GLACIER SLIDING DOWN AN INCLINED WAVY BED WITH FRICTION

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ABSTRACT. The effects of frictional tangential traction combined with regelation on the basal sliding of a temperate glacier down an inclined wavy bed are examined. Two friction models are treated. First, a Coulomb law model having the assumptions that sliding occurs everywhere and that the tangential traction is proportional to the normal pressure. Secondly, a velocity power law in which the tangential traction is proportional to a power of the slip velocity. The ice motion is approximated by steady slow Newtonian flow and the bed undulation about a mean bed-line has a maximum slope $\epsilon \ll 1$. Flow solutions are constructed as perturbations (in powers of ϵ) of the plane laminar flow corresponding to non-slip on the mean bed-line, assuming that the ice remains everywhere in contact with the bed; that is, no cavitation takes place. If the normal traction is predicted to be tensile over part of the bed, implying that cavitation has occurred, then a new solution is needed in which the ice base over cavities is traction-free. Since the cavity sections and profile of the free ice base are then part of the overall solution, an intricate mixed boundary-value problem is set up for the flow and the present analysis is inadequate.

For a sinusoidal bed the perfect-slip (zero tangential traction) solution predicts compressive normal traction everywhere on the bed provided that the mean bed-line inclination α (to the horizontal) is less than a critical value α_c which is of order ϵ. For greater values of α, including a range of order ϵ, the normal traction is tensile on some parts of the bed, and a solution with cavitation is needed. If the tensile sections are relatively small it is expected that the resulting cavitation will not change the overall solution significantly. However, the Coulomb friction solution has extensive zones of tensile traction for all values of α, so that extensive cavitation would occur. In contrast, the velocity-power friction solution has compressive traction everywhere on the bed for $\alpha \leqslant \alpha_c = O(1)$ provided that the ice depth is not too large, and also for deep glaciers for $\alpha \leqslant \alpha_c = O(\epsilon)$. Furthermore, the predicted basal sliding velocity varies much less with the length scale of the bed undulation than in the perfect-slip solution, and is smaller.

RÉSUMÉ. *Glissement d'un glacier avec frottement sur un lit incliné ondulé.* On a examiné les effets de la traction tangentielle de frottement combinée avec le regel sur le lit d'un glacier tempéré glissant sur un fond incliné avec des ondulations. Deux modèles de frottement sont envisagés: le premier est une loi de Coulomb dans laquelle la traction tangentielle est proportionnelle à la pression normale, le second est une loi-puissance en fonction de la vitesse dans laquelle la traction tangentielle est proportionnelle à une puissance de la vitesse de glissement. Le mouvement de la glace est assimilé à un écoulement stationnaire lent Newtonien et les ondulations du lit autour d'une direction moyenne, ont une pente maximum $\epsilon \ll 1$. Les solutions pour l'écoulement sont construites comme des perturbations en puissances de ϵ autour de l'écoulement plan laminaire correspondant au non-glissement sur le lit plan moyen, avec l'hypothèse que la glace reste partout au contact du lit, c'est-à-dire qu'il n'y a pas cavitation. Si l'on prévoit que l'effort normal sur une partie du lit sera une traction, ce qui implique que la cavitation peut se produire, alors il faut chercher une nouvelle solution dans laquelle la glace de la base au-dessus des cavités n'est pas soumise à une traction. Dès lors que les profil et sections des cavités de la glace libre de la base font partie de la solution générale, un problème inextricable de valeurs aux limites est soulevé par la détermination de l'écoulement et la présente analyse est inadéquate.

Pour un lit sinusoïdal, la solution du glissement parfait (traction tangentielle nulle) prévoit un effort normal de compression partout sur le lit pourvu que la pente moyenne α (sur l'horizontale) soit inférieure à une valeur critique α_c qui est de l'ordre de ϵ. Pour des valeurs supérieures de α, y compris un ordre de grandeur voisin de ϵ, il y a un effort de traction normal quelque part sur le lit, et il est nécessaire de chercher une solution avec cavitation. Si les sections avec traction sont relativement petites, on s'attend à ce que la cavitation résultante ne change pas la solution générale de manière significative. Cependant, la solution de la friction de Coulomb présente de larges zones avec efforts normaux de traction pour toutes les valeurs de α, si bien qu'une cavitation importante peut s'y produire. Par contre, la solution avec frottement en puissance de la vitesse est valable partout si $\alpha \leqslant \alpha_c$ avec une valeur $\alpha_c = O(1)$, pourvu que la profondeur du glacier ne soit pas excessive, et aussi pour les glaciers profonds lorsque $\alpha \leqslant \alpha_c$ avec $\alpha_c = O(\epsilon)$. En conséquence, la vitesse de glissement à la base qui est prédite, varie beaucoup moins avec l'échelle des longueurs d'onde des irrégularités du lit que dans la solution à glissement parfait et est plus faible.

