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BEST PROXIMITY POINTS AND FIXED POINTS WITH *R*-FUNCTIONS IN THE FRAMEWORK OF *w*-DISTANCES

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Abstract

We study best proximity points in the framework of metric spaces with w-distances. The results extend, generalise and unify several well-known fixed point results in the literature.

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1. Introduction and preliminaries

In this paper, we introduce a new class of contractions involving R-functions in the framework of complete metric spaces with a w-distance. Our main results (Theorems 2.3 and 2.5) give the existence and uniqueness of best proximity points of such mappings. Our results continue earlier work of Kostić *et al.* [8], where a similar problem has been investigated using the simulation functions of Khojasteh *et al.* [7]. However, as noted by Găvruţa *et al.* [1], the $\mathcal{Z}$-contractions (involving simulation functions) introduced in [7] are a special case of Meir–Keeler (MK) contractions [9]. The R-contractions introduced by Roldán López de Hierro and Shahzad [12] are a true generalisation of MK contractions. Our best proximity results for R-proximal contractions therefore generalise some earlier results such as those of Jleli *et al.* [4]. Moreover, our results hold in a more general setting than the usual metric space.

Definition 1.1. Let $\mathbb{A} \subseteq \mathbb{R}$ be a nonempty subset and let $\varrho : \mathbb{A} \times \mathbb{A} \to \mathbb{R}$ be a mapping. We say that ϱ is an R-function if the following two properties hold.

(a) $a_n \to 0$ for every sequence $\{a_n\} \subset (0, \infty) \cap \mathbb{A}$ such that $\varrho(a_{n+1}, a_n) > 0$ for all $n \in \mathbb{N}$.

(b) For any two sequences $\{a_n\}, \{b_n\} \subset (0, \infty) \cap \mathbb{A}$ such that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = L \geq 0$ with $L < a_n$ and $\varrho(a_n, b_n) > 0$ for all $n \in \mathbb{N}$, we have $L = 0$.

If, additionally, the following property is satisfied, then ϱ is called a strong R-function.

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(c) If $\{a_n\}, \{b_n\} \subset (0, \infty) \cap \mathbb{A}$ are two sequences such that $b_n \to 0$ and $\varrho(a_n, b_n) > 0$ for all $n \in \mathbb{N}$, then $a_n \to 0$.

The concept of R-functions was proposed by Roldán López de Hierro and Shahzad [12] in 2015, inspired by the simulation functions of Khojasteh *et al.* [7]. Since then, various authors have contributed to the study of fixed points, as well as best proximity points via R-functions (see, for example, [3, 6, 10, 11, 15]).

We recall some basic results and fundamental definitions. Meir and Keeler [9] proved the following theorem, which is a generalisation of the Banach contraction principle.

Theorem 1.2. *Let (X, d) be a complete metric space and let $T : X \to X$ be a mapping such that, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that*

$$\varepsilon \le d(x, y) < \varepsilon + \delta \Rightarrow d(Tx, Ty) < \varepsilon$$

for all $x, y \in X$. Then there exists a unique point $z \in X$ which is a fixed point of the mapping T, and $T^n x_0 \to z$ when $n \to \infty$ for every $x_0 \in X$.

From Theorem 1.2, we derive the notion of an MK-function.

Definition 1.3. A function $\phi : [0, \infty) \to [0, \infty)$ is called an MK-function if it satisfies:

(a) $\phi(0) = 0$;
(b) $\phi(t) > 0$ for all $t > 0$; and
(c) for all $\varepsilon > 0$ there exists a $\delta > 0$ such that $\phi(t) < \varepsilon$ for all $t \in [\varepsilon, \varepsilon + \delta)$.

The next definition recalls the notion of a simulation function which was introduced by Khojasteh *et al.* [7].

Definition 1.4. A mapping $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ is a simulation function if

(a) $\zeta(0, 0) = 0$;
(b) $\zeta(t, s) < s - t$ for $t, s > 0$; and
(c) if $\{t_n\}$ and $\{s_n\}$ are two sequences in $(0, \infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = \ell > 0$, then $\limsup_{n\to\infty} \zeta(t_n, s_n) < 0$.

