RADICAL Q-ALGEBRAS

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1. Introduction. The purpose of this paper is to exhibit various Q-algebras (quotients of uniform algebras) which are Jacobson radical. We begin by noting easy examples of nilpotent Q-algebras and Q-algebras with dense nil radical. Then we describe two ways of constructing semiprime, Jacobson radical Q-algebras. The first is by directly constructing a uniform algebra and an ideal. This produces a nasty Q-algebra as the quotient of a nice uniform algebra (in the sense that it is a maximal ideal of $R(X)$ for some $X \subseteq \mathbb{C}$). The second way is by using results of Craw and Varopoulos to show that certain weighted sequence algebras are Q-algebras. In fact we show that a weighted sequence algebra is Q if the weights satisfy (i) $w(n+1)/w(n) \downarrow 0$ and (ii) $(w(n+1)/w(n)) \in l^p$ for some $p \geqq 1$, but may be non-Q if either (i) or (ii) fails. This second method produces nice Q-algebras which are quotients of rather horrid uniform algebras as constructed by Craw's Lemma.

We summarize the terminology to be used. If X is a compact Hausdorff space, then $C(X)$ denotes the Banach algebra of all continuous complex-valued functions on X, with the sup norm. A *uniform algebra* is a closed subalgebra of some $C(X)$. A *Q-algebra* is a Banach algebra A which is bicontinuously isomorphic with the quotient of a uniform algebra by a closed ideal. If the isomorphism is isometric, then A is said to be an *IQ-algebra*. The complex numbers are denoted by $\mathbb{C}$, and $\Delta = \{z \in \mathbb{C} : |z| \leqq 1\}$. The *disc algebra* $A(\Delta)$ is the subalgebra of $C(\Delta)$ consisting of functions analytic on $\operatorname{int}(\Delta)$. For any algebra A, we write A^n for the linear span of the products of length n in A, and we say that A is *nilpotent* if $A^n = \{0\}$ for some positive integer n. An algebra A is *semiprime* if it has no non-zero nilpotent (two-sided) ideals. An element $x \in A$ is *nilpotent* if $x^n = 0$ for some n; A is *nil* if every element of A is nilpotent; and the *nil radical* of an algebra A is the largest nil ideal of A.

If A is a Banach algebra, the nil radical of A is the sum of the nilpotent ideals of A, ([4]). Thus, when we look for examples of non-nilpotent, Jacobson radical Q-algebras, the two extreme cases to be considered are algebras with dense nil radical and semiprime algebras.

2. Nilpotent Q-algebras.

(2.1) REMARK. Let $M = \{f \in A(\Delta) : f(0) = 0\}$. Then M is a uniform algebra and, for any integer $n > 1$, M^n is a closed ideal of M. Thus $A = M/M^n$ is an IQ-algebra, with $A^n = \{0\}$.

3. Q-algebras with dense nil radical. In view of (2.1), we can construct non-nilpotent Q-algebras with dense nil radical by taking a type of direct sum of the algebras M/M^n $(n \geqq 2)$.

(3.1) DEFINITION. Let $\{A_i\}_{i \in I}$ be a family of Banach algebras. By the *c_0-direct sum* $A = c_0\text{-}\bigoplus_{i \in I} A_i$ we shall mean the subalgebra of the (unrestricted) direct product $\prod_{i \in I} A_i$ consisting of those families $\{f_i\}_{i \in I}$ such that for every $\varepsilon > 0$ the set $\{i \in I : \|f_i\| > \varepsilon\}$ is finite. The norm on A is the sup norm $\|\{f_i\}\| = \sup\{\|f_i\| : i \in I\}$.

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(3.2) Lemma. *The c_0-direct sum of any family of IQ-algebras is an IQ-algebra.*

Proof. Let $\{A_i\}_{i\in I}$, $\{U_i\}_{i\in I}$, $\{J_i\}_{i\in I}$ be such that, for each $i\in I$, J_i is a closed ideal of the uniform algebra U_i and A_i is isometrically isomorphic with U_i/J_i. Then c_0-$\oplus A_i$ is isometrically isomorphic with c_0-$\oplus U_i/c_0$-$\oplus J_i$.

