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SYSTEMS OF CONGRUENCES

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An interesting problem is to discuss the solutions of the congruences in n variables $(x)=(x_1, \ldots, x_n)$,

$$\frac{P_n}{x_r}+a \equiv 0 \pmod{x_r}, \qquad r = 1, 2, \ldots, n, \tag{1}$$

where

$$P_n = x_1x_2\cdots x_n, \qquad a = \pm 1.$$

The case $n=3$ for positive x and $a=1$, was proposed as Problem 179 by G. E. J. Barbeau in the Canadian Mathematical Bulletin **14** (1971), p. 129.

It is obvious that every two of the x are relatively prime. It follows immediately that (1) is equivalent to the single congruence,

$$\frac{P_n}{x_1}+\cdots+\frac{P_n}{x_n}+a \equiv 0 \pmod{P_n}. \tag{2}$$

For if (2) holds, then $P_n/x_r+a\equiv 0 \pmod{P_n}$ for $r=1, 2, \ldots, n$. If (1) holds,

$$\frac{P_n}{x_1}+\cdots+\frac{P_n}{x_n}+a \equiv 0 \pmod{x_r}, \qquad r = 1, 2, \ldots, n,$$

and so (2) follows. Then from (2),

$$\frac{P_n}{x_1}+\cdots+\frac{P_n}{x_n}+a = y\cdot P_n, \tag{3}$$

where y is an integer. A trivial solution is given by $|x_1|=\cdots=|x_n|=1$. Further if for s of the x we have $|x|=1$, then (1) reduces to the corresponding problem in $n-s$ variables. Hence we may exclude without further mention the cases when some of the $|x|$ equal one. When $y=0$, it is difficult to find all the solutions of (3) when $n\geq 4$ though one can do so when $n=3$ on putting $x_2+x_3=t$ where x_3 is arbitrary and t is a divisor of $a-x_3^2$. We shall not hereafter consider the solution with $y=0$. I find all the other solutions for $2\leq n\leq 5$. There is no theoretical difficulty when $n\geq 6$ but much detailed work is involved.

Suppose then that $2\leq n\leq 5$. Write

$$y = \frac{1}{x_1}+\cdots+\frac{1}{x_n}+\frac{a}{x_1\cdots x_n}. \tag{4}$$

[1] Professor Mordell died on 12th March, 1972.

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We may assume that $|x_1|<|x_2|<\cdots<|x_n|$, and since the x's are relatively prime in pairs, that

$$|x_1|\geq 2,\quad |x_2|\geq 3,\quad |x_3|\geq 5,\quad |x_4|\geq 7,\quad |x_5|\geq 11.$$

Furthermore $|x_1 x_2\cdots x_n|\geq 6$. Hence

$$|y|\leq \tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{5}+\tfrac{1}{7}+\tfrac{1}{11}+\tfrac{1}{6}<2,$$

and so $y=\pm 1$, and

$$(5)\qquad \pm 1=\frac{1}{x_1}+\cdots+\frac{1}{x_n}+\frac{a}{x_1\cdots x_n}$$

We show now that $|x_1|=2$. For if $|x_1|\geq 3$, then

$$|x_2|\geq 5,\quad |x_3|\geq 7,\quad |x_5|\geq 11,\quad |x_7|\geq 13.$$

Then from (5),

$$1\leq \tfrac{1}{3}+\tfrac{1}{5}+\tfrac{1}{7}+\tfrac{1}{11}+\tfrac{1}{13}+\tfrac{1}{15},$$

and this is false. We shall consider only the solution with $x_1=2$, since those with $x_1=-2$ can be found by writing $-x$ for x and $(-1)^{n-1}$ for a.

We show now that $|x_2|=3$ if $n\leq 4$. For if $|x_2|>3$, then from (5)

$$1\leq \tfrac{1}{2}+\tfrac{1}{5}+\tfrac{1}{7}+\tfrac{1}{9}+\tfrac{1}{70},$$

which is false.

We now consider the various values of n.

$n=2$.

Here

$$2+x_2+a=2yx_2.$$

Since $x_2 \mid 2+a$, we have $a=1$ and $x_2=\pm 3$, but $x_2=-3$ corresponds to $y=0$ which is not being discussed.

Then $x_1=2$, $x_2=\pm 3$, $a=1$ is a solution.

$n=3$. We mention again, once and for all, that there are solutions with $x_1=-2$, and also with $|x_1|=1$ etc.

Here $x_2\neq -3$, since from (5)

$$1\leq \frac{1}{2}-\frac{1}{3}+\frac{1}{|x_3|}+\frac{1}{6},$$

which is false. Now from (1),

$$6+a\equiv 0 \pmod{x_3},$$

and so $|x_3|=5$ if $a=-1$, $|x_3|=7$ if $a=1$.