ZUSAMMENFASSUNG. *Gletschergleiten über ein geneigtes und gewelltes Bett mit Reibung.* Untersucht wird die Auswirkung tangentialen Reibungszuges in Kombination mit Regelation auf das Gleiten eines temperierten Gletschers über ein geneigtes und gewelltes Bett. Zwei Reibungsmodelle werden herangezogen: erstens ein Coulomb-Gesetz, bei dem die tangentiale Reibung proportional zum Normaldruck anwächst; zweitens ein Geschwindigkeitspotenzgesetz, bei dem die tangentiale Reibung proportional zu einer Potenz der Gleitgeschwindigkeit ist. Die Eisbewegung wird durch stationäres, langsames Newtonsches Fliessen angenähert; die Bettundulation um ein mittleres Bettprofil hat eine Maximalneigung von $\epsilon \ll 1$. Die Lösungen für das Strömungsfeld werden als Störungen der ebenen laminaren Strömung, bei der kein Gleiten auf dem mittleren Bettprofil stattfindet, nach Potenzen von ϵ entwickelt; dabei wird angenommen, dass das Eis überall mit dem Bett in Berührung bleibt, d.h. keine Kavitation stattfindet. Für den Fall, dass sich eine Normalkomponente der Kraft über einem Teil des Bettes in Zugrichtung ergibt, braucht man eine neue Lösung, die Kavitation

einschliesst, und bei der die Eisuntergrenze über den Hohlräumen reibungsfrei ist. Da die Hohlraum-abschnitte und das Profil der freien Eisuntergrenze dann einen Teil der Gesamtlösung bilden, erhält man für das Fliessen ein verwickeltes vermischtes Randwertproblem und die vorliegende Untersuchung ist unzulänglich.

Für ein sinusförmiges Bettprofil liefert die Lösung für vollkommenes Gleiten (ohne tangentiale Reibung) überall im Bett eine Normalkraft in Druckrichtung, vorausgesetzt, die Neigung α des mittleren Bettprofils gegenüber der Waagrechten ist kleiner als ein kritischer Wert α_c (in der Grösse von ϵ). Für höhere Werte von α, einschliesslich eines Bereiches in der Grösse von ϵ, ist die Normalkraft an einigen Stellen des Bettes in Zugrichtung, und man braucht eine Lösung mit Kavitation. Wenn die Abschnitte unter Zug verhalt-nismässig klein sind, wird angenommen, dass die entstehende Kavitation die Gesamtlösung nicht entscheidend verändert. Die Lösung für die Coulomb-Reibung hat jedoch für alle Werte von α ausgedehnte Zonen unter Zug, so dass entspechende ausgedehnte Kavitation auftreten wird. In Gegensatz dazu gilt die Lösung mit Reibung nach dem Geschwindigkeitspotenzgesetz überall, wenn $\alpha \leqslant \alpha_c = O(1)$, vorausgesetzt, dass die Eisdicke nicht zu gross ist, und auch für mächtige Gletscher, wenn $\alpha \leqslant \alpha_c = O(\epsilon)$. Des weiteren schwankt die vorausgesagte Gleitgeschwindigkeit weit weniger mit der Längenausdehnung der Bettundulation als in der Lösung des vollkommenen Gleitens und ist niedriger.

