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ON THE EXISTENCE OF NONINNER AUTOMORPHISMS OF ORDER TWO IN FINITE 2-GROUPS

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Abstract

In this paper we prove that every nonabelian finite 2-group with a cyclic commutator subgroup has a noninner automorphism of order two fixing either $\Phi(G)$ or $Z(G)$ elementwise. This, together with a result of Peter Schmid on regular p-groups, extends our result to the class of nonabelian finite p-groups with a cyclic commutator subgroup.

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1. Introduction

In 1966 Gaschütz [8], using cohomological techniques, showed that every nonabelian finite p-group, for p prime, possesses a noninner automorphism of order a power of p. A long-standing conjecture closely related to Gaschütz's result asks whether every nonabelian finite p-group G admits a noninner automorphism of order p (see, for example, [11, Problem 4.13]). By a reduction theorem, Deaconescu and Silberberg in [6] reduced the verification of the conjecture to the case where $C_G(Z(\Phi(G))) = \Phi(G)$. As an application of this result, it is seen, by a cohomological result of Schmid [12], that every regular p-group has a noninner automorphism of order p fixing $\Phi(G)$ elementwise (see [2]). The conjecture has also been established for some other classes of nonabelian finite p-groups. Liebeck [10] proved that if p is an odd prime, then every finite p-group G of class 2 has a noninner automorphism of order p fixing the Frattini subgroup of G elementwise. Using the above-mentioned reduction theorem, Abdollahi [1] showed that Liebeck's result remains true in the case where $p = 2$ by showing that every finite 2-group G of class 2 has a noninner automorphism of order two leaving either $\Phi(G)$ or $\Omega_1(Z(G))$ fixed elementwise. Recently, he extended the result of [2] to a wider family of p-groups containing the finite p-groups of class 2. In fact, he proves that if G is a nonabelian finite p-group such that $G/Z(G)$ is powerful then G has a noninner automorphism which fixes either $\Phi(G)$ or $\Omega_1(Z(G))$ elementwise. Also in a paper appearing in this issue of the *Bulletin* [13], Shabani-Attar proves the conjecture for another special class of finite p-groups.

The object of the present paper is to verify the validity of the conjecture for another class of finite p-groups, namely the class of finite p-groups with cyclic commutator subgroup. This class of p-groups has been studied in [5, 7], for example. It is worth mentioning here that we need only treat the challenging case where $p = 2$ because it is well known that every finite p-group, p odd, with cyclic commutator subgroup is regular.

In order to meet our aim we first improve the main results of [1, 2] for the case $p = 2$ and thereby prove the existence of noninner automorphisms of order two for the case of nonabelian finite 2-groups having cyclic commutator subgroup. We shall derive the following theorem.

Theorem 1.1. *Let G be a nonabelian finite p-group with cyclic commutator subgroup. Then G has a noninner automorphism of order p fixing $\Phi(G)$ elementwise whenever $p > 2$, and fixing either $\Phi(G)$ or $Z(G)$ elementwise whenever $p = 2$.*

In this paper, p is always a prime and all the groups considered are finite. The notation used is standard. In particular, for a p-group G and an integer $i \geq 0$, we write $\Omega_i(G) = \langle g \in G \mid g^{p^i} = 1\rangle$ and $\mho_i(G) = \langle g^{p^i} \mid g \in G\rangle$. We use $\Gamma_i(G)$ and $Z_i(G)$ for the ith terms of the lower and upper central series of G, respectively. The notation $d(G)$ is used to denote the minimal number of generators of G. We also recall that a group G is called an (internal) central product of its subgroups $G_1, \ldots, G_n$ if $G = G_1 \ldots G_n$ and $[G_i, G_j] = 1$ for all $1 \leq i < j \leq n$; in this situation we shall write $G = G_1 * \cdots * G_n$.

2. Powerful p-groups

In this section our aim is to state stronger forms of the main theorems of [1, 2] in the case $p = 2$. The main result of the present paper relies upon these stronger versions.

To avoid constant repetition, it will be our convention that the term *special Gaschütz automorphism of G* will mean a noninner automorphism of order p which fixes $\Phi(G)$ elementwise whenever $p > 2$ and fixes either $\Phi(G)$ or $Z(G)$ elementwise whenever $p = 2$.

