

COMPLETE EMBEDDINGS OF GROUPS

MARTIN R. BRIDSON  and HAMISH SHORT 

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For our friend and coauthor Chuck Miller in his ninth decade, with deep respect and affection

Abstract

Every countable group G can be embedded in a finitely generated group G^* that is hopfian and *complete*, that is, G^* has trivial centre and every epimorphism $G^* \rightarrow G^*$ is an inner automorphism. Every finite subgroup of G^* is conjugate to a finite subgroup of G . If G has a finite presentation (respectively, a finite classifying space), then so does G^* . Our construction of G^* relies on the existence of closed hyperbolic 3-manifolds that are asymmetric and non-Haken.

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1. Introduction

In 1971, Miller and Schupp [14] used small cancellation theory to prove that every countable group G can be embedded in a finitely generated group G^* that is *hopfian* and *complete* (asymmetric). They construct G^* as a quotient of a free product $G * U(p, q)$, where $U(p, q)$ is a free product of finite cyclic groups. In particular, each of the enveloping groups that they construct has torsion. The purpose of this note is to present an alternative construction of G^* that does not introduce torsion. It also preserves finiteness properties of G .

THEOREM A. *Every countable group G can be embedded in a finitely generated group G^* such that:*

- (1) G^* is hopfian and complete;
- (2) every finite subgroup of G^* is conjugate to a finite subgroup of G ;
- (3) if G has a finite presentation (respectively, a finite classifying space of dimension $d \geq 3$), then so does G^* .

Our construction of G^* is more explicit than that of Miller and Schupp. It is nontrivial in that it relies on the existence of asymmetric hyperbolic groups with



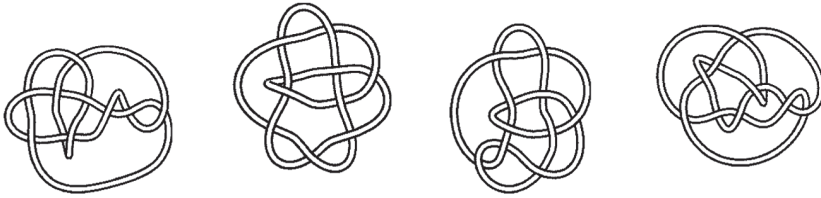


FIGURE 1. The knots 10_{102} , 10_{106} , 10_{107} and 10_{110} .

additional properties, but the basic idea behind it is straightforward: after some gentle preparation, we are able to assume that G is generated by two finitely generated free subgroups $F_1, F_2 < G$; we then *rigidify* G by attaching certain complete (asymmetric) groups A_1 and A_2 to it along F_1 and F_2 ; the groups A_i are constructed so that any epimorphism ψ of the resulting amalgam G^* must preserve the decomposition $G^* = A_1 *_{F_1} G *_{F_2} A_2$, sending each A_i to itself; by ensuring that the A_i are hopfian, as well as complete, we force the restrictions $\psi|_{A_i}$ to be the identity; and since $G^* = \langle A_1, A_2 \rangle$, we conclude that ψ is the identity.

Groups A_i with the properties that we need can be found among the fundamental groups of closed hyperbolic 3-manifolds obtained by Dehn surgery on the knots shown in Figure 1, as we shall explain.

2. Preliminaries

We remind the reader of some terminology. A group G is termed *hopfian* if every epimorphism $G \rightarrow G$ is an isomorphism, and *cohopfian* if every monomorphism $G \rightarrow G$ is an isomorphism. G is said to be *complete* (or *asymmetric*) if it has trivial centre and every automorphism is inner. A subgroup $H < G$ is *malnormal* if $H^g \cap H \neq 1$ implies $g \in H$. A group G is said to *split over a free group* if G can be decomposed as an amalgamated free product $G = A *_F B$ or Higman–Neumann–Neumann (HNN) extension $G = B *_F$ with F free. A group has Serre’s property FA if it fixes a point whenever it acts on a simplicial tree.