1. INTRODUCTION

A thin water layer produced by the melting and refreezing of basal ice in a temperate glacier on up- and down-stream faces of bed protuberances provides lubrication for the basal sliding which may be a significant part of the overall motion. The shear stress in such a thin layer is negligible and so there is no resistive tangential traction to the ice motion over the bed. The bed drag is the resultant, along the mean bed line, of the pressure distribution over the protuberances. This perfect-slip model is the basis of flow solutions obtained by Nye (1969, 1970), Kamb (1970), and more recently, Morland (1976). It is assumed that the ice can be approximated as an incompressible Newtonian fluid of high viscosity in slow steady flow, and that the bed profile is periodic with small maximum slope ϵ relative to the bed line. Solutions are obtained as power series expansions in ϵ, assuming that the ice base remains everywhere in contact with the bed. For a given glacier depth h, an inclination α of the mean bed line to the horizontal, and a profile shape, the plane flow solution determines the basal-sliding velocity, which is defined as the tangential velocity along the mean bed line. A calculation for a sinusoidal bed shows that the basal-sliding velocity is sensitive to the length scale of the bed undulation. Furthermore, the normal traction on the bed remains compressive everywhere only if $\alpha \leqslant \alpha_c$ for some critical angle α_c which is of order ϵ, and hence the solution predicts tensile tractions on part of the bed for α of the order of one and for some range of α of order ϵ. In these situations cavitation must occur, and a valid formula-tion must incorporate cavity sections over which the ice base is traction free, these sections and the profile of the ice base being part of the solution. This intricate problem involving, as it does, such mixed boundary values, has not yet been attempted. It is, however, expected that when the tensile sections are relatively small, the resulting cavitation will not have a significant effect on the overall solution.

The condition of zero tangential traction requires the existence of a continuous water layer. Pinching-out of the water layer at the crests of protuberances will cause local failure of this condition, but debris protruding from the basal ice makes frictional contact with the bed across the large areas over which it is carried (Boulton, 1974, p. 41; 1975, p. 7). The inter-action of such debris with the bed is complex, as is its dependence on bed profile (which reveals how it is transported and deposited). Data obtained by G. S. Boulton, A. Armstrong and E. M. Morris from field work carried out on the Glacier d'Argentière in 1973 and 1975 are being analysed in an attempt to construct "simple" friction laws exhibiting the main features observed in different situations. To ensure a manageable analysis one must assume that the mean effect of the individual debris contacts can be described by friction laws which apply continuously over the bed surface. It is with such an analysis in mind that any possible qualitative effects of bed friction on basal sliding are explored here, by solving the plane flow problem for two conventional friction laws.

First, the Coulomb law

$$t_s = -\nu t_n, \tag{1}$$

where t_s and t_n are the tangential and normal tractions on the ice base, with s defining a tangential coordinate in the direction of flow and n a normal coordinate directed into the ice, ν is the friction coefficient assumed to be of order unity. The limiting friction form of Equation (1) presupposes that sliding occurs everywhere. It is reasonable to assume that any non-slip zones will be of limited extent and should not influence significantly the overall basal sliding.

Secondly, a velocity power law

$$t_s = E V_s^{1/m}, \tag{2}$$

where V_s is the tangential velocity of the base ice, and E and m are constants. The form of Equation (2) has been used by Nye (1959) in a different context, and is also inferred by Weertman (1957, 1964) as a global relation between mean drag and basal-sliding velocity in his regelation theory which takes a non-linear viscous law for the ice. Weertman suggested that m is approximately 2 or a little larger, but gave no explicit value for E. In the present analysis, however, Equation (2) is considered as a possible qualitative relation arising from debris friction, in contrast with Equation (1), and also in an exploration of its effects on basal-sliding velocity and cavitation. A value of $m = 2$ is used in the calculations, and a range of values for E compatible with steady flow down the inclined bed is considered.

If part of the work done by the basal friction is released as a surface heat flux, and not used solely for surface crushing or other mechanical effects, it contributes to the thermal balance in the regelation mechanism and a non-linear thermo-mechanical coupling is introduced into the boundary conditions. When only a small part is released as heat this coupling may be neglected, and it is also shown here that, under moderate restrictions on the glacier depth and bed profile slope, the entire work contribution is small compared with the latent-heat terms. The solutions are therefore derived neglecting any friction contribution to the thermal balance, so that the surface distribution of heat sources at the ice base per unit area, is $L V_n$ (Nye, 1969), where $L = 2.8 \times 10^8$ J m^{-3} is the latent heat and V_n is the normal velocity of the basal ice. Thus, at the bed

$$-k_i \frac{\partial T}{\partial n} + k_b \frac{\partial S}{\partial n} = L V_n, \tag{3}$$

