M_A(E, c_0)^\perp$) whose structure we wish to unravel is precisely ${}^t\gamma \circ \bar{\tau}_A \circ {}^t\gamma^{-1}$.

A direct computation shows that

$${}^t\gamma \circ \bar{\tau}_A \circ {}^t\gamma^{-1}(\phi)(u) = \sum_{n \in A} \phi((e_n \otimes e_n') \circ u)$$

Remark: Corollary 3.9 in the case $A = N$ was first established by Mach and Ward [9] using the 3-balls-property. A different proof was given by Saatkamp in [10].

4. **Further Remarks on the Dual of $\ell_\infty(E)$.** In a recent paper [6] the authors have determined certain natural topological decompositions of the strong dual of $L_b(Z, E)$ where Z, E are Hausdorff locally convex spaces and $L_b(Z, E)$ carries the topology of uniform convergence on the bounded subsets of Z. It is shown that if E is quasi-complete and id_E has a suitable resolution into quasi-compact maps then $K_b(Z, E)^\perp$ is topologically complemented in $L_b(Z, E)'$ where $K_b(Z, E)$ is the space of continuous

linear maps $u:Z \to E$ which take bounded sets to relatively compact sets (such maps are called quasi-compact). $K_b(Z,E)$ is always a closed subspace of $L_b(Z,E)$ whenever E is quasi-complete, and $K_b(Z,E)$ coincides with the space of compact linear maps when Z, E are Banach spaces. It follows that for any Banach space Z and a large class of Banach spaces E [see definition 4.1] $K(Z,E)^\perp$ is complemented in $L(Z,E)'$. In particular $K(\ell_1,E)^\perp$ is topologically complemented in $L(\ell_1,E)'$ and $k_\infty(E)^\perp$ is complemented in $\ell_\infty(E)'$. In this section we explicitly construct projections on $\ell_\infty(E)'$ with kernels $k_\infty(E)^\perp$ and study the relationship between these projections and the L-projection τ on $\ell_\infty(E)'$ defined in section 3.

For simplicity we make the following definition

4.1 Definition: A Banach space E is admissible if there exists a sequence of compact operators $\zeta_n:E \to E$ such that

(4.1.1) $$x = \Sigma_n \zeta_n(x) \qquad (x \in E)$$

and

(4.2.1) $$\sup\left\{\left\|\sum_{n=1}^{m} \beta_n\zeta_n\right\| \,\middle|\, \beta = (\beta_n) \in \ell_\infty, \|\beta\|_\infty \leqq 1, m \geqq 1\right\} < \infty.$$

Examples of admissible Banach spaces are afforded by Banach spaces of the type $(\Sigma_n \otimes X_n)_{\ell_p}$, $p \geqq 1$ or $(\Sigma_n \otimes X_n)_{c_0}$ where $\dim X_n < \infty$.

Let E be an admissible Banach space. Define $\chi:\ell_\infty(E)' \to \ell_\infty(E)'$ by

(4.2) $$\chi(\phi)(x) = \Sigma_n \phi(\zeta_n x_1, \zeta_n x_2, \ldots) \qquad (x \in \ell_\infty(E),\ \phi \in \ell_\infty(E)').$$

From 4.1.2 it follows that the series in 4.2 is absolutely convergent and in turn that χ is a continuous operator on $\ell_\infty(E)'$. A direct computation shows that for $\phi \in \ell_\infty(E)'$, $\mu(\chi(\phi)) = \mu(\phi)$ and hence $\phi|k_\infty(E) = \chi(\phi)|k_\infty(E)$. Thus $\chi^2 = \chi$ and $\ker \chi = k_\infty(E)^\perp$. Furthermore $\tau \circ \chi = \chi \circ \tau = \tau$ where τ is the L-projection defined in section 3. Consequently $\tau, \chi - \tau$ and $\mathrm{id} - \chi$ are mutually orthogonal projections on $\ell_\infty(E)'$ and thus we obtain the topological decomposition

(4.3) $$\ell_\infty(E)' = \ker(\mathrm{id} - \tau) \oplus \ker(\mathrm{id} - \chi + \tau) \oplus \ker \chi$$

where $\ker(\mathrm{id} - \tau)$ is isometrically isomorphic to $\ell_1(E')$ and $\ker(\mathrm{id} - \chi + \tau)$ is isometrically isomorphic to $bva_0(\mathcal{F}, E')$, the space of E'-valued vector measures which vanish on the finite subsets of N. Note that card $bva_0(\mathcal{F}, E') \geqq 2^c$.