It is easily seen from (5), that $x_3\neq -5$, $x_3\neq -7$, and so we have the solutions,

$$x_1=2,\quad x_2=3,\quad x_3=5,\quad a=-1,$$
$$x_1=2,\quad x_2=3,\quad x_3=7,\quad a=1.$$

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$\underline{n=4}$. As before, $x_2 \neq -3$. Since

$$6x_3+a \equiv 0 \pmod{x_4}, \qquad 6x_4+a \equiv 0 \pmod{x_3},$$

we have

$$6(x_3+x_4)+a = z_1x_3x_4,$$

where z_1 is an integer, i.e.

$$6\left(\frac{1}{x_3}+\frac{1}{x_4}\right)+\frac{a}{x_3x_4} = z_1.$$

Since $|x_3| \geq 5$, $|x_4| \geq 7$, $|z_1| < 3$ and so $|z_1| = 1$.

Now $z_1x_3x_4 \equiv a \pmod 3$, and $2x_3x_4+a \equiv 0 \pmod 3$. Hence $z_1 \equiv 1 \pmod 3$ and so $z_1=1$. Hence

$$(x_3-6)(x_4-6) = 6^2+a,$$

and we have the following solutions. If $a=1$

$$x_3-6 = \pm 1, \qquad x_4-6 = \pm 37;$$

and so

$$(x_3, x_4) = (5, -31); \quad (7, 43).$$

If $a=-1$,

$$x_3-6 = \pm 1, \pm 5; \qquad x_4-6 = \pm 35, \pm 7$$

and so

$$(x_3, x_4) = (5, -29), (7, 41), (11, 13).$$

$\underline{n=5}$. We show that $|x_2|=3$ or 5. If $|x_2|>5$, then

$$|x_2| \geq 7, \qquad |x_3| \geq 9, \qquad |x_4| \geq 11, \quad |x_5| \geq 13,$$

From (5),

$$1 \leq \frac{1}{2}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11}+\frac{1}{13}+\frac{1}{2\cdot 7\cdot 9\cdot 11\cdot 13},$$

and this is false. If $|x_2|=5$, we can reject $x_2=-5$, from (5), and so

$$|x_3| \geq 7, \qquad |x_4| \geq 9, \qquad |x_5| \geq 11.$$

If $|x_3|>7$, then $|x_4| \geq 11$, $|x_5| \geq 13$, whence

$$1 \leq \frac{1}{2}+\frac{1}{5}+\frac{1}{9}+\frac{1}{11}+\frac{1}{13}+\frac{1}{2\cdot 5\cdot 9\cdot 11\cdot 13},$$

and this is false. Hence $|x_3|=7$ and we can reject $x_3=-7$, and so the solution is

$$(2, 5, 7, x_4, x_5), \qquad |x_4| \geq 9, \qquad |x_5| \geq 11.$$

Also from (1),

$$70(x_4+x_5)+a = z_2x_4x_5,$$

$$z_2 = 70\left(\frac{1}{x_4}+\frac{1}{x_5}\right)+\frac{a}{x_4x_5},$$

$$|z_2| \leq 70(\tfrac{1}{9}+\tfrac{1}{11})+\tfrac{1}{99} < 15.$$

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Also

$$z_2x_4x_5 \equiv a \pmod{35},$$

and

$$10x_4x_5+a \equiv 0 \pmod 7, \qquad 14x_4x_5+a \equiv 0 \pmod 5.$$

Hence

$$10+z_2 \equiv 0 \pmod 7, \qquad 14+z_2 \equiv 0 \pmod 5,$$

and so $z_2=11$. Next,

$$(11x_4-70)(11x_5-70) = 70^2+11a.$$

This has no solution since

$$4900+11 = 3{\cdot}1637, \qquad 4900-11 = 4889,$$

and

$$1637,\ 4889 \text{ are primes.}$$

We deal finally with $|x_2|=3$ and can reject $x_2=-3$ as usual. We have two cases $|x_3|=5$, $|x_3|\geq 5$.

If $|x_3|=5$, we can exclude $x_3=-5$ since

$$1 \leq \tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{5}+\tfrac{1}{7}+\tfrac{1}{11}+\tfrac{1}{100}$$

is false. Hence we must investigate the solution $(2, 3, 5, x_4, x_5)$ with $|x_4|\geq 7$, $|x_5|\geq 11$. Since $30x_4+a\equiv 0 \pmod{x_5}$, $30x_5+a\equiv 0 \pmod{x_4}$ we have

$$30(x_4+x_5)+a = z_3x_4x_5,$$

$$30\left(\frac{1}{x_4}+\frac{1}{x_5}\right)+\frac{a}{x_4x_5} = z_3,$$

where z_3 is an integer.