We shall assume that the reader is familiar with the rudiments of Bass–Serre theory [18] and the homology of groups.

2.1. The mild preparation of G . A classical construction of B. H. Neumann embeds a countable group G in a finitely generated group \tilde{G} by means of HNN extensions and amalgamated free products (see [12, page 188]). The finite subgroups of \tilde{G} are conjugate to subgroups of G ; in particular, \tilde{G} is torsion-free if G is torsion-free. Thus, in our attempts to construct G^* , there is no loss of generality in assuming that G is finitely generated.

Replacing G by $G * \mathbb{Z}$ if necessary, we may also assume that our group has a generating set $\{a_0, a_1, \dots, a_n\}$ where the a_i each have infinite order: given $G = \langle b_1, \dots, b_n \rangle$ and x generating \mathbb{Z} , define $a_0 = x$ and $a_i = xb_i$. We assume that this modification

has been made. The normal-form theorem for free products then yields the following results.

LEMMA 2.1. *In $G * \langle s, t \rangle$, the subsets $\{s, a_0t, \dots, a_n(s^{-n}ts^n)\}$ and $\{s, t, a_0t\}$ generate free subgroups of ranks $n + 2$ and 3, respectively.*

COROLLARY 2.2. *Replacing G by $G * \langle s, t \rangle$ if necessary, we may assume that $G = \langle F_1, F_2 \rangle$, where F_1 and F_2 are finitely generated free groups with noncyclic intersection and the centraliser of $F_1 \cap F_2$ in G is trivial.*

2.2. Asymmetric hyperbolic manifolds.

LEMMA 2.3. *Given integers $r > 0$ and $d \geq 3$, one can construct a complete, torsion-free, cohopfian (Gromov) hyperbolic group $A(d)$ and a malnormal free subgroup $L < A(d)$ of rank r such that:*

- (1) $A(d)$ has a finite classifying space of dimension d ;
- (2) $A(d)$ does not split over any free group; and
- (3) $A(d)/\langle\langle L \rangle\rangle$ is infinite.

PROOF. By Mostow rigidity, if M is a closed orientable hyperbolic manifold of dimension $d \geq 3$, then $\pi_1 M$ is complete (asymmetric) if and only if M is asymmetric, that is, M has no nontrivial isometries. There exist such manifolds in every dimension $d \geq 3$: Kojima [10] constructed examples in dimension 3 (along with manifolds that have any prescribed finite group of symmetries) and, inspired by arguments of Long and Reid [11], Belolipetsky and Lubotzky [1] constructed examples in each dimension $d \geq 3$. Let M be such a d -manifold and let $A(d) = \pi_1 M$. Note that M is a classifying space for $A(d)$.

If $A(d)$ splits nontrivially as an amalgamated free product, say $A(d) = D *_C B$, then there is a Mayer–Vietoris exact sequence for integral homology groups

$$\dots H_d D \oplus H_d B \rightarrow H_d M \rightarrow H_{d-1} C \dots$$

By Poincaré duality, $H_d M \cong \mathbb{Z}$ and (as $D, B < \pi_1 M$ are of infinite index) $H_d D = H_d B = 0$. Thus, $H_{d-1} C$ is infinite. In particular, since $d \geq 3$, the group C cannot be free. A similar argument shows that $\pi_1 M$ does not split as an HNN extension over a free group either.

Any subgroup of infinite index in $A(d)$ has lower cohomological dimension than $A(d)$, and a subgroup of finite index cannot be isomorphic to $A(d)$ by Mostow rigidity. Thus, $A(d)$ is cohopfian.