Lemma 2.1. *Let G be a finite 2-group of class 2. If G' is cyclic, then G has a special Gaschütz automorphism.*

Proof. Since G' is cyclic, by [2, Lemma 2.1], we may suppose that $Z(G)$ is cyclic. Now we argue as in [1, Theorem] to complete the proof. □

Lemma 2.2. *Let*

$$G = \langle a, b \mid a^{2^r} = b^{2^s}, b^{2^{s+t}} = 1, b^a = b^{2^t+1}\rangle,$$

where r, s, t are integers with $2 \leq t < s \leq r$. Then G has a special Gaschütz automorphism.

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Proof. We have $Z(G) = \langle a^{2^s}, b^{2^s}\rangle$. Define the mappings α , β on G such that $\alpha(a) = a^{1-2^{r-1}} b^{2^{s-1}}$ and $\alpha(b) = b$ whenever $r > s$, and $\beta(a) = a^{1-2^{r-1}} b^{2^{s-1}}$ and $\beta(b) = a^{-2^{r-1}} b^{1+2^{s-1}}$ whenever $r = s$. It can be shown that both α and β are noninner automorphisms of G of order two fixing $Z(G)$ elementwise. □

The following theorem extends [2, Theorem 2.6].

Theorem 2.3. *Let G be a nonabelian finite p-group. If $G/Z(G)$ is powerful, then G has a special Gaschütz automorphism.*

Proof. The theorem holds for $p > 2$ by [2, Theorem 2.6]. Let $p = 2$. In this case, we proceed by means of arguments already used in [2, Theorem 2.6] to observe that G has a special Gaschütz automorphism unless perhaps if

$$G = \langle a, b \mid a^{2^r} = b^{2^s}, b^{2^{s+t}} = 1, b^a = b^{2^t+1}\rangle,$$

where r, s, t are integers with $2 \le t \le s \le r$. If $t = s$, then $G' \le Z(G)$ and the result follows from Lemma 2.1. Otherwise, Lemma 2.2 can be applied to yield the result. □

The following result is a special case of Theorem 2.3 improving the main result of [1].

Corollary 2.4. *Let G be a finite p-group of class 2. Then G has a special Gaschütz automorphism.*

3. *p*-groups with cyclic commutator subgroups

In this section we shall establish a number of fundamental lemmas which, when taken together, give a proof for our main result (Theorem 1.1).

For the sake of brevity, we say that a nonabelian finite p-group G is a *$\mathcal{G}$-group* if every nonabelian subgroup H of G has a noninner automorphism of order p fixing either $Z(H)$ or $\Phi(H)$ elementwise. In view of Corollary 2.4, every finite p-group of class 2 is a $\mathcal{G}$-group.

Lemma 3.1. *Let G be a finite p-group which is a central product of two subgroups H and K. If H is a $\mathcal{G}$-group, then G has a noninner automorphism of order p fixing either $Z(G)$ or $\Phi(G)$ elementwise.*

Proof. Let H_0 be a minimal element of the set

$$\{L \le H \mid G = L * K, L \text{ is nonabelian}\}.$$

If $Z(H_0) \le \Phi(H_0)$, then H_0 has a noninner automorphism of order p fixing $Z(H_0)$ elementwise, and the result is proved by [1, Remark 2.5]. Now suppose that $Z(H_0) \nleq \Phi(H_0)$. Hence H_0 has a maximal subgroup M_0 such that $Z(H_0) \nleq M_0$. So $H_0 = M_0 Z(H_0)$. Since H_0 is nonabelian, so is M_0. It follows, by the minimality of H_0,

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that M_0K is a proper subgroup of G. Let M be a maximal subgroup of G containing M_0K. Then

$$MZ(G) = MZ(H_0)Z(K) \geq M_0KZ(H_0) = M_0Z(H_0)K = H_0K = G.$$

Hence G has a noninner automorphism of order p fixing $\Phi(G)$ elementwise by [6, Theorem]. □

The following lemma plays a key role in the proof of our main theorem (Theorem 1.1).