Hence

$$|z_3| \leq 30(\tfrac{1}{7}+\tfrac{1}{11})+\frac{1}{7.11} < 8.$$

Also

$$z_3x_4x_5 \equiv a \pmod{15},$$

$$6x_4x_5+a \equiv 0 \pmod 5,$$

$$10x_4x_5+a \equiv 0 \pmod 3.$$

$$z_3 \equiv -6 \pmod 5, \qquad z_3 \equiv -10 \pmod 3, \qquad z_3 = -1.$$

We note that we need only satisfy $x_4x_5+a\equiv 0 \pmod 5$ and $\pmod 3$. Since when $a=1$, $x_4x_5\equiv -1 \pmod 3$ and $\pmod 5$, we have solution $(2, 3, 5, -31, -929)$, $(2, 3, 5, -29, 869)$, $(2, 3, 5, -59, -61)$. When $a=-1$, $x_4x_5\equiv +1$, $\pmod 3$ and $\pmod 5$, we have solution $(2, 3, 5, -13, 23)$, $(2, 3, 5, -31, -931)$, $(2, 3, 5, -29, 871)$, $(2, 3, 5, -47, -83)$

$|x_3|>5$

$$|x_3| \geq 7, \qquad |x_4| \geq 11, \qquad |x_5| \geq 13.$$

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We can exclude $|x_3| \geq 17$ since

$$1 < \tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{17}+\tfrac{1}{19}+\tfrac{1}{23}+\tfrac{1}{100}$$

is false. Hence we may have $|x_3|=7, 11, 13$.

We may exclude the case $x_3<0$ since

$$1 < \frac{1}{2}+\frac{1}{3}+\frac{1}{x_3}+\frac{1}{x_4}+\frac{1}{x_5}+\frac{1}{6x_3x_4x_5}$$

is false when $x_3=-7, -11, -13$.

We have an equation

$$6x_3(x_4+x_5)+a = z_4x_4x_5,$$

where z_4 is an integer and so

$$a \equiv z_4x_4x_5 \pmod{x_3},$$

Since

$$6x_4x_5+a \equiv 0 \pmod{x_3}, \qquad z_4+6 \equiv 0 \pmod{x_3}.$$

From

$$6x_3\left(\frac{1}{x_4}+\frac{1}{x_5}\right)+\frac{a}{x_4x_5} = z_4,$$

and $|x_4| \geq x_3+2$, $|x_5| \geq x_3+4$, we have $|z_4| \leq 12$, and so since $(z_4, 6)=1$, $|z_4|=1, 5, 7, 11$. Hence we have $(x_3, z_4)=(7, 1), (11, 5), (13, 7)$.

$\underline{x_3=7}$ and so

$$(x_4-42)(x_5-42) = 42^2+a.$$

If $a=1$, $(x_4-42)(x_5-42)=1\cdot 1765=5\cdot 353$. This gives the solutions

$$a = 1, (2, 3, 7, 43, 1807), (2, 3, 7, 41, -1723), (2, 3, 7, 47, 395), (2, 3, 7, 37, -311)$$

If $a=-1$, $(x_4-42)\,(x_5-42)=1\cdot 1763=41\cdot 43$
and this gives the solutions

$$a = -1, (2, 3, 7, 43, 1805), (2, 3, 7, 41, -1721), (2, 3, 7, 83, 85),$$

$\underline{x_3=11}$

$$66(x_4+x_5)+a = 5x_4x_5,$$

$$(5x_4-66)(5x_5-66) = 66^2+5a.$$

If $a=1$,

$$5x_4-66 = \pm 1, \pm 7^2, \pm 7,$$

$$5x_5-66 = \pm 4361, \pm 89, \pm 623,$$

and so

$$x_4 = 13, \qquad x_5 = -859,$$

or

$$x_4 = 23, \qquad x_5 = 31.$$

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If $a=-1$

$$5x_4-66 = \pm 1, \pm 19$$

$$5x_5-66 = \pm 4351, \pm 229$$

$$x_4 = 13, 17, \qquad x_5 = -857, 59$$

$\underline{x_3=13}$

$$78(x_4+x_5)+a = 7x_4x_5$$

$$(7x_4-78)(7x_5-78) = 78^2+7a$$

If $a=1$,

$$7x_4-78 = \pm 1, \qquad 7x_5-78 = \pm 6091$$

$$x_4 = 11, \qquad x_5 = -859$$

If $a=-1$,

$$7x_4-78 = \pm 1, \pm 59$$

$$7x_5-78 = \pm 6077, \pm 103$$

$$x_4 = 11, \qquad x_5 = -857.$$

This completes the investigation.

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