Example 1.5. The following examples of R-functions are taken from [3, 6, 7, 12, 15]:

(a) $\varrho(t, s) = s\varphi(s) - t$, where $\varphi : [0, \infty) \to [0, 1)$ is a mapping such that $\limsup_{t\to s+} \varphi(t) < 1$ for all $s \in (0, \infty)$;
(b) $\varrho(t, s) = s\varphi(s) - t$, where $\varphi : [0, \infty) \to [0, 1)$ is a mapping such that $\lim_{n\to\infty} \varphi(t_n) = 1$ implies that $\lim_{n\to\infty} t_n = 0$ for every sequence $\{t_n\} \subseteq [0, \infty)$;
(c) $\varrho(t, s) = \varphi(s) - t$, where $\varphi : [0, \infty) \to [0, \infty)$ is an MK-function (Definition 1.2);
(d) $\varrho(t, s) = \zeta(t, s)$, where $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ is a simulation function;
(e) $\varrho(t, s) = \psi(s) - \varphi(s) - \psi(t)$, where $\psi, \varphi : [0, \infty) \to [0, \infty)$ are two functions such that ψ is nondecreasing and continuous from the right, while φ is lower semicontinuous and $\varphi^{-1}(\{0\}) = \{0\}$;

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(f) $\varrho(t, s) = s/(t + 1) - t$;
(g) $\varrho(t, s) = se^{-t} - t$; and
(h) $\varrho(t, s) = \ln(s + 1) - t$.

In 1996, Kada *et al.* [5] introduced a new generalised distance, the w-distance, which they used to extend and improve some well-known fixed point results, most notably Caristi's theorem, Ekeland's variational principle and the minimisation theorems of Takahashi.

Definition 1.6. Let (X, d) be a metric space and let $p : X \times X \to [0, \infty)$ be a function. Then p is called a w-distance on X if

(a) $p(x, y) \leq p(x, z) + p(y, z)$ for every $x, y, z \in X$;
(b) for any $x \in X$, the function $p(x, \cdot) : X \to [0, \infty)$ is lower semicontinuous; and
(c) for any $\varepsilon > 0$ there exists a $\delta > 0$ such that $p(z, x) \leq \delta,\ p(z, y) \leq \delta \Rightarrow d(x, y) \leq \varepsilon$ holds for all $x, y, z \in X$.

By adding the condition of semicontinuity with respect to the second variable in Definition 1.6, we propose a new notion of w_0-distance.

Definition 1.7. Let (X, d) be a metric space. A w-distance function $p : X \times X \to [0, \infty)$ is called a w_0-distance on X if, additionally, it fulfils the following condition:

(d) $p(\cdot, y) : X \to [0, \infty)$ is a lower semicontinuous function for any $y \in X$.

Remark 1.8. In general, accounts of w-distance (see, for example, [13, 15]) assume the symmetry condition, $p(x, y) = p(y, x)$ for all $x, y \in X$. We note that every symmetric w-distance is a w_0-distance in the sense of Definition 1.7, but the converse is not true.

Example 1.9. Let (X, d) be a metric space and let $p : X \times X \to [0, \infty)$ be a function. Kada *et al.* [5] gave the following examples of w-distances on X:

(1) $p(x, y) = d(x, y)$;
(2) $p(x, y) = c$, where c is a positive real number;
(3) if $(X, \|\cdot\|)$ is a normed space, then $p(x, y) = \|x\| + \|y\|$ is a w-distance on X;
(4) if $(X, \|\cdot\|)$ is as in (3), then $p(x, y) = \|y\|$ is also a w-distance on X;
(5) $p(x, y) = \max\{d(Tx, y), d(Tx, Ty)\}$, where $T : X \to X$ is a continuous mapping;
(6) if $X = \mathbb{R}$ with the standard metric d, then $p(x, y) = |\int_x^y f(u)\, du|$ is a w-distance on X, where $f : X \to [0, \infty)$ is a continuous function such that $\inf_{x \in X} \int_x^{x+r} f(u)\, du > 0$ for any $r > 0$; and
(7) if F is a closed bounded subset of X and $c \geq \operatorname{diam} F$, then

$$p(x, y) = \begin{cases} d(x, y) & \text{for all } x, y \in F, \\ c & \text{for all } x \notin F \text{ or } y \notin F. \end{cases}$$

It is clear that the w-distances defined in each of these examples are lower semicontinuous with respect to both variables. Hence all of the examples (1)–(7) are, in fact, examples of w_0-distances. Moreover, examples (1)–(3) and (7) are symmetric w-distances.

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EXAMPLE 1.10. Here we give an example of a w-distance which is not a lower semicontinuous function of the first variable when the other one is fixed.