Every nonelementary hyperbolic group contains proper, normal subgroups of infinite index, and Kapovich [8] shows that inside such a subgroup one can find a malnormal free subgroup of rank 2, and inside that one can find a malnormal subgroup of any finite rank. □

3. The main argument

Given a countable group G , we modify it to arrive in the situation $G = \langle F_1, F_2 \rangle$ described in Corollary 2.2. Let r_i be the rank of the free group F_i and construct groups $A_i = A(d_i)$ with malnormal free subgroups $L_i < A_i$ of rank r_i as in Lemma 2.3. We may assume that A_1 is not isomorphic to A_2 (Remark 3.2). Let

$$G^* = A_1 *_F_1 G *_F_2 A_2,$$

where the amalgamation identifies $F_i < G$ with $L_i < A_i$. Note that since $G = \langle F_1, F_2 \rangle$, we have $G^* = \langle A_1, A_2 \rangle$.

THEOREM 3.1. *G^* is complete.*

PROOF. The centre of G^* is trivial because the centre of any amalgamated free product lies in the intersection of the edge groups, and the edge groups in the defining decomposition of G^* are centreless.

Let $\phi : G^* \rightarrow G^*$ be an automorphism; we must argue that ϕ is inner. As A_i does not split over a free group, the action of $\phi(A_i)$ on the Bass–Serre tree of the given splitting of G^* must fix a vertex. Thus, each of $\phi(A_1)$ and $\phi(A_2)$ is contained in a conjugate of A_1, A_2 or G .

We have chosen the A_i so that $Q_i := A_i / \langle\langle F_i \rangle\rangle$ is infinite. Note that $Q_1 \cong G^* / \langle\langle G, A_2 \rangle\rangle$ and $Q_2 \cong G^* / \langle\langle G, A_1 \rangle\rangle$. Thus, a pair of conjugates of A_1, A_2 or G can only generate G^* if one of the pair is a conjugate of A_1 and the other is a conjugate of A_2 . If ϕ maps A_1 into a conjugate of A_2 , then ϕ^2 would map A_1 to a conjugate of itself, and since A_1 is cohopfian, the image would be the whole of this conjugate. This forces $\phi(A_1)$ to be equal to the conjugate of A_2 containing it, which is impossible since we have chosen A_1 and A_2 to be not isomorphic. We conclude that ϕ maps A_i isomorphically onto a conjugate of itself for $i = 1, 2$.

Now, since all automorphisms of A_1 are assumed to be inner, we can compose ϕ with an inner automorphism of G^* to assume that $\phi|_{A_1} = \text{id}_{A_1}$, while $\phi(A_2) = A_2^\gamma$ for some $\gamma \in G^*$.

Consider the Bass–Serre tree for the splitting $G^* = A_1 *_F_1 G *_F_2 A_2$. We refer to the vertices as being of type A_1, A_2 or G , according to whether they are in the G^* orbit of the vertices (identity cosets) A_1, A_2 or G , respectively. Since $F_i < A_i$ is malnormal for $i = 1, 2$, no arc of length greater than 2 in this tree has nontrivial stabiliser, and any arc of length 2 with nontrivial stabiliser must be centred at a vertex of type G . In particular, the subtree fixed by $F_1 \cap F_2$, which contains the vertices A_1, G and A_2 , has diameter 2 and centroid G . The centraliser of $F_1 \cap F_2$ in G^* leaves this subtree and its centroid invariant, and hence is contained in G . But, by construction, the centraliser of $F_1 \cap F_2$ in G is trivial, and hence so is its centraliser in G^* .

As $\phi(A_2) = A_2^\gamma$, we have an isomorphism $\text{ad}(\gamma)^{-1} \circ \phi|_{A_2} : A_2 \rightarrow A_2$. As A_2 is complete, this isomorphism is conjugation by some $a \in A_2$, so $\phi|_{A_2}$ is conjugation by $\gamma a \in G^*$. But ϕ restricts to the identity on $F_1 \cap F_2 < A_2$, and we know that the centraliser of $F_1 \cap F_2$ in G^* is trivial, so $\gamma = a^{-1}$ and $\phi|_{A_2} = \text{id}_{A_2}$.