(3.3) Proposition. *There exists a non-nilpotent Q-algebra with dense nil radical.*

Proof. Take the algebra c_0-$\oplus_{n=2}^{\infty} M/M^n$.

Here, again, both the Q-algebra and its uniform algebra (in this case c_0-$\oplus_{n=2}^{\infty} M$) are easily accessible objects. In the next two sections only one of these will be at all tractable.

4. Semiprime, Jacobson radical Q-algebras. First method. Our first construction for semiprime, Jacobson radical Q-algebras exhibits such an algebra as a quotient of a maximal ideal of $R(X)$ for a certain plane set X.

(4.1) Definition. If X is a compact plane set, then we denote by $R_0(X)$ the algebra of all rational functions on X with poles off X, and by $R(X)$ the closure of $R_0(X)$ in $C(X)$.

(4.2) Theorem. *There is a Jacobson radical Q-algebra with no divisors of zero which may be realized as the quotient of a maximal ideal of $R(X)$, for some compact $X \subseteq \mathbb{C}$.*

Proof. This construction is based heavily on the example of a non-trivial, normal uniform algebra due to McKissick [5], an account of which may also be found in [6, §27]. He proves the following lemma.

(4.3) Lemma ([5] Lemma 2, [6] Lemma 27.6). *Let D be an open disc in the complex plane. For every $\varepsilon > 0$, there exists a sequence $\{\Delta_k\}$ of open discs contained in D and a sequence $\{r_n\}$ of rational functions such that:*

(i) $\sum_{k=1}^{\infty}(\text{radius } \Delta_k) < \varepsilon$;
(ii) *the poles of r_n lie in $\bigcup\{\Delta_k : 1 \leqq k \leqq n\}$;*
(iii) *the sequence $\{r_n\}$ converges uniformly on the complement of $\bigcup\{\Delta_k : 1 \leqq k < \infty\}$ to a function which is identically zero outside D and is nowhere zero on $D\backslash\bigcup\{\Delta_k : 1 \leqq k < \infty\}$.*

Let $\{D_m\}$ be a sequential arrangement of all open discs in the plane having centres at points $x_m + iy_m$ (x_m, y_m rational, $x_m + iy_m \neq 0$) and having rational radii $\rho_m < \frac{1}{2}|x_m + iy_m|$. We can clearly arrange the numbering so that $\frac{1}{2}|x_m + iy_m| > 2^{-m}$ ($m = 1, 2, 3, \ldots$). Applying Lemma (4.3), with $D = D_m$ and $\varepsilon = m^{-m}$, we obtain a double sequence of discs $\Delta_{m,k}$ ($m, k = 1, 2, 3, \ldots$) of radii $\rho_{m,k}$, with $\sum_{k=1}^{\infty} \rho_{m,k} < m^{-m}$ ($m = 1, 2, 3, \ldots$). Let $\sigma_{m,k}$ be the distance of the set $\Delta_{m,k}$ from zero. Then $\sigma_{m,k} \geqq |x_m + iy_m| - \rho_m > 2^{-m}$, and so

$$\sum_{m,k=1}^{\infty} \rho_{m,k}\sigma_{m,k}{}^{-N} < \sum_{m=1}^{\infty} (m^{-1}2^N)^m < \infty \tag{1}$$

for all $N \geqq 1$. Furthermore, we have, as in [5], that if

$$X = \{x : |z| \leqq 1\}\backslash\bigcup\{\Delta_{m,k} : 1 \leqq m, k < \infty\},$$