Let (X, d) be a metric space endowed with the w-distance p, defined as in Example 1.9(7). Let $x_0 \in X$ be an accumulation point of X and let $\alpha : X \to [0, \infty)$ be a function defined by

$$\alpha(x) = \begin{cases} 3c & \text{for } x = x_0, \\ 2c & \text{for all } x \neq x_0. \end{cases}$$

The function $P : X \times X \to [0, \infty)$ defined by

$$P(x, y) = \max\{\alpha(x), p(x, y)\}$$

is also a w-distance on X [5, Lemma 3]. However, P is not a w_0-distance on X.

Indeed, since x_0 is an accumulation point of X, there exists a sequence $\{x_n\} \subseteq X$ such that $x_n \to x_0$ and $x_n \neq x_0$ for all $n \in \mathbb{N}$. Then

$$P(x_0, y) = \max\{\alpha(x_0), p(x_0, y)\} = 3c > 2c = \liminf_{n\to\infty} P(x_n, y)$$

for any $y \in X$, which means that $P(\cdot, y)$ is not a lower semicontinuous function.

Basic properties of a w_0-distance are the same as those of a w-distance, as described in the next lemma due to Kada *et al.* [5].

LEMMA 1.11 (Kada *et al.* [5]). *Let (X, d) be a metric space with a w-distance p. Also, let $\{x_n\}, \{y_n\}$ be two sequences in X and let $\{\alpha_n\}, \{\beta_n\}$ be two sequences of real numbers converging to zero. Then the following properties hold for all $x, y, z \in X$:*

(a) *(for all $n \in \mathbb{N}$) $p(x_n, y) \leq \alpha_n, p(x_n, z) \leq \beta_n \Rightarrow y = z$ and, in particular, $p(x, y) = p(x, z) = 0 \Rightarrow y = z$;*
(b) *(for all $n \in \mathbb{N}$) $p(x_n, y_n) \leq \alpha_n, p(x_n, z) \leq \beta_n \Rightarrow y_n \to z$;*
(c) *(for all $m, n \in \mathbb{N}, m > n$) $p(x_n, x_m) \leq \alpha_n \Rightarrow \{x_n\}$ is a Cauchy sequence; and*
(d) *(for all $n \in \mathbb{N}$) $p(y, x_n) \leq \alpha_n \Rightarrow \{x_n\}$ is a Cauchy sequence.*

We also recall the following standard notation in the setting of a metric space (X, d): for $\emptyset \neq A, B \subseteq X$,

$$d(A, B) = \inf\{d(x, y) : x \in A, y \in B\},$$
$$A_0 = \{x \in A : (\exists y \in B)\, d(x, y) = d(A, B)\},$$
$$B_0 = \{y \in B : (\exists x \in A)\, d(x, y) = d(A, B)\}.$$

In the next section, we introduce the notion of R-proximal contractions and investigate whether such mappings yield the existence and uniqueness of best proximity points (and also fixed points) in the context of a complete metric space with a w_0-distance.

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2. Main results

In this section, we introduce the notions of R-proximal contractions and prove our main results. For all $x, y \in X$, where (X, d) is a metric space with a w_0-distance p, define a function $q : X \times X \to [0, \infty)$ by

$$q(x, y) = \max\{p(x, y), p(y, x)\}.$$

It is easily checked that the function q is symmetric and satisfies the triangle inequality and $q(x, y) = 0$ implies that $x = y$ for all $x, y \in X$.

Definition 2.1. Let (X, d) be a metric space with a w_0-distance p and $\emptyset \neq A, B \subseteq X$. Let $\varrho : \mathbb{A} \times \mathbb{A} \to [0, \infty)$ be a strong R-function and assume that $\{p(x, y) : x, y \in X\} \subseteq \mathbb{A}$. A mapping $T : A \to B$ such that

$$d(u, Tv) = d(x, Ty) = d(A, B) \Rightarrow \varrho(q(u, x), q(y, v)) > 0$$

holds for all $u, v, x, y \in A$ is called an R-proximal contraction of the first kind.

In the same setting, the mapping T is called an R-proximal contraction of the second kind if

$$d(u, Tv) = d(x, Ty) = d(A, B) \Rightarrow \varrho(q(Tu, Tx), q(Tv, Ty)) > 0$$

for every $u, v, x, y \in A$.

