Various perspectives on a pair of simple inequalities for $\log x$

G. J. O. JAMESON

The basic inequalities

The inequalities in question are

$$\log x \leqslant \frac{1}{2}\left(x - \frac{1}{x}\right) \tag{1}$$

and

$$\log x \geqslant \frac{2(x-1)}{x+1} \tag{2}$$

for $x > 1$.

Ahead of the proofs we record some preliminary remarks. Write

$$f(x) = \frac{1}{2}\left(x - \frac{1}{x}\right), \qquad g(x) = \frac{2(x-1)}{x+1};$$

Then the inequalities say $g(x) \leqslant \log x \leqslant f(x)$ for $x > 1$. Now $f\left(\frac{1}{x}\right) = -f(x)$, $g\left(\frac{1}{x}\right) = -g(x)$ and $\log\left(\frac{1}{x}\right) = -\log x$, so it follows that $f(x) \leqslant \log x \leqslant g(x)$ for $0 < x < 1$ (with all terms negative). Of course equality holds when $x = 1$. Also,

$$\frac{f(x)}{g(x)} = \frac{x^2-1}{2x}\,\frac{x+1}{2(x-1)} = \frac{(x+1)^2}{4x},$$

so (2) can be reformulated as follows in terms of $f(x)$:

$$f(x) \leqslant \frac{(x+1)^2}{4x}\log x. \tag{3}$$

Together with (1), this implies that $\dfrac{f(x)}{\log x} \to 1$ as $x \to 1$, since $\dfrac{(x+1)^2}{4x}$ tends to 1. (Recall that it is tempting, but wrong, to say $f(x) \to \log x$ as $x \to 1$; in fact, both tend to 0.) Similarly, (1) can be reformulated in terms of $g(x)$:

$$g(x) \geqslant \frac{4x}{(x+1)^2}\log x$$

and $\dfrac{g(x)}{\log x} \to 1$ as $x \to 1$. Clearly, the inequalities are most effective for x close to 1; they say nothing of interest for large x.

Applied to $x^{1/2}$, inequality (1) says

$$\log x \leqslant x^{1/2} - x^{-1/2}. \tag{4}$$

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We will see later that this is a better approximation than (1), with applications of its own. By writing $x^{1/2} - x^{-1/2}$ as $x^{-1/2}(x - 1)$, we see that (4) strengthens the well-known inequality $\log x \leqslant x - 1$ when $x > 1$. (Meanwhile, the reversed (2) strengthens this inequality for $0 < x < 1$.)

We now present four completely different proofs of (1) and (2). There is no need to choose a 'best' one. They are all (more or less) equally elementary, but each one deploys a different set of ideas, and each sheds light on the inequalities in a distinctive way. Our basic objective is comparison between these proofs, rather than the inequalities themselves.

Method 1: differentiation, mean-value theorem, positivity of squares.

Write $\ell(x) = \log x$ (this is to avoid notation like log-(x), which I find unappealing). Then

$$f'(x) - \ell'(x) = \frac{1}{2} + \frac{1}{2x^2} - \frac{1}{x} = \frac{1}{2}\left(1 - \frac{1}{x}\right)^2 \geqslant 0,$$

so $f(x) - \ell(x)$ is increasing for all $x > 0$. Since $f(1) - \ell(1) = 0$, inequality (1) follows. As always with such applications of the mean-value theorem, the same reasoning can be expressed in integral form:

$$f(x) - \ell(x) = \int_1^x \left(f'(t) - \ell'(t)\right) dt = \frac{1}{2}\int_1^x \left(1 - \frac{1}{t}\right)^2 dt \geqslant 0.$$

(We will refrain from counting this as a fifth method!)

Writing $g(x)$ as $2 - \dfrac{4}{x + 1}$, we see that

$$\ell'(x) - g'(x) = \frac{1}{x} - \frac{4}{(x + 1)^2} = \frac{(x + 1)^2 - 4x}{x(x + 1)^2} = \frac{(x - 1)^2}{x(x + 1)^2} \geqslant 0,$$

so $\ell(x) - g(x)$ is increasing for all $x > 0$, hence (2).