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then, for every non-zero $x \in X$, there exists an $f \in R(X)$ such that f vanishes on a neighbourhood of zero, but $f(x) \neq 0$. We define measures μ_N ($N = 1, 2, 3, \ldots$) as follows (c.f. [6], proof of Lemma 24.1). For $n = 1, 2, 3, \ldots$, let

$$X_n = \{z: |z| \leqq \varepsilon\} \backslash \bigcup \{\Delta_{m,k}: 1 \leqq m, k \leqq n\},$$

where $0 < \varepsilon \leqq 1$. Let $\mu_{N,n}$ be measures on ∂X_n defined by

$$\mu_{N,n}(f) = \int_{\partial X_n} z^{-N} f(z)\, dz \quad (f \in C(\Delta)),$$

where ∂X_n is the boundary of X_n taken in the positive direction. Then, by (1), the $\mu_{N,n}$ are norm-bounded, uniformly in n, for each N. Hence, we may find a sequence of integers $n_1, n_2, n_3, \ldots$ and measures μ_N on X such that μ_{N,n_i} converges weak* to μ_N as $i \to \infty$, for each N. The measures μ_N depend on ε. However, by Cauchy's Theorem, $\mu_N(f)$ is independent of ε for $f \in R_0(X)$ and hence, by continuity, for all $f \in R(X)$. Let

$$I_N = \{f \in R(X): \mu_n(f) = 0 \quad (1 \leqq n \leqq N)\},$$
$$I = \bigcap \{I_N : 1 \leqq N < \infty\}.$$

The I_N and I are closed subspaces of $R(X)$. We shall show that they are ideals and that the algebra I_1/I is a Jacobson radical algebra with no divisors of zero. First, let us note that, for $f \in R_0(X)$, we have $\mu_N(f) = \mu_{N,n_i}(f)$ for all sufficiently large i, ($N = 1, 2, 3, \ldots$) and so $\mu_N(f) = f^{(N-1)}(0)$, the $(N-1)$th derivative of f at 0. (In particular, we have $\mu_1(f) = f(0)$ for all $f \in R(X)$, so I_1 is just the maximal ideal associated with the point 0.) By Leibniz' theorem,

$$\mu_n(fg) = \sum_{r=0}^{n-1} \binom{n-1}{r} \mu_{r+1}(f)\mu_{n-r}(g),$$

for $f, g \in R_0(X)$ and hence, since the μ_i are continuous, for all $f, g \in R(X)$. From this we obtain: first, that the I_N and I are ideals; and, secondly, that if $f \in I_{n-1} \backslash I_n$, $g \in I_{m-1} \backslash I_m$, then $fg \notin I_{m+n-1}$. Hence, I_1/I has no divisors of zero. Suppose I_1/I is not Jacobson radical. Then $R(X)/I$ has a maximal ideal other than I_1/I. This ideal must be of the form M/I, where M is a maximal ideal of $R(X)$ containing I and associated with a point of X (= Spec$(R(X))$) other than 0. This is impossible, since, for every non-zero $x \in X$ there is an $f \in R(X)$ which vanishes on a neighbourhood of 0, but which does not vanish at x. This has $\mu_N(f) = 0$ ($N = 1, 2, 3, \ldots$), by taking ε suitably small in the definition of μ_N, and so $f \in I$. However, $f(x) \neq 0$ implies $f \notin M$. Thus I_1/I is a Jacobson radical Q-algebra with no divisors of zero.

5. Semiprime, Jacobson radical Q-algebras. Second method: weighted sequence algebras. One large and highly accessible class of semiprime, Jacobson radical algebras are the weighted sequence algebras $W(w)$, defined below, with rapidly decreasing weight functions w. In this section, we show that the results of Varopoulos [7] give a simple sufficient condition on w for $W(w)$ to be a Q-algebra. Unfortunately, although these Q-algebras are of fairly simple structure, the uniform algebras of which they are quotients appear only through the complicated construction in the proof of Craw's Lemma ([3], Lemma (3.1); [1], §50

Proposition 5). It would be interesting to know if they are expressible as quotients of simpler uniform algebras; e.g., uniform algebras on plane sets.