where T and S denote temperature in the ice and bedrock respectively, and k_i and k_b are the thermal conductivities of ice and bedrock respectively. $k_i = 2.1$ J m^{-1} s^{-1} K^{-1} and $k_b = r k_i$ where r ranges from 1 to 2 for typical bed rocks; the value $r = 1.6$ (appropriate to granite) is used in later calculations. In the regelation model the basal ice is everywhere at its melting point, and if (p_0, T_0) is a pressure-melting point on the bed-line, then at the bed

$$T - T_0 = S - T_0 = -C(p - p_0). \tag{4}$$

Such a linear relation presupposes that T and S remain close to T_0 at the bed, and $C = 0.7 \times 10^{-7}$ K/(N m^{-2}). Following Kamb (1970) the first-order dependence is on pressure p only with no contribution from deviatoric stress changes.

Figure 1 shows the overall plane-flow problem, with coordinates (x, y) respectively along and normal to a bed line which is inclined at angle α to the horizontal. U_s is the surface velocity at $y = h$, in the x-direction, and the basal-sliding velocity U_b is defined as the leading-order term of the x-velocity in a flow continued onto $y = 0$. The bed profile is

$$y_b = f(x), \tag{5}$$

where $f(x)$ is smooth and extends periodically as $x \to \pm \infty$. It is assumed that the bed slope is everywhere small, thus

$$\epsilon = |f'(x)|_{\max} \ll 1. \tag{6}$$
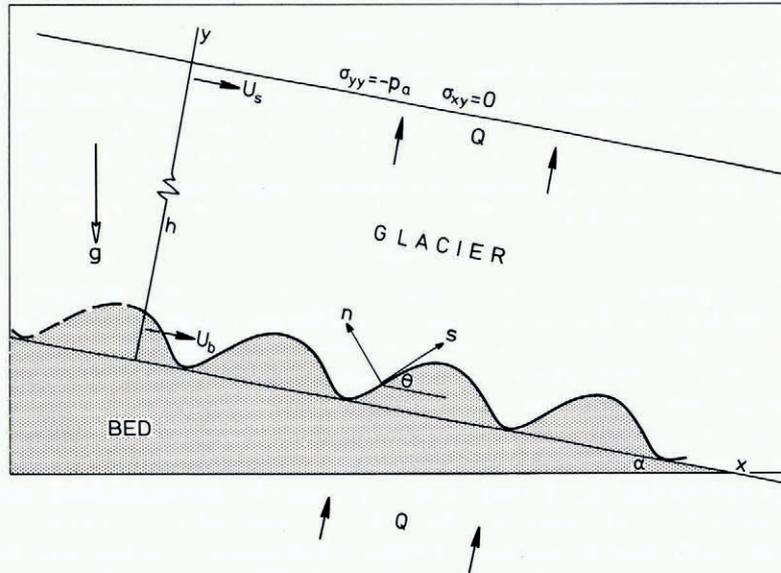
Fig. 1. *Glacier flow over a wavy bed.*

The bed conditions (Equations (1) or (2)) and Equations (3) and (4) are applied on $y = y_b(x)$. On the upper free surface the stress $\boldsymbol{\sigma}$ satisfies

$$y = h: \qquad \sigma_{yy} = -p_a, \qquad \sigma_{xy} = 0, \tag{7}$$

where p_a is atmospheric pressure, and where it is anticipated that the normal velocity V_y on $y = h$ is zero to the required order in ϵ. The exponential decay of V_y with height demonstrated in the solutions justifies the surface prescription $y \equiv h$ under a very weak restriction on h. A geothermal heat flux Q normal to the bed line, with a typical value of 4×10^{-2} J m^{-2} s^{-1} (Paterson, 1969) is included, requiring

$$T \approx -\frac{Q}{k_i}y \quad \text{as } y \to \infty, \qquad S \approx -\frac{Q}{k_b}y \quad \text{as } y \to -\infty, \tag{8}$$

in the half-plane solutions determined for T and S.

2. FLOW EQUATIONS

The construction of velocity and temperature fields, in terms of complex potentials, and the development of boundary-condition expansions in powers of ϵ, have been described by Morland (1976). Equations from this earlier paper are prefixed by the letter "M", only the main steps are repeated here.