As G^* is generated by $A_1 \cup A_2$, we conclude that ϕ (previously adjusted by a conjugacy to ensure that $\phi_{A_1} = \text{id}_{A_1}$) is the identity, and the theorem is proved. \square

REMARK 3.2. In the theorem above, we required $A_1 \not\cong A_2$. An easy way to arrange this is to take $A_i = A(d_i)$ from Lemma 2.3 with $d_1 \neq d_2$. But one is also free to take both A_i to have the same dimension $d \geq 3$, appealing to [1] or Theorem 4.1 below. With this second choice, if G has geometric (or cohomological) dimension D , then G^* will have geometric (or cohomological) dimension $\max\{D, d\}$.

4. Asymmetric hyperbolic 3-manifolds

A closed orientable 3-manifold M is Haken if it is irreducible and contains a closed incompressible surface, that is, a closed surface of positive genus such that the inclusion map $S \hookrightarrow M$ induces a monomorphism of groups $\pi_1 S \hookrightarrow \pi_1 M$. If M contains such a surface, then $\pi_1 M$ acts without a fixed point on the tree T that is obtained from the universal covering $p : \tilde{M} \rightarrow M$ as follows: the vertex set of T is the set of connected components of $\tilde{M} \setminus p^{-1}(S)$; two vertices are connected by an edge if the components that they represent abut along a component of $p^{-1}(S)$; and the action of $\pi_1 M$ on T is induced by the action of $\pi_1 M$ on \tilde{M} by deck transformations.

In the opposite direction, John Stallings proved that if M is irreducible and $\pi_1 M$ acts without a fixed point on a simplicial tree, then M contains an incompressible surface (see [19]). In particular, a closed orientable 3-manifold M that is aspherical will be non-Haken if and only if $\pi_1 M$ has Serre's property FA.

Our purpose in this section is to explain how well-known facts about hyperbolic 3-manifolds imply the following result, which will be familiar to experts.

THEOREM 4.1. *There exist infinitely many distinct, closed, asymmetric, hyperbolic 3-manifolds M such that $\pi_1 M$ has property FA.*

PROOF. In the light of the preceding discussion, what we must show is that there are infinitely many distinct hyperbolic 3-manifolds that are asymmetric and non-Haken. The manifolds that we shall describe are obtained by Dehn surgery on the knots shown in Figure 1. These are the only four knots K with at most 10 crossings that have the three properties that we are interested in: first, each K is prime and alternating (and nontorus), hence hyperbolic [13]; second, the knot complement $\mathbb{S}^3 \setminus K$ has no nontrivial symmetries, so by Mostow rigidity its fundamental group is complete (asymmetric); and third, $\mathbb{S}^3 \setminus K$ is small, that is, contains no closed incompressible surface other than the tori parallel to the boundary of a regular neighbourhood of K . The first of these properties is immediately visible in the diagrams, the second is established in the census of Henry and Weeks [6] who calculated the symmetry groups of all knots up to 10 crossings (alternatively Kodama and Sakuma [9]), and the third is recorded in the census [4] of Burton *et al.*, who calculated all knots with at most 12 crossings that are small.

Thurston's celebrated Dehn surgery theorem [20] states that all but finitely many of the closed manifolds obtained by Dehn surgery on these knots will support a

hyperbolic metric. And Hatcher's theorem on boundary slopes [5] implies that all but finitely many of these closed manifolds will be non-Haken. Thus, we will be done if we can argue that all but finitely many of these closed hyperbolic manifolds are asymmetric. And this is what Kojima's arguments show [10], as we shall now explain.