Lemma 3.2. *Let G be a nonabelian finite 2-group with cyclic commutator subgroup. If $\Gamma_3(G) \leq \mho_2(G')$, then G has a special Gaschütz automorphism.*

Proof. According to [7], we may write $G = A_1 * A_2 * \cdots * A_n * B$, where B is an abelian subgroup, $A_1, A_2, \ldots, A_n$ are 2-generator subgroups, and the classes of $A_2, \ldots, A_n$ are equal to 2. If $n > 1$, then on setting $H = A_2 * \cdots * A_n * B$, by Corollary 2.4 and Lemma 3.1, we see that G has a special Gaschütz automorphism. Therefore we assume that $n = 1$. In this case, $G' = A_1'$ and $\Gamma_3(G) = \Gamma_3(A_1)$. Hence, by [5], we may choose the generators x, y for A_1 such that $y^xZ(A_1) = y^{1+2^s}Z(A_1)$, where $s \geq 2$. Since $A_1' = \langle [x, y] \rangle$, we have $A_1'Z(A_1) \leq \mho_2(A_1)Z(A_1)$. Now $Z(G) = Z(A_1)B$ implies that

$$G'Z(G) = A_1'Z(A_1)B \leq \mho_2(A_1)Z(A_1)B \leq \mho_2(G)Z(G).$$

It follows that $G/Z(G)$ is powerful and the result holds by Theorem 2.3. □

In what follows, we shall show that the result of Lemma 3.2 remains true in the case of nonabelian finite 2-groups satisfying $\Gamma_3(G) \nleq \mho_2(G')$. This case is much more complicated. We shall proceed by a series of lemmas which lead to the main theorem of the paper.

Lemma 3.3. *Let G be a finite 2-group with cyclic commutator subgroup such that $\Gamma_3(G) \nleq \mho_2(G')$. Suppose that G has no noninner automorphism of order two fixing $\Phi(G)$ elementwise. Then:*

(i) *$Z(G)$ is cyclic;*
(ii) *$\Gamma_i(G) = \mho_{i-2}(G')$ for all $i \geq 2$, and hence $|G' \cap Z(G)| = 2$.*

Proof. (i) By [2, Lemma 2.1], $\Omega_1(Z(G)) \leq G'$. Therefore $Z(G)$ is cyclic since G' is cyclic.

(ii) We assume that $G' = \langle g \rangle$ and $|G'| = 2^m$. It follows from $\Gamma_3(G) \nleq \mho_2(G')$ that $1 \neq \Gamma_3(G) = \langle g^2 \rangle$. Now since $\exp(\Gamma_{i+1}(G)/\Gamma_{i+2}(G))$ divides $\exp(\Gamma_i(G)/\Gamma_{i+1}(G))$ for all i, we deduce that $\Gamma_i(G) = \langle g^{2^{i-2}} \rangle$ for all $i \geq 2$. We set $I = Z(G) \cap G'$ and observe that $I = \Gamma_{j+2}(G)$ for some $j < m$, whence

$$\langle g^{2^{j+1}} \rangle = \Gamma_{j+3}(G) = [G, I] = 1,$$

and we conclude that $j = m - 1$. Thus $|I| = 2$, as required. □

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Lemma 3.4. *Let G be a finite 2-group with cyclic commutator subgroup such that $\Gamma_3(G) \nleq \mho_2(G')$. Suppose that G has no special Gaschütz automorphism. Then:*

(i) $Z_2(G) \leq \Phi(G)$;
(ii) $Z_2(G)$ *is abelian;*
(iii) $\Omega_1(Z_2(G)) = \bigcup_{M \in \mathcal{M}} \Omega_1(Z(M))$, *where $\mathcal{M}$ denotes the set of all maximal subgroups of G.*

Proof. (i) By [6, Remark 1], $Z(G) \leq \Phi(G)$. Let $x \in Z_2(G) \setminus Z(G)$. It follows that $1 \neq [x, G] \leq G' \cap Z(G)$, and hence $|G : C_G(x)| = 2$ by Lemma 3.3. Consequently, $M = C_G(x)$ is a maximal subgroup of G, and we have $x \in C_G(M) \leq C_G(\Phi(G)) \leq C_G(Z(\Phi(G))) = \Phi(G)$, the latter equality holding by virtue of [6, Theorem]. Thus $Z_2(G) \leq \Phi(G)$.

(ii) Assume that $x_1, x_2 \in Z_2(G) \setminus Z(G)$. Then $1 \neq [x_1, G] \leq Z(G) \cap G'$, and hence $|G : C_G(x_1)| = 2$ because $|G' \cap Z(G)| = 2$. This implies that $M_1 = C_G(x_1)$ is maximal in G. Similarly, $M_2 = C_G(x_2)$ is maximal in G. By [6, Theorem],

$$x_1 \in C_G(M_1) \leq C_G(\Phi(G)) \leq C_G(Z(\Phi(G))) = \Phi(G) \leq M_2 = C_G(x_2).$$

Hence $[x_1, x_2] = 1$ and $Z_2(G)$ is abelian.