A length scale of the bed undulation λ is defined by

$$\lambda = f_m/\epsilon, \qquad \text{where } f_m = |f(x)|_{\max}, \tag{9}$$

and dimensionless coordinates (X, Y) are defined by

$$x = \lambda X, \qquad y = \lambda Y, \tag{10}$$

in which the bed profile becomes

$$\left. \begin{array}{c} Y_b = f(\lambda X)/\lambda = \epsilon F(X); \\[2mm] |F(X)| \leqslant 1, \qquad |F'(X)| \leqslant 1. \end{array} \right\} \tag{11}$$

The following velocity, pressure and temperature decompositions are formulated:

$$\frac{V_x}{U_s} = 1 - \kappa\left(1 - \frac{\lambda Y}{h}\right)^2 + u, \qquad \frac{V_y}{U_s} = v, \tag{12}$$

$$p = p_0 - \rho g \cos \alpha \, \lambda Y + \frac{2\mu U_s}{\lambda} P, \qquad p_0 = p_a + \rho g \cos \alpha \, h, \tag{13}$$

$$T = T_0 - \frac{\lambda Q}{k_i} Y + \frac{2\mu U_s C}{\lambda} \tilde{T}, \qquad S = T_0 - \frac{\lambda Q}{k_b} Y + \frac{2\mu U_s C}{\lambda} \tilde{S}, \tag{14}$$

where μ is the ice viscosity and

$$\kappa = \frac{\rho g \sin \alpha \, h^2}{2\mu U_s}. \tag{15}$$

A value for μ of 3×10^{12} N m^{-2} s is used in the calculations. For given values of α and h, Equation (15) determines U_s once κ is known. The velocities u and v are dimensionless with a scale unit U_s, P is a dimensionless pressure with unit $2\mu U_s/\lambda$ ($\boldsymbol{\Sigma}$ will denote a dimensionless stress tensor with the same unit), $\tilde{T}$ and $\tilde{S}$ are dimensionless temperatures with unit $2\mu U_s C/\lambda$. The free surface conditions of Equation (7) and the flux conditions of Equation (8) are satisfied exactly if

$$\left.\begin{array}{c} Y = h/\lambda, \qquad u = v = P = 0; \\[2mm] \tilde{T} \to 0 \quad \text{as } Y \to \infty, \qquad \tilde{S} \to 0 \quad \text{as } Y \to -\infty. \end{array}\right\} \tag{16}$$

The condition of momentum balance is satisfied for slow viscous flow if

$$\left.\begin{array}{c} \nabla^2 P = 0, \\[3mm] \dfrac{\partial P}{\partial X} = \tfrac{1}{2}\nabla^2 u, \\[3mm] \dfrac{\partial P}{\partial Y} = \tfrac{1}{2}\nabla^2 v, \end{array}\right\} \tag{17}$$

and

so that P, u and v are the pressure and velocity fields for viscosity 0.5 in the absence of any body force (Langlois, 1964). ∇^2 is the two-dimensional Laplacian in (X, Y) coordinates. Steady heat conduction in the ice and bed respectively requires that

$$\nabla^2 \tilde{T} = 0 \qquad \text{and} \qquad \nabla^2 \tilde{S} = 0, \tag{18}$$

if we neglect the motion of the ice (Kamb, 1970). Solutions of the biharmonic relations (Equation (17)) and the harmonic relations (Equations (18)) can be expressed in terms of analytic functions of a complex variable $z = X + iY$ (equations (M43) to (M47)). Let the values of u, v, $\tilde{T}$ and $\tilde{S}$ on $Y = 0$ (continued analytically where $Y = 0$ lies outside the domain) be denoted by

$$\left.\begin{array}{l} u(X, 0) = U(X), \\ v(X, 0) = V(X), \\ \tilde{T}(X, 0) = \Theta(X), \\ \tilde{S}(X, 0) = \Omega(X), \end{array}\right\} \tag{19}$$

and also assume that U, V, Θ and Ω vanish or behave sinusoidally as $X \to \pm\infty$. Then, assuming that the complex potentials vanish at infinity in a way consistent with Equation (16), the potentials may be represented as Cauchy integrals of U, V, Θ and Ω (equations (M50)–(M53)).