(5.1) DEFINITION. A *weight function* is a real-valued function w on $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ satisfying (i) $w(x) > 0$ and (ii) $w(x+y) \leqq w(x)w(y)$, for all $x, y \in \mathbb{Z}^+$. Such a weight function is said to be *rapidly decreasing* if $w(n)^{1/n} \to 0$ as $n \to \infty$. The *weighted sequence algebra* $W(w)$ is defined to be the convolution algebra of complex-valued functions f on $\mathbb{Z}^+$ such that

$$\|f\| = \sum_{n=1}^{\infty} w(n)\,|f(n)| < \infty.$$

This algebra has no divisors of zero and is Jacobson radical if and only if w is rapidly decreasing.

(5.2) DEFINITION. A Banach algebra A is said to be *injective* if the map of the algebraic tensor product $A \otimes A$ into A induced by the multiplication on A is continuous when $A \otimes A$ is given the injective tensor product norm (i.e. the least crossnorm).

Varopoulos ([7, Theorem 1]) shows that every commutative injective algebra is a Q-algebra. (The converse is false: l^p with pointwise multiplication is Q for $1 \leqq p \leqq \infty$, but injective only for $p = 1$ and $p = \infty$.) Our main theorem is a sufficient condition for the injectivity of $W(w)$.

(5.3) THEOREM. *If w is a weight function such that*

(i) $\dfrac{w(n+1)}{w(n)} \downarrow 0$ *and*

(ii) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{w(n+1)}{w(n)}\right)^p < \infty$, for some $p \geqq 1$,

then $W(w)$ is injective.

Note that $w(n+1)/w(n) \downarrow 0$ implies that w is rapidly decreasing, so $W(w)$ is Jacobson radical.

Proof. The methods used in [7] to establish conclusion (ii) of the lemma on p. 6 apply here to show that $W(w)$ is injective if

$$\sup_n \sum_{m=1}^{n} \left(\frac{w(m+n)}{w(m)w(n)}\right)^2 < \infty. \tag{1}$$

Suppose w satisfies (i) and (ii). For $n \geqq N = [p/2]+1$, we have

$$\frac{w(m+n)}{w(m)w(n)} = \frac{w(m+N)}{w(m)w(N)} \prod_{r=N+1}^{n} \frac{w(m+r)w(r-1)}{w(m+r-1)w(r)}$$

$$\leqq \frac{w(m+N)}{w(m)w(N)} \qquad \text{by (i)}$$

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$$= \frac{1}{w(N)} \frac{w(m+1)}{w(m)} \frac{w(m+2)}{w(m+1)} \cdots \frac{w(m+N)}{w(m+N-1)}$$

$$\leqq \frac{1}{w(N)} \left(\frac{w(m+1)}{w(m)} \right)^N \qquad \text{by (i)}$$

$$\leqq \frac{1}{w(N)} \left(\frac{w(m+1)}{w(m)} \right)^{p/2}.$$

Then

$$\sup_{n \geqq N} \sum_{m=1} \left(\frac{w(m+n)}{w(m)w(n)} \right)^2 \leqq \frac{1}{w(N)^2} \sum_{m=1}^{\infty} \left(\frac{w(m+1)}{w(m)} \right)^p < \infty.$$

But

$$\sup_{n<N} \sum_{m=1}^{n} \left(\frac{w(m+n)}{w(m)w(n)} \right)^2 \leqq N-1,$$

so (1) holds and so $W(w)$ is injective.

We conclude this paper with two examples to show that neither (i) nor (ii) is, by itself, sufficient to make $W(w)$ a Q-algebra. Notice that, by Corollary 3 of **[2]**, both these examples produce Arens regular, non-Q algebras $W(w)$.