Lemma 2.2. *Let (X, d) be a metric space with w_0-distance p and let $\{x_n\}$ be a sequence in X such that*

$$\lim_{n\to\infty} q(x_n, x_{n+1}) = 0. \tag{2.1}$$

Then one of the following conditions holds:

(i) $\lim_{m,n\to\infty} q(x_n, x_m) = 0$; *or*

(ii) *there exist $\varepsilon > 0$ and two subsequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ of $\{x_n\}$ with $m_k > n_k$ for all $k \in \mathbb{N}$ such that $q(x_{n_k}, x_{m_k}) \geq \varepsilon$ for all $k \in \mathbb{N}$ and*

$$\lim_{k\to\infty} q(x_{n_k}, x_{m_k}) = \lim_{k\to\infty} q(x_{n_k-1}, x_{m_k-1}) = \varepsilon.$$

Proof. Suppose that (i) is not true. Then there exist $\varepsilon > 0$ and two sequences $\{m_k\}, \{n_k\} \subseteq \mathbb{N} \cup \{0\}$ with $m_k > n_k$ such that

$$q(x_{n_k}, x_{m_k}) \geq \varepsilon \tag{2.2}$$

for all $k \in \mathbb{N}$. We can assume that m_k is a minimal index for which (2.2) holds. Then

$$q(x_{n_k}, x_{m_k-1}) < \varepsilon \tag{2.3}$$

for any $k \in \mathbb{N}$. Using the triangle inequality for q, together with (2.2) and (2.3),

$$\varepsilon \leq q(x_{n_k}, x_{m_k}) \leq q(x_{n_k}, x_{m_k-1}) + q(x_{m_k-1}, x_{m_k}) < \varepsilon + q(x_{m_k-1}, x_{m_k}).$$

Passing to the limit when $k \to \infty$, by (2.1),

$$\lim_{k\to\infty} q(x_{n_k}, x_{m_k}) = \varepsilon. \tag{2.4}$$

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Next, we show that

$$\lim_{k\to\infty} q(x_{n_k-1}, x_{m_k-1}) = \varepsilon. \tag{2.5}$$

Letting $k \to \infty$ in the inequalities

$$q(x_{n_k-1}, x_{m_k-1}) \le q(x_{n_k-1}, x_{n_k}) + q(x_{n_k}, x_{m_k}) + q(x_{m_k}, x_{m_k-1})$$

and

$$q(x_{n_k}, x_{m_k}) \le q(x_{n_k}, x_{n_k-1}) + q(x_{n_k-1}, x_{m_k-1}) + q(x_{m_k-1}, x_{m_k}),$$

by (2.1) and (2.4),

$$\lim_{k\to\infty} q(x_{n_k-1}, x_{m_k-1}) \le \varepsilon$$

and

$$\varepsilon \le \lim_{k\to\infty} q(x_{n_k-1}, x_{m_k-1}),$$

respectively, which together imply (2.5). □

Now we can formulate our first main result.

THEOREM 2.3. *Let (X, d) be a complete metric space with a w_0-distance p and let $\emptyset \neq A, B \subseteq X$ such that A_0 is nonempty and closed. Let $T : A \to B$ and $g : A \to A$ be two mappings satisfying the following conditions:*

(a) *T is an R-proximal contraction of the first kind;*
(b) *$T(A_0) \subseteq B_0$;*
(c) *$p(x, y) = p(gx, gy)$ for all $x, y \in A$;*
(d) *g is continuous; and*
(e) *$A_0 \subseteq g(A_0)$.*

Then there is a unique point $z \in A$ such that $d(gz, Tz) = d(A, B)$ and $p(z, z) = 0$. Moreover, starting with an arbitrary $x_0 \in A_0$, we can construct a sequence $\{x_n\} \subset A_0$ such that $d(gx_{n+1}, Tx_n) = d(A, B)$ for every $n \in \mathbb{N} \cup \{0\}$ and $x_n \to z$ when $n \to \infty$.

REMARK 2.4. Theorem 2.3 extends and generalises several best proximity point (and also fixed point) theorems. We give a number of examples which can be obtained by specialising the parameters in Theorem 2.3.