In Dehn filling, one starts from the manifold M_K with torus boundary obtained by removing an open tubular neighbourhood of a hyperbolic knot K in \mathbb{S}^3 . A framing of the knot gives an identification $\mathbb{Z}^2 = \pi_1 \partial M_K$. Given a reduced fraction p/q , one attaches a thickened 2-disc D to an annular neighbourhood of a simple closed curve on ∂M_K representing the homotopy class $(p, q) \in \mathbb{Z}^2$. The boundary of the resulting manifold $M_K \cup D$ is a 2-sphere, which one caps off with a 3-ball to obtain the manifold $M_K(p, q)$. (Note that the union of D and this 3-ball is a solid torus.) For all but finitely many choices of p/q , Thurston [20] provides a hyperbolic metric on $M_K(p, q)$, and by Mostow rigidity this metric is unique. As $|p| + |q| \rightarrow \infty$ this metric converges to the complete hyperbolic metric on M_K , and when $|p| + |q|$ is sufficiently large, the unique shortest geodesic in $M_K(p, q)$ is the core of the solid torus added during Dehn filling [20]. Thus, by restriction, we obtain a homomorphism $\text{Isom}(M_K(p, q)) \rightarrow \text{Homeo}(M'_K)$, where M'_K is the interior of M_K . By Mostow rigidity, every homeomorphism of M'_K is homotopic to a unique isometry of the complete hyperbolic metric on M'_K , so, passing to homotopy classes, we have a homomorphism $\rho : \text{Isom}(M_K(p, q)) \rightarrow \text{Isom}(M'_K)$ (where the isometries of M'_K are with respect to its complete hyperbolic metric, not the restriction of the metric on $M_K(p, q)$). Mostow rigidity also tells us that, for any complete, finite-volume hyperbolic 3-manifold N , the natural map $\text{Isom}(N) \rightarrow \text{Out}(\pi_1 N)$ is an isomorphism. Thus, an isometry ϕ of $M_K(p, q)$ will lie in the kernel of ρ only if its restriction to M'_K induces an inner automorphism of $\pi_1 M_K$. But if this is the case, then the map that ϕ induces on $\pi_1 M_K(p, q)$ (a quotient of $\pi_1 M'_K$) will also be inner, and therefore ϕ is trivial. Thus, $\rho : \text{Isom}(M_K(p, q)) \rightarrow \text{Isom}(M'_K)$ is injective. In particular, for K with $\pi_1 M_K$ complete, the triviality of $\text{Isom}(M'_K) \cong \text{Out}(\pi_1 M')$ implies that $\text{Out}(\pi_1 M_K(p, q)) \cong \text{Isom}(M_K(p, q))$ is trivial when $|p| + |q|$ is sufficiently large. \square

We shall also need the following lemma. Note that a finitely generated group with property FA has finite abelianisation.

LEMMA 4.2. *Let M be a closed hyperbolic 3-manifold. If $H_1 M$ is finite, then every nontrivial homomorphism $f : \pi_1 M \rightarrow \pi_1 M$ is an isomorphism.*

PROOF. Scott's compact core theorem [16] implies that every nontrivial subgroup of infinite index in $\pi_1 M$ has infinite abelianisation, and therefore cannot be the image of f . In more detail, if the finitely generated group $G = f(\pi_1 M)$ has infinite index, then the corresponding covering space $M' = \tilde{M}/G$ of M has a compact submanifold C such that $C \hookrightarrow M'$ is a homotopy equivalence. A standard argument using Poincaré–Lefschetz duality ('half lives, half dies') shows that the rank of $H_1(\partial C)$ is twice the rank of the image of $H_1(\partial C)$ in $H_1 M'$. Since C has at least one component of positive genus, both $H_1 \partial C$ and $H_1 M'$ are infinite.