From 3.5 τ is always an L-projection. If χ is also an L-projection then for each $\phi \in \ell_\infty(E)'$

$$\begin{aligned}\|\phi\| &= \|\chi(\phi)\| + \|\phi - \chi(\phi)\| \\ &= \|\tau(\chi(\phi))\| + \|\chi(\phi) - \tau(\chi(\phi))\| + \|\phi - \chi(\phi)\| \\ &= \|\tau(\phi)\| + \|\chi(\phi) - \tau(\phi)\| + \|\phi - \chi(\phi)\|\end{aligned}$$

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If $E = c_0$ then χ is an L-projection. This follows from the direct calculation ${}^t\rho_{c_0}^{-1} \circ \sigma \circ {}^t\rho_{c_0} = \chi$ where ρ_{c_0} is the isometry of $L(\ell_1, c_0)$ onto $\ell_\infty(c_0)$ and $\sigma = {}^t\gamma \circ \bar{\tau} \circ {}^t\gamma^{-1}$ with γ and $\bar{\tau}$ defined as in section 3 (with $E = \ell_1$). Consequently, in this case we have the interesting decomposition

$$\ell_\infty(c_0)' = \text{im}\,\tau \oplus_{\ell_1} \text{im}\,(\chi - \tau) \oplus_{\ell_1} k_\infty(c_0)^\perp$$

where im τ is isometrically isomorphic to $\ell_1(\ell_1)$ and im $\tau \oplus$ im $(\chi - \tau)$ is isometrically isomorphic to $bva(\mathcal{F}, \ell_1)$ which is in turn isometrically isomorphic to $\ell_1(\ell'_\infty)$.

The functionals in $k_\infty(c_0)^\perp$ are of a more exotic nature than those in $bva(\mathcal{F}, \ell_1)$. To obtain examples of such functionals let η, ξ be elements of ℓ'_∞ where η extends the limit functional on c and $\xi \in c_0^\perp$. Then $\xi \otimes \eta \in \ell_\infty(\ell_\infty)'$ where $\xi \otimes \eta(x) = \xi((\langle x_n, \eta\rangle)_n)$ for $x = (x_n) \in \ell_\infty(\ell_\infty)$. Clearly $\xi \otimes \eta$ vanishes on $c_0(\ell_\infty)$. Let

$$\pi = \rho_{\ell_\infty} \circ t \circ \rho_{c_0}^{-1} : \ell_\infty(c_0) \to \ell_\infty(\ell_\infty)$$

where $t: L(\ell_1, c_0) \to L(\ell_1, \ell_\infty)$ is the transpose map and $\rho_{c_0}: L(\ell_1, c_0) \to \ell_\infty(c_0)$, $\rho_{\ell_\infty}: L(\ell_1, \ell_\infty) \to \ell_\infty(\ell_\infty)$ are the canonical isometries. From 3.8 $\pi(k_\infty(c_0)) = c_0(\ell_\infty)$ and hence $\xi \otimes \eta \circ \pi \in k_\infty(c_0)^\perp$. Moreover the map $\xi \to \xi \otimes \eta \circ \pi$ is injective for if $\beta \in \ell_\infty$ and if $x_n = \Sigma_{i=1}^n \beta_i e_i$ for each n then $\xi \otimes \eta \circ \pi((x_n)) = \xi(\beta)$. It follows that card $k_\infty(c_0)^\perp \geqq$ card $c_0^\perp = 2^c$.

REMARKS: Let $j_\infty(c_0)$ denote the closed linear span of all forms of the type $\xi \otimes \eta \circ \pi$, $\xi \in c_0^\perp$ and $\eta \in \ell'_\infty$ extending the limit functional. It would be of interest to describe the quotient space $k_\infty(c_0)^\perp / j_\infty(c_0)$.

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