- If $\varrho : [0, \infty) \times [0, \infty) \to \mathbb{R}$ is a simulation function, $g = id_A$ and $p = d$, we obtain Corollary 2.1 of Tchier *et al.* [14].
- If p is a symmetric w-distance on X and $A = B = X$, we obtain Theorem 9 of Zarinfar *et al.* [15].
- If $\varrho(t, s) = \phi(s) - t$, where $\phi : [0, \infty) \to [0, \infty)$ is an MK-function, we obtain a generalisation of the best proximity point results of Jleli *et al.* [4]. Moreover, the conditions imposed on the sets A and B are also relaxed.
- If $\varrho(t, s)$ is defined as in Example 1.5(b), and $A = B = X$, we obtain the fixed point theorem of Geraghty [2] extended to spaces with a w_0-distance.

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Proof of theorem 2.3. Let $x_0 \in A_0$. Then conditions (b) and (e) imply that there is an $x_1 \in A_0$ such that $d(gx_1, Tx_0) = d(A, B)$. Continuing in the same manner, for any $x_n \in A_0$, we can find an $x_{n+1} \in A_0$ such that $d(gx_{n+1}, Tx_n) = d(A, B)$.

If there exists $n_0 \in \mathbb{N}$ such that $q(x_{n_0-1}, x_{n_0}) = 0$, then $x_{n_0-1} = x_{n_0}$, which means that $d(gx_{n_0-1}, Tx_{n_0-1}) = d(A, B)$: that is, x_{n_0-1} is a best proximity point of T under the mapping g and the proof is finished.

Hence we can assume that $q(x_{n-1}, x_n) > 0$ for all $n \in \mathbb{N}$.

Let us prove that the sequence $\{x_n\}$ converges. Since T is an R-proximal contraction of the first kind,

$$0 < \varrho(q(gx_n, gx_{n+1}), q(x_{n-1}, x_n)) = \varrho(q(x_n, x_{n+1}), q(x_{n-1}, x_n))$$

for every $n \in \mathbb{N}$. By property (a) of Definition 1.1,

$$\lim_{n\to\infty} q(x_{n-1}, x_n) = 0.$$

Next, we show that

$$\lim_{m,n\to\infty} q(x_n, x_m) = 0. \tag{2.6}$$

Suppose, on the contrary, that the limit in (2.6) is not zero. Then, by Lemma 2.2, there exist an $\varepsilon > 0$ and two sequences $\{m_k\}, \{n_k\} \subseteq \mathbb{N}$ with $m_k > n_k$ for all $k \in \mathbb{N}$ such that

$$q(x_{n_k}, x_{m_k}) \geq \varepsilon \tag{2.7}$$

for all $k \in \mathbb{N}$ and

$$\lim_{k\to\infty} q(x_{n_k}, x_{m_k}) = \lim_{k\to\infty} q(x_{n_k-1}, x_{m_k-1}) = \varepsilon. \tag{2.8}$$

Since T is an R-proximal contraction of the first kind and condition (c) holds,

$$\varrho(q(gx_{n_k}, gx_{m_k}), q(x_{n_k-1}, x_{m_k-1})) = \varrho(q(x_{n_k}, x_{m_k}), q(x_{n_k-1}, x_{m_k-1})) > 0$$

for all $k \in \mathbb{N}$. Now put $a_k := q(x_{n_k}, x_{m_k})$ and $b_k := q(x_{n_k-1}, x_{m_k-1})$ for $k \in \mathbb{N}$. By the last inequality and Definition 1.1(b), together with (2.7) and (2.8),

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} b_k = 0,$$

which is a contradiction. Hence (2.6) holds.

From (2.6) and Lemma 1.11(c), $\{x_n\} \subset A_0$ is a Cauchy sequence. But (X, d) is a complete metric space and $A_0 \subseteq X$ is closed, so there exists $\lim_{n\to\infty} x_n = z \in A_0$. Conditions (c) and (d) also yield $\lim_{n\to\infty} gx_n = gz \in A_0$. On the other hand, $Tz \in B_0$ by condition (b), which means that there is a $u \in A$ such that $d(u, Tz) = d(A, B)$.

To complete the proof, we need to show that $u = gz$ and $p(z, z) = 0$.

If $u = gx_n$ for infinitely many $n \in \mathbb{N}$, then $u = gz$. Hence we assume that $u \neq gz$, in which case there exists $n_0 \in \mathbb{N}$ such that $u \neq gx_n$ for all $n \geq n_0$. If $q(gx_n, u) = 0$ for some $n \geq n_0$, then $gx_n = u$, so we must have $q(gx_n, u) > 0$ for all $n \geq n_0$. Also, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $q(x_{n_k}, z) > 0$ for every $k \in \mathbb{N}$ (if that is not true, then there exists $N \in \mathbb{N}$ such that $q(x_n, z) = 0$ for all $n \geq N$, and then $q(x_{n-1}, x_n) = 0$

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for all $n > N$, which is contrary to our assumption). Also, $q(x_{n_k}, u) > 0$ for every $k \in \mathbb{N}$ such that $n_k \geq n_0$. For convenience, from now on we will identify $\{x_{n_k}\}$ with the whole sequence $\{x_n\}$.