Method 2: integrals of convex functions.

This method has the merit of exhibiting a geometrical interpretation of (1) and (2). Also, it actually finds the bounds for us instead of requiring us to know them in advance. We use the fact that $\log x = \int_1^x \frac{1}{t}\, dt$. Recall that a function with increasing derivative is *convex*. In particular, $1/t$ is convex for $t > 0$. A convex function $u(t)$ lies below its chords and above its tangents. Consequently, its integral on $[a, b]$ is not greater than the trapezium estimate:

$$\int_a^b u(t)\, dt \leqslant \frac{1}{2}\left(u(a) + u(b)\right)(b - a).$$

Also, if $c = \frac{1}{2}(a + b)$, then

$$\int_a^b u(t)\, dt \geqslant u(c)(b - a):$$

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this is the area below the tangent at c. These inequalities are geometrically compelling, and an analytic proof is straightforward (but we won't spell it out here). Applied to $\frac{1}{t}$, they give at once

$$\log x \leqslant \frac{1}{2}\left(1 + \frac{1}{x}\right)(x - 1) = \frac{1}{2}\left(x - \frac{1}{x}\right) = f(x)$$

and

$$\log x \geqslant \frac{x - 1}{\frac{1}{2}(x + 1)} = g(x).$$

Method 3: substitution, hyperbolic functions.

Substitute $x = e^t$, so that $\log x = t$ and $f(x) = \sinh t$. Then $t > 0$ when $x > 1$ and $t \to 0$ when $x \to 1$. The inequality $\log x \leqslant f(x)$ is equivalent to $t \leqslant \sinh t$, which is clear (for $t > 0$) from the defining series

$$\sinh t = t + \frac{t^3}{3!} + \frac{t^5}{5!} + \dots .$$

A beautifully neat and quick proof! (It also delivers the limit $\lim\limits_{x \to 0} \dfrac{f(x)}{\log x} = 1$ without involving $g(x)$.) For $g(x)$, substitute instead $x = e^{2t}$. Then

$$g(x) = \frac{2\left(e^{2t} - 1\right)}{e^{2t} + 1} = \frac{2 \sinh t}{\cosh t}.$$

The inequality $g(x) \leqslant \log x$ is equivalent to $\sinh t \leqslant t \cosh t$, which is seen at once by termwise comparison of the series for $\sinh t$ with the series

$$t \cosh t = t + \frac{t^3}{2!} + \frac{t^5}{4!} + \dots .$$

Method 4: substitution, logarithmic series.

Substitute $x = \dfrac{1 + y}{1 - y}$, so that $y = \dfrac{x - 1}{x + 1}$. Then $x > 1$ implies that $0 < y < 1$. By the logarithmic series,

$$\frac{1}{2} \log x = \frac{1}{2} \log \frac{1 + y}{1 - y} = y + \frac{y^3}{3} + \frac{y^5}{5} + \dots .$$

This time (2) follows instantly: $\frac{1}{2} \log x \geqslant y = \frac{1}{2} g(x)$. To derive (1), observe that

$$\frac{1}{2} \log x \leqslant y + y^3 + y^5 + \dots = \frac{y}{1 - y^2}.$$

Now $1 + y = \dfrac{2x}{x + 1}$ and $1 - y = \dfrac{2}{x + 1}$, so

$$\frac{2y}{1 - y^2} = \frac{1}{1 - y} - \frac{1}{1 + y} = (x + 1)\left(\frac{1}{2} - \frac{1}{2x}\right) = \frac{1}{2}\left(x - \frac{1}{x}\right),$$

hence (1).

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We remark that all the methods actually establish strict inequality: $g(x) < \log x < f(x)$ for $x > 1$.