(5.4) Example. *There is a weight function w with $w(n+1)/w(n) \downarrow 0$ such that $W(w)$ is not a Q-algebra.*

Proof. We define w by induction. More precisely, we define an increasing sequence of integers $\{r_N\}$ and the values $w(1), \ldots, w(r_N - 1)$, by induction on N. First, $r_1 = 2$ and $w(1) = 1$. Now suppose $w(1), \ldots, w(r_N - 1)$ have been defined so that

$$w(s+t) \leqq w(s)w(t) \quad (1 \leqq s, t, s+t < r_N) \tag{1}$$

and

$$\frac{w(s+1)}{w(s)} \geqq \frac{w(t+1)}{w(t)} \quad (1 \leqq s \leqq t < r_N). \tag{2}$$

We define

$$w(r_N) = \frac{1}{N} w(r_N - 1)^2, \tag{3}$$

and, temporarily, we define

$$w(r_N + k) = w(r_N)^{k+1} \quad (1 \leqq k < \infty).$$

With this definition of a weight function w, the algebra $W(w)$ is not Arens regular, by **[2**, Theorem 1], and so not a Q-algebra (c.f. **[2]**, **[3]**). Therefore by **[7**, p. 1], $W(w)$ has the property $(\mathscr{P}_N)$: there exists a positive integer $p \geqq 1$, elements $x_1, \ldots, x_p$ of norm $\leqq 1$ and a homogeneous polynomial P of positive degree in p variables such that

$$\| P(x_1, \ldots, x_p) \| > N^{\deg P} \| P \|_\infty,$$

where $\| P \|_\infty = \sup\{ |P(z_1, \ldots, z_p)| : z_j \in \mathbb{C}, |z_j| \leqq 1, \ (1 \leqq j \leqq p)\}$.

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We approximate each of the x_j by an x'_j such that $\|x'_j\| \leqq 1$ and $x'_j(n) = 0$ for all sufficiently large n, and we make this approximation so close that we still have

$$\| P(x'_1, \ldots, x'_p) \| > N^{\deg P} \| P \|_\infty. \tag{4}$$

We choose an integer r_{N+1} so large that $x'_j(n) = 0$, $P(x'_1, \ldots, x'_p)(n) = 0$ $(1 \leqq j \leqq p, n \geqq r_{N+1})$ Now, we can redefine $w(n)$ $(n \geqq r_{N+1})$ and still have (4) true, and so $(\mathscr{P}_N)$. We make our temporary definition of $w(r_N), \ldots, w(r_{N+1}-1)$ permanent, and this completes the induction step in the definition of w, the inequalities (1) and (2) being easily verified. The resulting, inductively defined, function w is a weight function such that $W(w)$ has the property $(\mathscr{P}_N)$ for all N. Hence, by [7, p. 1], $W(w)$ is not a Q-algebra, but $w(n+1)/w(n) \downarrow 0$, by (2) and (3).

(5.5) Example. *There is a rapidly decreasing weight function w with*

$$\sum_{n=1}^{\infty} \frac{w(n+1)}{w(n)} < \infty$$

such that $W(w)$ is not a Q-algebra.

This follows immediately from the following result.

(5.6) Theorem. *For every positive function g on $\mathbb{Z}^+$, there exists a rapidly decreasing weight function w with $w(n+1)/w(n) \leqq g(n)$ for all n, such that $W(w)$ is not a Q-algebra.*

The proof of this theorem will be based on the following sufficient condition for $W(w)$ to be non-Q.

(5.7) Theorem. *Let w be a weight function such that, for every integer $R \geqq 1$, there exists $q \in \mathbb{Z}^+$ with $w(rq) = w(q)^r$ $(r = 1, 2, \ldots R)$. Then $W(w)$ is not a Q-algebra.*

The proof of (5.7) is based on two lemmas.