From (2.6), for any $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that $q(x_n, x_m) < \varepsilon$ for all $m > n \geq N_\varepsilon$. For a fixed $n \in \mathbb{N}$ with $n \geq \max\{n_0, N_\varepsilon\}$, the function $p(x_n, \cdot)$ is lower semicontinuous so that

$$p(x_n, z) \leq \liminf_{m\to\infty} p(x_n, x_m) < \varepsilon.$$

Thus $\lim_{n\to\infty} p(x_n, z) = 0$. Similarly, $\lim_{n\to\infty} p(z, x_n) = 0$. Combined with the previous inequality, this yields

$$\lim_{n\to\infty} q(x_n, z) = \lim_{n\to\infty} q(gx_n, gz) = 0. \tag{2.9}$$

Take $a_n := q(gx_{n+1}, u)$ and $b_n := q(x_n, z)$ for $n \in \mathbb{N}$ in Definition 1.1(c). Then (2.9) gives

$$\lim_{n\to\infty} q(gx_{n+1}, u) = 0. \tag{2.10}$$

Finally, from (2.9) and (2.10) we conclude that $gz = u$ by Lemma 1.11(a). Uniqueness is proved using Definition 1.1(a) by taking $a_n := q(gv, gz) = q(v, z)$ to be a constant sequence, where $v \in A$ is such that $d(gv, Tv) = d(A, B)$. That $p(z, z) = q(z, z) = 0$ is proved similarly. □

Our second main result is a best proximity point theorem for R-proximal contractions of the second kind.

Theorem 2.5. *Let (X, d) be a complete metric space with a w_0-distance p and let $\emptyset \neq A, B \subseteq X$ such that $T(A_0)$ is nonempty and closed. Let $T : A \to B$ and $g : A \to A$ be two mappings with the following properties:*

(a) *T is an R-proximal contraction of the second kind;*
(b) *$T(A_0) \subseteq B_0$;*
(c) *T is injective on A_0;*
(d) *$p(Tx, Ty) = p(Tgx, Tgy)$ for all $x, y \in A$;*
(e) *g is continuous; and*
(f) *$A_0 \subseteq g(A_0)$.*

Then there is a unique point $z \in A$ such that $d(gz, Tz) = d(A, B)$ and $p(Tz, Tz) = 0$. Moreover, starting with an arbitrary $x_0 \in A_0$ we can construct a sequence $\{x_n\} \subset A_0$ such that $d(gx_{n+1}, Tx_n) = d(A, B)$ for every $n \in \mathbb{N} \cup \{0\}$ and $x_n \to z$ when $n \to \infty$.

Proof. Let $x_0 \in A_0$. By similar reasoning to that in the proof of Theorem 2.3, using conditions (b) and (f) we can construct a sequence $\{x_n\} \subseteq A_0$ such that $d(gx_{n+1}, Tx_n) = d(A, B)$ for all $n \in \mathbb{N} \cup \{0\}$.

Suppose there exists $n_0 \in \mathbb{N}$ such that $q(Tx_{n_0-1}, Tx_{n_0}) = 0$. Then $Tx_{n_0-1} = Tx_{n_0}$ and $x_{n_0-1} = x_{n_0}$ because T is injective on A_0. But then $d(gx_{n_0-1}, Tx_{n_0}) = d(gx_{n_0}, Tx_{n_0}) = d(A, B)$ and x_{n_0} is the best proximity point of T under the mapping g.

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Now suppose that $q(Tx_{n-1}, Tx_n) > 0$ for all $n \in \mathbb{N}$.