A reformulation: the logarithmic mean

The *logarithmic mean* of two distinct positive numbers x_1, x_2 is defined to be

$$L(x_1, x_2) = \frac{x_2 - x_1}{\log x_2 - \log x_1}.$$

We assume notation chosen so that $x_1 < x_2$. Of course, the arithmetic mean is $A(x_1, x_2) = \frac{1}{2}(x_1 + x_2)$ and the geometric mean is $G(x_1, x_2) = \sqrt{x_1 x_2}$. The basic relationship between these quantities is

$$G(x_1, x_2) \leqslant L(x_1, x_2) \leqslant A(x_1, x_2). \tag{5}$$

This, perhaps, explains why it is reasonable to regard $L(x_1, x_2)$ as a "mean" of some kind. We show here that (5) is essentially equivalent to (1) and (2).

Write $\dfrac{x_2}{x_1} = x$. Then $x > 1$ and

$$L(x_1, x_2) = x_1 \frac{x - 1}{\log x} = x_1 L(x, 1).$$

Similarly, it is clear that $A(x_1, x_2) = x_1 A(x, 1)$ and $G(x_1, x_2) = x_1 G(x, 1)$, so (5) is equivalent to $G(x, 1) \leqslant L(x, 1) \leqslant A(x, 1)$, in other words

$$\sqrt{x} \leqslant \frac{x - 1}{\log x} \leqslant \frac{x + 1}{2}.$$

The right-hand inequality is clearly equivalent to (2). The left-hand inequality is equivalent to $\log x \leqslant x^{1/2} - x^{-1/2}$, which we saw in (4).

Let us just write L for $L(x, 1)$, similarly A and G. In fact, the estimation (5) can be strengthened to

$$G^{2/3} A^{1/3} \leqslant L \leqslant \tfrac{2}{3} G + \tfrac{1}{3} A. \tag{6}$$

A proof of (6) using Cauchy's mean-value theorem was given in [1], together with some historical references. In fact, the ordinary mean-value theorem works just as well, but, either way, some fairly heavy manipulation is required. A much neater proof is delivered by a development of our Method 3. It was set out, along with some further results of this sort, in [2]. Here we will just reproduce the proof of the right-hand inequality.

We do not need to know the factor $\frac{1}{3}$ in advance: we can let it emerge from the reasoning. We look for an inequality $L \leqslant (1 - p)G + pA$, where p is to be found. In other words, we want

$$\frac{x - 1}{\log x} \leqslant (1 - p)\sqrt{x} + \frac{p}{2}(x + 1).$$

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The substitution $x = e^{2t}$ transforms this into

$$\frac{e^{2t-1}}{2t} \leqslant (1 - p)e^{t} + \frac{p}{2}\left(e^{2t} + 1\right),$$

equivalently

$$\frac{\sinh t}{t} \leqslant (1 - p) + p\cosh t.$$

Now

$$\frac{\sinh t}{t} = 1 + \frac{t^2}{3!} + \frac{t^4}{5!} + \dots ,$$

while

$$(1 - p) + p\cosh t = 1 + p\left(\frac{t^2}{2!} + \frac{t^4}{4!} + \dots\right).$$

By choosing $p = \frac{1}{3}$ we make the t^2 terms coincide, and comparing the coefficients of t^{2n}, we see that $\dfrac{1}{(2n + 1)!} < \dfrac{1}{3(2n)!}$ for $n \geqslant 2$, so the desired inequality holds.

Results concerning $f(x^a)$ and $g(x^a)$

Applying (1) and (2) to x^a, where $x > 1$ and $a > 0$, we obtain $g(x^a) \leqslant \log x^a \leqslant f(x^a)$ so

$$\frac{1}{a}g(x^a) \leqslant \log x \leqslant \frac{1}{a}f(x^a).$$

For $f(x)$, we have already stated the case $a = \frac{1}{2}$ in (4).

With x fixed, $\frac{1}{a}f(x^a)$ tends to $\log x$ as $a \to 0^+$: in fact, by (3), applied to x^a,

$$\frac{1}{a}f(x^a) \leqslant \frac{(x^a + 1)^2}{4x^a}\log x,$$

and $\dfrac{(x^a + 1)^2}{4x^a} \to 1$ as $a \to 0^+$, since $x_a \to 1$. By taking (for example) $a = 1/2^n$, we obtain a sequence convergent to $\log x$. Similar remarks apply to $\frac{1}{a}g(x^a)$.