(iii) According to Lemma 3.3, $Z(G)$ is cyclic. In order to complete the proof, it therefore suffices to show that

$$\Omega_1(Z_2(G)) \setminus Z(G) = \bigcup_{M \in \mathcal{M}} \Omega_1(Z(M)) \setminus Z(G).$$

On setting $\Omega_1(Z(G)) = \langle z_0 \rangle$, we see that $G' \cap Z(G) = \langle z_0 \rangle$ since $|G' \cap Z(G)| = 2$ by Lemma 3.3. Let $a \in \Omega_1(Z_2(G)) \setminus Z(G)$. Then $1 \neq [a, G] \leq Z(G) \cap G' = \langle z_0 \rangle$. Thus $|G : C_G(a)| = 2$, which implies that the subgroup $M = C_G(a)$ is maximal. Now since $a \in Z(M)$, we conclude that $a \in \Omega_1(Z(M)) \setminus Z(G)$. Suppose next that $M \in \mathcal{M}$ and $x \in \Omega_1(Z(M)) \setminus Z(G)$. For any y in $G \setminus M$, $1 = [x^2, y] = [x, y]^2$. Hence $[x, y] \in \Omega_1(G')$. Moreover, $M = C_G(x)$ shows that $[x, y] \neq 1$, from which we find that $[x, y] = z_0 \in Z(G)$. Therefore $x \in \Omega_1(Z_2(G)) \setminus Z(G)$ as required. □

Lemma 3.5. *Let G be a 2-generator finite 2-group with cyclic commutator subgroup. Suppose that $\Gamma_3(G) \nleq \mho_2(G')$ and $|Z(G)| = 2$. Then G has a special Gaschütz automorphism.*

Proof. Suppose to the contrary that G has no special Gaschütz automorphism. It follows from $\Gamma_3(G) \nleq \mho_2(G')$ that $|G'| \geq 4$. We therefore assume that $|G'| = 2^m$, where $m \geq 2$. Since G is a 2-generator 2-group, G has exactly three maximal subgroups. Indeed, there are elements a_0, a_1, a_2 in G such that the subgroups $M_i = \langle \Phi(G), a_i \rangle$ for $0 \leq i \leq 2$ are all maximal subgroups of G. Evidently the set $\{a_i, a_j\}$, where $0 \leq i < j \leq 2$, is a generating set for G. At least one of these sets, say B, has the property that $[b, G'] \nleq \mho_2(G')$ for all b in B. Without loss of generality, we may assume that $B = \{a_1, a_2\}$. We let $G' = \langle u_0 \rangle$ and $u_1 = u_0^{2^{m-2}}$. For every x in G,

$$[x, u_1] = [x, u_0^{2^{m-2}}] = [x, u_0]^{2^{m-2}} \in \langle u_0^{2^{m-1}} \rangle.$$

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Consequently $|G : C_G(u_1)| \leq 2$. Now since u_1 is of order four, $u_1 \notin Z(G)$, which shows that $|G : C_G(u_1)| = 2$. It follows that $C_G(u_1)$ is a maximal subgroup of G. Thus there exists an element b_0 in $\{a_0, a_1, a_2\}$ such that $C_G(u_1) = \langle b_0, \Phi(G)\rangle$. Since $[b_0, u_0]^{2^{m-2}} = [b_0, u_1] = 1$, we have $[b_0, G'] \leq \mho_2(G')$. So $b_0 \notin B$, and thus $b_0 = a_0$. We put $v_1 = [a_0, a_1]$ and observe that $G' = \langle v_1\rangle$. There exists an odd integer t such that $[v_1, a_1] = v_1^{2t}$. Evidently $(v_1^{2t})^k = v_1^{-2}$ for some odd integer k. Now we define the mapping α_1 by setting $\alpha_1(a_0) = a_0 v_1^k$ and $\alpha_1(a_1) = a_1$. By [4, Theorem 3.2], α_1 extends to an automorphism of G. We have