We proceed to prove that the sequence $\{x_n\}$ is convergent. Since T is an R-proximal contraction of the second kind,

$$0 < \varrho(q(Tgx_n, Tgx_{n+1}), q(Tx_{n-1}, Tx_n)) = \varrho(q(Tx_n, Tx_{n+1}), q(Tx_{n-1}, Tx_n))$$

for all $n \in \mathbb{N}$, which by Definition 1.1(a) implies that

$$\lim_{n\to\infty} q(Tx_{n-1}, Tx_n) = 0.$$

Let us show that

$$\lim_{m,n\to\infty} q(Tx_n, Tx_m) = 0. \tag{2.11}$$

Assume, to the contrary, that (2.11) does not hold. In that case, by Lemma 2.2 there exist an $\varepsilon > 0$ and two sequences $\{m_k\}, \{n_k\} \subseteq \mathbb{N}$ with $m_k > n_k$ for all $k \in \mathbb{N}$ such that

$$q(Tx_{n_k}, Tx_{m_k}) \geq \varepsilon \tag{2.12}$$

for all $k \in \mathbb{N}$ and

$$\lim_{k\to\infty} q(Tx_{n_k}, Tx_{m_k}) = \lim_{k\to\infty} q(Tx_{n_k-1}, Tx_{m_k-1}) = \varepsilon. \tag{2.13}$$

Since T is an R-proximal contraction of the second kind,

$$\varrho(q(Tgx_{n_k}, Tgx_{m_k}), q(Tx_{n_k-1}, Tx_{m_k-1})) = \varrho(q(Tx_{n_k}, Tx_{m_k}), q(Tx_{n_k-1}, Tx_{m_k-1})) > 0$$

for all $k \in \mathbb{N}$. Take $a_k := q(Tx_{n_k}, Tx_{m_k})$ and $b_k := q(Tx_{n_k-1}, Tx_{m_k-1})$ for $k \in \mathbb{N}$ in Definition 1.1(b). By (2.12) and (2.13), it follows that

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} b_k = 0,$$

which is a contradiction. Thus (2.11) is proved.

From (2.6) and Lemma 1.11(c), $\{Tx_n\} \subset T(A_0)$ is a Cauchy sequence. Since (X, d) is a complete metric space and $T(A_0) \subseteq X$ is closed, there exists $\lim_{n\to\infty} Tx_n = Tz \in T(A_0)$. By condition (b), $Tz \in T(A_0) \subseteq B_0$, so there exists a $u \in A_0$ such that $d(u, Tz) = d(A, B)$. Also, from condition (f), $u = gx$ for some $x \in A_0$. Hence $d(gx, Tz) = d(A, B)$.

Now we prove that $Tx = Tz$.

If $Tx_n = Tx$ holds for infinitely many values of $n \in \mathbb{N}$, then $Tz = Tx$. Therefore we can assume that there exists $n_0 \in \mathbb{N}$ such that $Tx_n \neq Tx$ for all $n \geq n_0$. Also, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ (which we can assume is the whole sequence) such that $q(Tx_{n_k}, Tz) > 0$ for all $k \in \mathbb{N}$.

Using (2.11), for any $\varepsilon > 0$ there exists an $N_\varepsilon \in \mathbb{N}$ such that $q(Tx_n, Tx_m) < \varepsilon$ for every $m > n \geq N_\varepsilon$. Then, from Definition 1.6(b),

$$p(Tx_n, Tz) \leq \liminf_{m\to\infty} p(Tx_n, Tx_m) < \varepsilon$$

for any fixed $n \geq \max\{n_0, N_\varepsilon\}$, which implies that $\lim_{n\to\infty} p(Tx_n, Tz) = 0$. Similarly, $\lim_{n\to\infty} p(Tz, Tx_n) = 0$, and so

$$\lim_{n\to\infty} q(Tx_n, Tz) = 0. \tag{2.14}$$

Now take $a_n := q(Tgx_{n+1}, Tgx) = q(Tx_{n+1}, Tx)$ and $b_n := q(Tx_n, Tz)$ for $n \in \mathbb{N}$ in Definition 1.1(c). By (2.14),

$$\lim_{n\to\infty} q(Tx_{n+1}, Tx) = 0. \tag{2.15}$$

Finally, from (2.14), (2.15) and Lemma 1.11(a), we conclude that $Tx = Tz$.

To prove the uniqueness, take $a_n := q(Tgv, Tgz) = q(Tv, Tz)$ for all $n \in \mathbb{N}$ in Definition 1.1(a), where $v \in A$ is such that $d(gv, Tv) = d(A, B)$. Then $q(Tv, Tz) = 0$, that is, $Tv = Tz$, and then condition (c) yields $v = z$. The proof of $p(Tz, Tz) = q(Tz, Tz) = 0$ is similar. □

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