The boundary values at $\Upsilon = \Upsilon_b(X)$, which arise in the bed conditions (Equations (1)–(4)), can be approximated by a truncated Taylor series in Υ about $\Upsilon = 0$. The series appears as a power expansion in ϵ by virtue of Equation (11). Evaluation of the various quantities and derivatives on $\Upsilon = 0$ is made from equations (M50)–(M53), these involve Hilbert transforms (Erdélyi, 1954). If $W(t)$ is continuous and vanishes or behaves sinusoidally at infinity, then its Hilbert transform is

$$H[W](X) = \frac{1}{\pi} \int\limits_{-\infty}^{\infty} \frac{W(t)}{t-X}\, \mathrm{d}t, \tag{20}$$

where $\int$ denotes the Cauchy principal value. Also

$$\frac{\mathrm{d}H[W]}{\mathrm{d}X} = H[W'] \left.\vphantom{\begin{array}{c}a\\b\end{array}}\right\}$$

and

$$H[H[W]] = -W, \tag{21}$$

the latter being the inversion theorem. The useful results when $F(x)$ is a truncated Fourier series are

$$\left.\begin{array}{l} H[1] = 0, \\ H\,[\sin kt] = \cos kX, \\ H\,[\cos kt] = -\sin kX; \end{array}\right\} \tag{22}$$

the constant function does not satisfy the inversion theorem.

Now the bed conditions represented by Equations (3) and (4) are given by equations (M65), (M63) and (M62), respectively. Neglecting terms of order $\epsilon^2(U, V, \Theta, \Omega)$, these are

$$-H[\Theta'+r\Omega']+\epsilon\{F(\Theta''-r\Omega'')+F'(\Theta'-r\Omega')\} = 2\omega^2\{V-\epsilon(1-\kappa)\,F'-\epsilon(FU)'\}, \tag{23}$$

$$\Theta+\epsilon FH[\Theta'] = U'+H[V']+\epsilon F\{B+H[U'']-V''\}, \tag{24}$$

$$\Theta - \Omega+\epsilon FH[\Theta'+\Omega'] = \epsilon DF. \tag{25}$$

The argument X is omitted for brevity. A natural length $\lambda_\star$, and later a length $\lambda_{\bigstar}$, occur, these are defined by

$$\left.\begin{array}{l} \lambda_\star = 2\left(\dfrac{\mu k_i C}{L}\right)^{\frac{1}{2}} = 0.077\ \mathrm{m}, \\[12pt] \lambda_{\bigstar} = \left(\dfrac{1+r}{2}\right)^{\frac{1}{2}}\lambda_\star = 0.088\ \mathrm{m}, \end{array}\right\} \tag{26}$$

with previous values, and the ratios

$$\omega = \lambda/\lambda_\star, \qquad \bar\omega = \lambda/\lambda_{\bigstar}, \tag{27}$$

are introduced for convenience. The constants B and D in Equations (24) and (25) are given by

$$\left.\begin{array}{l} B = \dfrac{\kappa}{\sin\alpha}\left(\cos\alpha+\dfrac{Q}{\rho g k_i C}\right)\left(\dfrac{\lambda}{h}\right)^2, \\[14pt] D = \dfrac{r-1}{r}\dfrac{\kappa}{\sin\alpha}\dfrac{Q}{\rho g k_i C}\left(\dfrac{\lambda}{h}\right)^2. \end{array}\right\} \tag{28}$$

The bed conditions given by Equations (1) and (2) require expansions for the dimensionless tractions Σ_s and Σ_n, and the dimensionless tangential velocity V_s/U_s. To the same approximation as Equations (23)–(25), these expansions are:

$$\Sigma_s = \kappa \frac{\lambda}{h}\left\{1-\frac{\lambda}{h}\epsilon F-2\epsilon^2(F')^2\right\}+H[U']-\epsilon\{FH[V'']+2(FU')'\}, \tag{29}$$

$$-\Sigma_n = \beta\kappa \frac{\lambda}{h}\left\{1-M\frac{\lambda}{h}\epsilon F\right\}-H[V']+2\kappa \frac{\lambda}{h}\epsilon F'\left\{1-\frac{\lambda}{h}\epsilon F\right\}+ \\ +\epsilon\{FH[U'']+2F'H[U']\}, \tag{30}$$

where

$$\frac{\lambda p_0}{2\mu U_s} = \beta\kappa \frac{\lambda}{h}, \qquad \beta = \frac{\cot \alpha}{M}, \tag{31}$$

$$M = \frac{\rho g \cos \alpha\, h}{p_a+\rho g \cos \alpha\, h} = O(1), \tag{32}$$

$$\frac{V_s}{U_s} = 1-\kappa+U+2\kappa \frac{\lambda}{h}\epsilon F+\epsilon\{2FH[U']-FV''+F'V\}. \tag{33}$$