$$\alpha_1^2(a_0) = \alpha_1(a_0 v_1^k) = a_0 v_1^k [a_0 v_1^k, a_1]^k = a_0 v_1^k (v_1^{v_1^k} v_1^{2tk})^k = a_0 (v_1^2 v_1^{2tk})^k = a_0,$$

whence α_1 has order two. Note that α_1 is an inner automorphism of G induced by some $g_1 \in G$ because G has no special Gaschütz automorphism. Clearly $g_1^2 \in Z(G) < G'$. Since M_1 is the only maximal subgroup of G containing $C_G(a_1)$, we have $g_1 \in M_1$. Also, since

$$\alpha_1(v_1) = [a_0 v_1^k, a_1] = v_1 [v_1^k, a_1] = v_1 v_1^{2tk} = v_1^{-1},$$

we conclude that $g_1 \notin C_G(u_1)$. Hence $g_1 \in M_1 \setminus \Phi(G)$. Similarly, we can find some $g_2 \in M_2 \setminus \Phi(G)$ such that $g_2^2 \in G'$. It is easy to see that $G = \langle g_1, g_2\rangle$ and $|G : G'| = 4$. Thus G has a special Gaschütz automorphism, by [3, Proposition 4.10] and [2, Corollary 2.4]. This contradiction completes the proof of the lemma. □

Lemma 3.6. *Let G be a 2-generator finite 2-group with cyclic commutator subgroup of order four. Then G has a special Gaschütz automorphism.*

Proof. Let u be a generator of G'. If $u \in Z(G)$, then the lemma follows from Corollary 2.4. So we assume that $u \notin Z(G)$. Thus $C_G(u)$ is a maximal subgroup of G and there are generators $a, b \in G$ outside $C_G(u)$, whence $u^a = u^b = u^{-1}$. We can suppose that $u = [a, b]$. Then $[a^2, b] = u^a u = 1$ and similarly $[a, b^2] = 1$, so $a^2, b^2 \in Z(G)$. Thus the group $G/Z(G)$ can be generated by two involutions and therefore is a dihedral group. Its commutator subgroup has order two, so $G/Z(G)$ in fact has order eight. By [6, Theorem] we may assume that $Z(G) \leq \Phi(G)$. Now, if $\bar{x} \in G/Z(G)$ is an element of order four, $\Phi(G) = \langle x^2\rangle Z(G)$ and $C_G(\Phi(G)) = \langle x\rangle Z(G)$, which is a maximal subgroup of G. We conclude that G has a special Gaschütz automorphism by [6, Theorem]. □

Lemma 3.7. *Let G be a 2-generator finite 2-group with cyclic commutator subgroup. Then for any x in $\Phi(G)$, there are two elements a and b in G such that $G = \langle a, b\rangle$ and $x = a^{2^n}[a, b]^k$ for some positive integers n and k.*

Proof. Let $G = \langle u, v\rangle$ and $x \in \Phi(G)$. Since $\Phi(G) = \mho_1(G)$, we may write $xG' = u^{2^n r} v^{2^m s} G'$, where r and s are positive odd integers and $m, n \geq 1$. Without loss of generality, one may assume that $m \geq n$. Now it is readily seen that the elements $a = u^r v^{2^{m-n} s}$ and $b = v$ satisfy the conditions of the lemma. □

Lemma 3.8. *Let G be a 2-generator finite 2-group with cyclic commutator subgroup. Suppose that $\Gamma_3(G) \nleq \mho_2(G')$. Then G has a special Gaschütz automorphism.*

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PROOF. Suppose to the contrary that G has no special Gaschütz automorphism. In view of [6, Theorem], Lemmas 3.3 and 3.5, we see that $Z(G)$ is a cyclic subgroup of $\Phi(G)$, $|Z(G) \cap G'| = 2$, and $|Z(G)| = 2^l > 2$. Let $Z(G) = \langle z \rangle$, $z_1 = z^{2^{l-2}}$, and $Z(G) \cap G' = \langle z_0 \rangle$. By Lemma 3.7, we may choose the elements a, b such that $G = \langle a, b \rangle$ and $z = a^{2^n} u^k$ for some positive integers n and k, where $u = [a, b]$. We have $G' = \langle u \rangle$ and $2^m = |u| \geq 8$ by Lemma 3.6. Let $u_1 = u^{2^{m-2}}$ and $N = \langle a \rangle G' Z(G) \cap \langle b \rangle G' Z(G)$. Evidently $N \leq \Phi(G)$ because any of two distinct maximal subgroups of G intersect in the Frattini subgroup of G. We shall obtain a contradiction in the following steps.