(5.8) Lemma. *Let A be a Q-algebra. Then there is a constant $C > 0$ such that, for all $f \in A$ with $\|f\| \leqq 1$ and all polynomials $P(z) = \sum_{n=1}^{m} a_n z^n$, we have*

$$\|P(f)\| \leqq C \sup\left\{ \left| \sum_{n=1}^{m} a_n n^{1/3} z^n \right| : |z| \leqq 1 \right\}.$$

Proof. By putting $C = 1$, $\alpha = 0$, $\varepsilon = 1/3$ in Theorem 3 of [7].

(5.9) Lemma. *For every $C > 0$ there exists a polynomial $P_C(z) = \sum_{n=1}^{N} a_n z^n$ such that*

$$\sum_{n=1}^{N} |a_n| > C \sup\left\{ \left| \sum_{n=1}^{N} a_n n^{1/3} z^n \right| : |z| \leqq 1 \right\}.$$

Proof. Let $a_n = (1/n) e^{in \log n}$. Then $\sum_{n=1}^{\infty} |a_n| = \infty$, but, by [8, V (4.2)],

$$\sup\{ |\textstyle\sum_{n=1}^{N} a_n n^{1/3} z^n | : |z| \leqq 1, N \in \mathbb{Z}^+ \} < \infty.$$

Proof of Theorem (5.7). Suppose $W(w)$ is a Q-algebra. Apply (5.8) to $A = W(w)$, obtaining a constant C, and then (5.9), with this same C, obtaining a polynomial P_C of degree $N(C)$. By the hypothesis on w, there exists q such that $w(nq) = w(q)^n$ $(1 \leqq n \leqq N(C))$. Now if $f \in W(w)$ is defined by $f(s) = 0$ $(s \neq q)$, $f(q) = 1/w(q)$, then $\|f\| = 1$ and

$$\|P_C(f)\| = \sum_{n=1}^{N} |a_n w(q)^{-n}| w(nq)$$
$$= \sum_{n=1}^{N} |a_n|.$$

Consequently, the conclusion of (5.8) for this f and $P = P_C$ contradicts the conclusion of (5.9). Therefore $W(w)$ cannot be a Q-algebra.

Proof of Theorem (5.6). We may assume that $g(n) \to 0$ as $n \to \infty$. We define the weight function w inductively by blocks, as in (5.4). Let $r_1 = 2$, $w(1) = 1$. Suppose $w(1), \ldots, w(r_N - 1)$ have been defined with

$$w(x+y) \leqq w(x)w(y) \ (1 \leqq x, y, x+y < r_N), \ w(x+1)/w(x) \leqq g(x) \ (1 \leqq x < r_N - 1),$$

and such that

$$(*) \text{ there exists } q \text{ with } w(rq) = w(q)^r \quad (1 \leqq r \leqq R),$$

holds for all $R < N$. We put $r_{N+1} = Nr_N + 1$ and define $w(r_N), \ldots, w(r_{N+1} - 1)$ so that (*) holds for $R = N$, with $q = r_N$.

Let us write $r_N = a$, and let $\eta = \min\{g(n): 1 \leqq n < r_{N+1}\}$. Then define

$$w(a) = \min\{w(x)w(y)\eta^a : 1 \leqq x, y < a\}.$$

(The main point of this construction is that $w(a)$ may be chosen very small compared with all previous $w(x)$.) For $1 \leqq s < N$, $0 \leqq t < a$ and $s = N$, $t = 0$ define

$$w(sa+t) = w(a)^s w(t)\eta^t$$

where $w(0) = 1$, formally. It is straightforward to check that

$$w(x+y) \leqq w(x)w(y) \quad (1 \leqq x, y, x+y \leqq r_{N+1})$$

and $w(x+1)/w(x) \leqq g(x)$ $(1 \leqq x < r_{N+1} - 1)$. This completes the induction step. The resulting function w is a weight function with $w(n+1)/w(n) \leqq g(n)$ for all n, and, since $g(n) \to 0$, it follows that w is rapidly decreasing. Further, w satisfies (*) for all R and so, by (5.7), $W(w)$ is not a Q-algebra.

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