The lemma now follows from the fact that every epimorphism from $\pi_1 M$ to a subgroup of finite index in itself is an isomorphism (see [2] or [17]). In more detail, an argument of Hirshon [7] shows that if a finitely generated, torsion-free group Γ is residually finite, then every homomorphism from Γ to a subgroup of finite index in itself is injective, and Mostow rigidity implies that $\pi_1 M$ (which is residually finite because it is linear, and torsion-free because M is aspherical) cannot be isomorphic to a subgroup of finite index in itself (because such a subgroup has greater covolume). \square

5. The proof of Theorem A

Towards proving items (2) and (3) of the theorem, first note that the process by which a given finitely generated group G was transformed into the conditioned state described in Corollary 2.2 involved only free products with free groups, so it preserves the finiteness properties in (3), and any finite subgroup of the conditioned group is conjugate to a subgroup of the original G . This last property is also true in the case of a countable group that is first embedded in a finitely generated group using Neumann's embedding (as we noted in Section 2).

In the main construction, we defined $G^* = A_1 *_{F_1} G *_{F_2} A_2$, where the A_i are torsion-free and the F_i are finitely generated free groups. Every finite subgroup of an amalgamated free product is conjugate to one of the vertex groups, and amalgamating groups that are finitely presented (respectively, have a finite classifying space of dimension at most d) along finitely generated free groups preserves these properties. Thus, (2) is proved, and (3) will be proved if we can argue that both A_1 and A_2 can be taken to be three-dimensional hyperbolic groups.

In the light of Theorems 3.1 and 4.1, the following proposition completes the proof of Theorem A.

PROPOSITION 5.1. *If the complete groups $A_1 \not\cong A_2$ used in the construction of G^* are the fundamental groups of hyperbolic 3-manifolds that have property FA, then G^* is hopfian.*

PROOF. We have $G^* = A_1 *_{F_1} G *_{F_2} A_2$. Let $\phi : G^* \rightarrow G^*$ be an epimorphism. The action of $\phi(A_1)$ on the Bass–Serre tree for G^* has a fixed point, as A_1 has property FA, so $\phi(A_1)$ lies in a conjugate of one of the vertex groups, A_1, G, A_2 . The same argument applies to A_2 .

As in the proof of Theorem 3.1, we use the fact that a pair of conjugates of A_1, A_2 or G can only generate G^* if one of the pair is a conjugate of A_1 and the other is a conjugate of A_2 . From this, and the surjectivity of ϕ , we deduce that $\phi^2(A_1)$ is a nontrivial subgroup of a conjugate of A_1 and $\phi^2(A_2)$ is a nontrivial subgroup of a conjugate of A_2 . Lemma 4.2 then forces $\phi^2|_{A_1}$ and $\phi^2|_{A_2}$ to be isomorphisms onto conjugates of A_1 and A_2 , respectively. We can then proceed exactly as in the proof of Theorem 3.1 to conclude that ϕ is an inner automorphism of G^* . \square

REMARK 5.2. (1). Let G be a group that has a finite classifying space and contains an element $\gamma \neq 1$ conjugate to its inverse. Our construction embeds G in a group G^* that retains these properties and is complete. This recovers the main theorem of [3]. To obtain a specific example, one can take $G = \langle x, y \mid xyx^{-1}y \rangle$.

(2) A second (less concise and explicit) proof of Theorem A can be obtained by means of a careful application of relative small cancellation theory, following Miller and Schupp's method in [14] but avoiding the use of finite groups. In outline, one constructs a two-generator perfect hyperbolic group B , takes a free product $(G \times \mathbb{Z}^2) * (B \times \mathbb{Z})$ and then forms the quotient by a set of relators satisfying a strong small cancellation condition [15]. A key point in [14] is that automorphisms preserve the different finite orders of elements, while here they preserve the different ranks of centralisers of elements.

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MARTIN R. BRIDSON, Mathematical Institute,
Andrew Wiles Building, ROQ, Woodstock Road, Oxford OX2 6GG, UK
e-mail: bridson@maths.ox.ac.uk

HAMISH SHORT, L.A.T.P., U.M.R. 6632, C.M.I.,
39 Rue Joliot–Curie, Université de Provence, F–13453 Marseille cedex 13, France
e-mail: hamish@cmi.univ-mrs.fr