It is supposed that

$$\frac{\lambda}{h} \leqslant O(\epsilon), \tag{34}$$

so that Equation (16) is satisfied to any order in ϵ with the exponential decay of u, v and P in Y. Also

$$\beta = O(1/\epsilon) \quad \text{if } \alpha = O(\epsilon), \qquad \beta = O(1) \quad \text{if } \alpha = O(1). \tag{35}$$

In order to balance the boundary conditions in powers of ϵ, let

$$\left.\begin{aligned} U &= U_0+U_1\epsilon+U_2\epsilon^2+ \cdots , \\ V &= V_0+V_1\epsilon+V_2\epsilon^2+ \cdots , \end{aligned}\right\} \tag{36}$$

$$\kappa(\lambda/h) = \gamma_0+\gamma_1\epsilon+\gamma_2\epsilon^2+ \cdots . \tag{37}$$

A determination of the coefficients γ_r gives κ in terms of λ/h and, in turn, U_s and U'_b. It is expected that $\kappa \leqslant O(1)$ so $\gamma_0 = 0$, but this will follow from the balance of the boundary conditions.

Finally, if a proportion j of the work done against basal friction is released as heat, then in Equation (3) V_n is replaced by $V_n+jt_sV_s/L$. For the Coulomb law (Equation (1)) t_s is of order ρgh (and possibly also for the velocity power law (Equation (2)) when α is of order 1) and generally (V_n/V_s) is of order ϵ, so neglect of the coupling term requires

$$jh/\epsilon \ll 3 \times 10^4. \tag{38}$$

This is satisfied for small j, or j approximately one, provided h is moderate. In fact, for Equation (2) the condition for no cavitation when α is of order one also requires that h be not too large. When α is of order ϵ, $t_s \leqslant O(\rho gh\alpha)$, and non-coupling only requires $jh \ll 3 \times 10^4$.

3. COULOMB LAW

The bed conditions are given by Equations (23) to (25) together with Equation (1) written in the form

$$\Sigma_s = -\nu\Sigma_n, \tag{39}$$

where Σ_s and Σ_n are given by Equations (29) and (30). First, consider

$$\alpha = O(\epsilon), \qquad \beta = \beta_1/\epsilon, \qquad \beta_1 = O(1), \tag{40}$$

and suppose ω is of order 1 in the series balance. Values of ω which are smaller or larger may be treated directly, or obtained as limits to the present solution.

The term in ϵ^{-1} gives immediately from Equation (39)

$$\gamma_0 = 0, \qquad (41)$$

and the ϵ^0 terms give

$$H[U_0'] = \nu(\beta_1\gamma_1 - H[V_0']),$$

which imply that

$$\gamma_1 = 0,$$

and

$$U_0 = -\nu V_0, \qquad \left.\right\} \qquad (42)$$

setting constant terms to zero in order to obtain the required behaviour at infinity. Now the ϵ^0 terms of Equations (23) to (25) give

$$\Theta_0 = \Omega_0 = H[V_0'] - \nu V_0', \qquad -(1+r)\,H[\Theta_0'] = 2\omega^2 V_0, \qquad (43)$$

leading to

$$V_0'' - \bar{\omega}^2 V_0 + \nu H[V_0''] = 0. \qquad (44)$$

A search for a periodic solution without constant term,

$$V_0 = \sum_{n=1}^{\infty} (a_n \cos \xi_n X + b_n \sin \xi_n X), \qquad \xi_n = 2\pi n/l, \qquad (45)$$

and the use of the results of Equation (22) shows that both a_n and b_n are zero, so

$$\begin{array}{l} U_0 = 0, \\ V_0 = 0, \\ \Theta_0 = 0, \\ \Omega_0 = 0. \end{array} \left.\right\} \qquad (46)$$

Similarly, the ϵ terms of Equations (39) show that

$$\gamma_2 = 0,$$

and

$$U_1 = -\nu V_1, \qquad \left.\right\} \qquad (47)$$

and Equations (24) and (25) show that

$$\Omega_1 = \Theta_1 - DF, \qquad \Theta_1 = BF + H[V_1'] - \nu V_1'. \qquad (48)$$