Step 1. $b^2 \in N$. Assume to the contrary that $b^2 \notin N$. Since $G/N = \langle aN \rangle \times \langle bN \rangle$, the mapping α_1 defined by $\alpha_1(a^i b^j x) = a^i (bz_1)^j x$, where $x \in N$, $0 \leq i < o(aN)$ and $0 \leq j < o(bN)$, is an automorphism of G. As $z_1 \in \Phi(G)$, $G = \langle a, bz_1 \rangle$. Thus by [4, Theorem 3.2], there is an automorphism α_2 of G fixing a and sending bz_1 to bz_1u_1. Let α be the composite automorphism $\alpha_2\alpha_1$. Then $\alpha(a) = a$ and $\alpha(b) = bz_1u_1$. Obviously α is noninner because $z_1 \notin G'$. On the other hand, $[u^k, a] = 1$ since $z^a = z$. Hence $[a, u]^k = [u, a]^k = 1$ and

$$\begin{aligned} \alpha(z) &= a^{2^n}[a, bz_1u_1]^k = a^{2^n}([a, z_1u_1][a, b]^{z_1u_1})^k = a^{2^n} u^k [a, u^{2^{m-2}}]^k \\ &= a^{2^n} u^k([a, u]^k)^{2^{m-2}} = a^{2^n} u^k = z. \end{aligned}$$

It follows that α fixes $Z(G)$ elementwise. Finally, we observe that

$$\alpha^2(b) = bz_1u_1z_1[a, bz_1u_1]^{2^{m-2}} = bz_1^2u_1(u[a, u]^{2^{m-2}})^{2^{m-2}} = bz_1^2u_1^2[a, u]^{2^{2m-4}}.$$

But $[a, u] \in \langle u^2 \rangle$ and $z_1^2 = u_1^2 = z_0$, so $\alpha^2(b) = b$. Therefore α has order 2, that is, α is a special Gaschütz automorphism, a contradiction.

Step 2. k is even. Suppose that k is odd. Then $G' = \langle za^{-2^n} \rangle \leq Z(G)\langle a \rangle = H$. Clearly H is abelian. Since $b^2 \in N \leq H$, we also have $|G : H| = 2$. Therefore, by [6, Theorem], G has a noninner automorphism of order two fixing $\Phi(G)$ elementwise, a contradiction.

Step 3. If $N = \Phi(G)$ then $a^2 \notin G'Z(G)$, and in particular $n \neq 1$. Moreover, there exists $c \in G \setminus G'Z(G)$ such that $c^2 \in G'Z(G)$ and

$$G/G'Z(G) = \langle aG'Z(G) \rangle \times \langle cG'Z(G) \rangle.$$

Let $N = \Phi(G)$. Since $a \notin \langle b \rangle G'Z(G)$ and $G'Z(G) \leq N$, we conclude that $N \leq \langle a^2 \rangle G'Z(G)$. Thus since $\langle a^2 \rangle G'Z(G) \leq \Phi(G) = N$, we have $\Phi(G) = \langle a^2 \rangle G'Z(G)$. So

$$|G : \langle a \rangle G'Z(G)| = \frac{|G : \Phi(G)|}{|\langle a \rangle G'Z(G) : \Phi(G)|} = 2.$$

Hence $\langle a \rangle G'Z(G)$ is a maximal subgroup of G. So $\langle aG'Z(G) \rangle$ is a cyclic maximal subgroup of the finite abelian 2-group $G/G'Z(G)$. Hence it follows from [9, Theorem 5.3.1] that there exists an element $c \in G$ such that

$$G/G'Z(G) = \langle aG'Z(G) \rangle \times \langle cG'Z(G) \rangle$$

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and $o(cG'Z(G)) = 2$. By [2, Corollary 2.3], $d(Z_2(G)/Z(G)) = 2$. Assume that $a^2 \in G'Z(G)$. This implies that $\Phi(G) = G'Z(G)$ and thus $\Phi(G)/Z(G)$ is cyclic. By Lemma 3.4, $Z_2(G)/Z(G) \leq \Phi(G)/Z(G)$, and hence $Z_2(G)/Z(G)$ is cyclic, a contradiction. It follows that $a^2 \notin G'Z(G)$. So $n \neq 1$, as required.