We illustrate this by recording successive approximations to $\log 2$ generated in this way. Recall that the actual value is approximately 0.693147.

a	$\frac{1}{a}g(2^a)$	$\frac{1}{a}f(2^a)$
1	$\frac{2}{3}$	$\frac{3}{4}$
$\frac{1}{2}$	$12 - 8\sqrt{2} \approx 0.6863$	$\sqrt{2} - 1/\sqrt{2} \approx 0.7071$
$\frac{1}{4}$	$8(2^{1/4} - 1)^2(\sqrt{2} + 1) \approx 0.6914$	$2(2^{1/4} - 2^{-1/4}) \approx 0.6966$

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Do these approximations always improve steadily as a gets closer to 1? First, consider the special case $a = \frac{1}{2}$. It is easy to show that $2f(\sqrt{2}) \leqslant f(x)$ for any $x > 1$. Equivalently, $2f(x) \leqslant f(x^2)$ for $x > 1$, seen as follows:

$$f(x^2) = \frac{1}{2}\left(x^2 - \frac{1}{x^2}\right) = \left(x + \frac{1}{x}\right)f(x)$$

and $x + \frac{1}{x} \geqslant 2$. Similarly, $g(x) \leqslant 2g(\sqrt{x})$, equivalently $g(x^2) \leqslant 2g(x)$, since

$$\frac{g(x^2)}{g(x)} = \frac{x^2 - 1}{x^2 + 1}\frac{x + 1}{x - 1} = \frac{(x + 1)^2}{x^2 + 1}$$

and $(x + 1)^2 \leqslant 2(x^2 + 1)$.

As all this suggests, the following is actually true.

Theorem

For fixed $x > 1$, $\frac{1}{a}f(x^a)$ increases with a for all $a > 0$, and $\frac{1}{a}g(x^a)$ decreases.

Again we present alternative proofs, illustrating three perfectly valid ways to establish inequalities of this sort.

Proof 1: differentiation with respect to x.

Fix a, b with $a > b > 0$. We have to show that $\frac{1}{a}f(x^a) \geqslant \frac{1}{b}f(x^b)$, equivalently

$$b(x^a - x^{-a}) - a(x^b - x^{-b}) \geqslant 0.$$

Denote this by $p(x)$, now regarding x as the variable. Then $p(1) = 0$ and

$$\begin{aligned} p'(x) &= ab(x^{a-1} + x^{-a-1}) - ab(x^{b-1} + x^{-b-1}) \\ &= \frac{ab}{x}\left[(x^a + x^{-a}) - (x^b + x^{-b})\right]. \end{aligned}$$

Now $y + \frac{1}{y}$ increases with y for $y > 1$ (the derivative $1 - \frac{1}{y^2}$ is positive) and $x^a > x^b$, so $x^a + x^{-a} > x^b + x^{-b}$, so $p'(x) > 0$, hence also $p(x) \geqslant 0$, for $x > 1$.

Similarly, the statement $\frac{1}{a}g(x^a) \leqslant \frac{1}{b}g(x^b)$ is equivalent to $q(x) \geqslant 0$, where

$$\begin{aligned} q(x) &= a(x^a + 1)(x^b - 1) - b(x^a - 1)(x^b + 1) \\ &= (a - b)x^{a+b} - (a + b)x^a + (a + b)x^b - a + b. \end{aligned}$$

Then $q(1) = 0$, and our statement follows if we can show that $q'(x) \geqslant 0$ for $x > 1$. Now

$$\frac{1}{a + b}q'(x) = (a - b)x^{a+b-1} - ax^{a-1} + bx^{b-1} = x^{b-1}r(x),$$

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where $r(x) = (a - b)x^a - ax^{a-b} + b$. Then $r(1) = 0$ and

$$r'(x) = a(a - b)\left(x^{a-1} - x^{a-b-1}\right) \geqslant 0$$

for $x > 1$. So $r(x) \geqslant 0$, hence $q'(x) \geqslant 0$, for $x > 1$.

Proof 2: differentiation with respect to a.

One might need to take a moment to get used to the idea that x is fixed and a is now the variable. Let

$$F(a) = \frac{1}{a}f\left(x^a\right) = \frac{1}{2a}\left(x^a - x^{-a}\right).$$

Recall that since $x^a = e^{a \log x}$, we have $\frac{d}{da}x^a = x^a \log x$. So

$$F'(a) = \frac{1}{2a}\left(x^a + x^{-a}\right)\log x - \frac{1}{2a^2}\left(x^a - x^{-a}\right).$$

By our inequality (2) applied to x^{2a},

$$2a \log x \geqslant \frac{2\left(x^{2a} - 1\right)}{x^{2a} + 1} = \frac{2\left(x^a - x^{-a}\right)}{x^a + x^{-a}}.$$

It follows that $F'(a) \geqslant 0$ for all $a > 0$.

In similar style, one finds that (1) (applied to x^a) is what is needed to show that $G'(a) \leqslant 0$, where $G(a) = \frac{1}{a}g\left(x^a\right)$. We leave it to the reader to verify the details.

Proof 3: substitution.

Substitute $x = e^t$. We are assuming $x > 1$, so $t > 0$. Then

$$\frac{1}{a}f\left(x^a\right) = \frac{1}{a}\sinh at = t + a^2\frac{t^3}{3!} + a^4\frac{t^5}{5!} + \dots .$$

This clearly increases with a: an instant proof again!

For g, as before, we modify the substitution to $x = e^{2t}$. Then

$$\frac{1}{a}g\left(x^a\right) = \frac{2\left(e^{2at} - 1\right)}{e^{2at} + 1} = \frac{2 \sinh at}{a \cosh at}.$$

It is not quite so transparent that this decreases with a for fixed t. Writing $at = y$, we see that the statement is equivalent to saying that $h(y)$ increases with y, where $h(y) = \dfrac{y \cosh y}{\sinh y}$. This is easily established by differentiation:

$$\begin{aligned} h'(y) &= \frac{1}{\sinh^2 y}\left((y \sinh y + \cosh y)\sinh y - y \cosh^2 y\right) \\ &= \frac{\cosh y \sinh y - y}{\sinh^2 y}. \end{aligned}$$

This is non-negative, since $\cosh y \sinh y - y = \frac{1}{2}\sinh 2y - y \geqslant 0$.

A completely different way to show that $h(y)$ is increasing (yet another alternative!) was given in [3]. By quite elementary methods, without differentiation, it provides a simple criterion for the ratio of two power series to be monotonic.

References

1. Peter R. Mercer, Cauchy's mean value theorem meets the logarithmic mean, *Math. Gaz.* **101** (March 2017) pp. 108-115.
2. Graham Jameson and Peter R. Mercer, The logarithmic mean revisited, *Amer. Math. Monthly* **126** (2019) pp. 641-645.
3. G. J. O. Jameson, Monotonic ratios of functions, *Math. Gaz.* **105** (March 2021) pp. 129-134.

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G. J. O. JAMESON
13 Sandown Road,
Lancaster LA1 4LN
e-mail: *pgjameson@talktalk.net*

The answers to the *Nemo* page from July 2023 on the hypotenuse were:

1.	James Joyce	Portrait of the Artist as a Young Man	Chapter 5
2.	Mary Roberts Rinehart	Tenting Tonight	Chapter 14
3.	Grant Allen	Charles Darwin	Chapter 5
4.	O. Henry	Schools and Schools	
5.	Michael Frayn	Collected Columns: Pas devant les enfants	
6.	HG Wells	The First Men in the Moon	Chapter 13

Congratulations to Lawrence Smallman on tracking all of these down. This month it is time to focus on the force of friction. Quotations are to be identified by reference to author and work. Solutions are invited to the Editor by 23rd January 2024.

1. Being by trade a mason, he wore a long linen apron reaching almost to his toes, corduroy breeches and gaiters, which, together with his boots, graduated in tints of whitish-brown by constant friction against lime and stone.

2. A busy little man he always is, in the polishing at harness-house doors, of stirrup-irons, bits, curb-chains, harness-bosses, anything in the way of a stable-yard that will take a polish: leading a life of friction.

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