Step 4. $N \neq \Phi(G)$. Assume the contrary. Then $n \geq 2$ and we can choose c as in Step 3. Clearly $G = \langle a, c\rangle$, and thus $u = [a, c]^{t_0}$, where t_0 is an odd integer. We set $t = t_0 k$, $v = [a, c]$ and $v_1 = v^{2^{m-2}}$. So $z = a^{2^n} v^t$. Note that $a^2 \notin G'Z(G)$, by Step 3. Then it is straightforward to verify that the mapping α_1 defined by $\alpha_1(a^i c^j x) = (az_1)^i c^j x$, where $x \in G'Z(G)$, $0 \leq i < o(aG'Z(G))$ and $0 \leq j < 2$, is an automorphism of G. As $z_1 \in \Phi(G)$, $G = \langle az_1, c\rangle$. Thus by [4, Theorem 3.2], there is an automorphism α_2 of G such that $\alpha_2(az_1) = az_1v_1$ and $\alpha_2(c) = c$. We put $\alpha = \alpha_2\alpha_1$. Then α is an automorphism of G with $\alpha(a) = az_1v_1$ and $\alpha(c) = c$. It follows from $z_1 \notin G'$ that α is noninner. Since $z^a = z$, we conclude that $[v^t, a] = 1$. Now we write $t = 2^r s$, where $r \geq 1$ and s is odd. First assume that $r \leq m - 2$. In this case it follows, from $v_1 \in \langle v^t\rangle$ and $[v^t, a] = 1$, that $[v_1, a] = 1$. We have

$$\alpha(z) = (az_1v_1)^{2^n}[az_1v_1, c]^t = a^{2^n} v^t [v, c]^{2^{m-2}t}.$$

Thus since $[v, c] \in \langle v^2\rangle$ and t is even, $\alpha(z) = z$. We also obtain

$$\begin{aligned}\alpha^2(a) &= \alpha(az_1v_1) = az_1v_1z_1[az_1v_1, c]^{2^{m-2}} = az_1^2v_1^2[v_1, c]^{2^{m-2}}\\ &= a[v, c]^{2^{2m-4}} = a.\end{aligned}$$

This shows that α is a special Gaschütz automorphism, a contradiction. Next we suppose that $r \geq m$, and so $z = a^{2^n}$. Since

$$[v_1, a] = [v^{2^{m-2}}, a] = [v, a]^{2^{m-2}} \in \Omega_1(G') = \langle z_0\rangle,$$

it follows that $(av_1)^4 = a^4$. Hence

$$\alpha(z) = (az_1v_1)^{2^n} = ((az_1v_1)^4)^{2^{n-2}} = (a^4)^{2^{n-2}} = a^{2^n} = z.$$

Moreover α has order two, again a contradiction.

It remains to consider the case where $r = m - 1$. In this case, we put $z' = zz_0$. Since $z_0 = v^t$, we observe that $Z(G) = \langle z'\rangle$ and $z' = a^{2^n}$. As above, we may construct the automorphism β defined by $\beta(a) = az'_1v_1$ and $\beta(b) = b$, where $z'_1 = (z')^{2^{l-2}}$, to reach a contradiction.

Step 5. $a^2 \notin N$. Suppose to the contrary that $a^2 \in N$. By Step 1, we have $b^2 \in N$, and hence G/N is elementary abelian. This implies that $N = \Phi(G)$, a contradiction.

Step 6. $n = 1$. Assume that this is not the case. So $n \geq 2$. Since k is even, we may write $k = 2^r s$, where s is odd and $r \geq 1$. Using arguments similar to those in the proof of Step 4, we observe that if $r \leq m - 2$ or $r \geq m$ then G has a special Gaschütz automorphism α such that $\alpha(a) = az_1u_1$ and $\alpha(b) = b$. Also, when $r = m - 1$, there exists a special Gaschütz automorphism β with $\beta(b) = b$ and $\beta(a) = a(zz_0)^{2^{l-2}}$.

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Step 7. The final contradiction. According to Step 6, $n = 1$, and so $a^2 = zu^{-k} \in N$ contradicting the fact that $a^2 \notin N$ as shown in Step 5. □

The next lemma improves Lemma 3.8.

LEMMA 3.9. *Let G be a finite 2-group with cyclic commutator subgroup. Suppose that $\Gamma_3(G) \nleq \mho_2(G')$. Then G has a special Gaschütz automorphism.*

PROOF. Assume to the contrary that G has no special Gaschütz automorphism. According to Lemmas 3.8 and 3.3, we may suppose that $d(G) > 2$, $Z(G)$ is cyclic, and $G' \cap Z(G) = \langle z_0 \rangle$ has order two. Let $\mathcal{M}$ be the set of all maximal subgroups of G. The following steps lead to a contradiction.

Step 1. For each $M \in \mathcal{M}$, $|\Omega_1(Z(M))| \leq 4$. Let $|\Omega_1(Z(M))| > 2$, and let h and h' be distinct elements from $\Omega_1(Z(M)) \setminus \langle z_0 \rangle$. It is clear that $[hh', x] = 1$ for all $x \in M$. Let $a \in G \setminus M$. Since $h \in Z(M)$ and $[h, a] \in M$, $1 = [h^2, a] = [h, a]^2$. Since $1 \neq [h, a] \in \Omega_1(G') = \langle z_0 \rangle$, we conclude that $[h, a] = z_0$. Similarly, we observe that $[h', a] = z_0$. Hence $[hh', a] = 1$, and so $[hh', G] = 1$. Thus $hh' \in \langle z_0 \rangle$, and therefore $|\Omega_1(Z(M)) \setminus \langle z_0 \rangle| = 2$. It follows that $|\Omega_1(Z(M))| = 4$.

Step 2. $\mathcal{M}$ contains two distinct elements M_1 and M_2 such that

$$|\Omega_1(Z(M_2))| = |\Omega_1(Z(M_1))| = 4.$$

For any M in $\mathcal{M}$, we set $\tilde{M} = \Omega_1(Z(M)) \setminus \langle z_0 \rangle$. It is readily seen that if M_i and M_j are distinct elements of $\mathcal{M}$, then $\tilde{M}_i$ and $\tilde{M}_j$ are disjoint. By Lemma 3.4,

$$\Omega_1(Z_2(G)) = \bigcup_{M \in \mathcal{M}} \Omega_1(Z(M)).$$

It follows that

$$|\Omega_1(Z_2(G))| = \sum_{M \in \mathcal{M}} |\tilde{M}| + 2.$$

On the other hand, by [2, Corollary 2.3], $r = d(Z_2(G)/Z(G)) = d(G) > 2$. Thus $d(Z_2(G)) > 2$, and so by Lemma 3.4 (ii) we conclude that $|\Omega_1(Z_2(G))| \geq 2^r \geq 8$. Now, if t is the number of elements M of $\mathcal{M}$ for which $|\Omega_1(Z(M))| = 4$, then $8 \leq 2t + 2$, and hence $t \geq 3$. In particular, there are at least two distinct elements M_1 and M_2 in $\mathcal{M}$ such that $|\Omega_1(Z(M_i))| = 4$, $i = 1, 2$.

Step 3. Obtaining a contradiction. Let M_1 and M_2 be as above, and set $N = M_1 \cap M_2$. We choose the elements a_1 and a_2 such that $M_i = \langle N, a_i \rangle$, $i = 1, 2$. Let $h_i \in \Omega_1(Z(M_i)) \setminus \langle z_0 \rangle$, $i = 1, 2$. It can be shown that the mapping

$$\alpha(x a_1^{i_1} a_2^{i_2}) = x(a_1 h_1)^{i_1} (a_2 h_2)^{i_2} \quad (x \in N, 1 \leq i_1, i_2 \leq 2)$$

is an automorphism of G. By Lemma 3.4, $h_1, h_2 \in \Phi(G) \leq N$, and so α has order two. Also, α is not inner. To see this simply note that if α were inner, then h_1 and h_2 would be in $\Omega_1(G')$, a contradiction. Thus α is a noninner automorphism of order two fixing $\Phi(G)$ elementwise. This contradiction completes the proof. □

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The next result follows immediately from Lemmas 3.2 and 3.9.

THEOREM 3.10. *Let G be a nonabelian finite 2-group with cyclic commutator subgroup. Then G has a special Gaschütz automorphism.*

As mentioned earlier, every regular p-group, for p odd, has a noninner automorphism of order p fixing $\Phi(G)$ elementwise. Theorem 1.1 now follows at once using the above theorem.

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