Using these results in the ϵ balance of Equation (23) gives

$$V_1'' - \bar{\omega}^2 V_1 + \nu H[V_1''] = -\bar{\omega}^2(1-\kappa)\{F' - AH[F']\}, \qquad (49)$$

where

$$B - \frac{r}{1+r}\,D = A(1-\kappa)\,\bar{\omega}^2. \qquad (50)$$

It was shown in Morland (1976) that A has a maximum value of 0.05 for extreme values of the glacier parameters, so contributions to the geothermal heat flux which arise solely from this coefficient are small. The bounded complementary solution of Equation (49) is zero, analogous to Equation (44). The determination of a particular periodic integral needs F to be specified. As an illustration of this consider a sinusoidal bed whose shape is described by the equation

$$F(X) = \sin X. \qquad (51)$$

The solution of Equation (49) is now

$$V_1 = a \cos X + b \sin X, \tag{52}$$

$$[(1+\bar\omega^2)^2+\nu^2] \begin{pmatrix} a \\ b \end{pmatrix} = (1-\kappa)\,\bar\omega^2 \begin{pmatrix} 1+\bar\omega^2-\nu A \\ A(1+\bar\omega^2)+\nu \end{pmatrix}. \tag{53}$$

As ν tends to zero, U_1 tends also to zero, and

$$a \to \frac{(1-\kappa)\,\bar\omega^2}{1+\bar\omega^2}, \qquad b \to \frac{A(1-\kappa)\,\bar\omega^2}{1+\bar\omega^2}; \tag{54}$$

thus we recover the solution for perfect slip (Morland, 1976).

The leading velocity terms U_1 and V_1 are now determined but, in contrast with the perfect slip solution, $\gamma_2 = 0$ and so κ, U_s and U_b are still unknown. From the ϵ^2 terms of Equation (39)

$$H[U_2'] - FH[V_1''] - 2(FU_1')' = \nu\{\beta_1\gamma_3 - H[V_2'] + FH[U_1''] + 2F'H[U_1']\}. \tag{55}$$

Thus, the second-order velocity coefficient U_2 is given by the balance of periodic terms whereas $\beta_1\gamma_3$ is equal to the constant term of

$$-\{FH[U_1''] + 2F'H[U_1'] + \nu^{-1}FH[V_1'']\}. \tag{56}$$

For the profile represented by Equation (51)

$$\beta_1\gamma_3 = -\frac{a}{2\nu}\,(1+\nu^2) < 0 \qquad \text{for } \nu A < 1+\bar\omega^2, \tag{57}$$

which, since κ and β_1 are positive, implies that U_s is less than zero. This contradictory result stems from the application of Equation (39) to the regions of negative pressure ($\Sigma_n > 0$) given in this solution, when the friction ($\Sigma_s < 0$) is directed up-glacier. From Equation (30) the leading normal pressure term is

$$\begin{aligned} -\Sigma_n &\sim -\epsilon H[V_1'] \\ &= \epsilon(a \cos X + b \sin X), \end{aligned} \tag{58}$$

which oscillates equally between positive and negative values. Thus there is no balance without cavitation for Equations (39) and (40).

Now, α is of order one as are both β and β_0. If we assume that $\nu\beta_0$ is not equal to one, then the ϵ^0 terms of Equation (39) give

$$\left. \begin{aligned} \gamma_0 &= 0, \\ U_0 &= -\nu V_0, \end{aligned} \right\} \tag{59}$$

and

and so Equations (23) to (25) lead to Equations (43) to (46) again. Similarly, the ϵ terms give

$$\left. \begin{aligned} \gamma_1 &= 0, \\ U_1 &= -\nu V_1, \end{aligned} \right\} \tag{60}$$

and also Equations (48) to (54), while the ϵ^2 terms of Equation (39) give Equation (56) for a left-hand side of $\gamma_2(\beta_0-\nu^{-1})$. Thus

$$\gamma_2 = \frac{a(1+\nu^2)}{2(1-\nu\beta_0)}, \tag{61}$$

which tends to $(1-\kappa)\,\bar\omega^2/(2(1+\bar\omega^2))$ as ν tends to zero; we recover the perfect slip solution in this case also. Provided ν is less than $1/\beta$ then we are able to predict a positive surface velocity U_s. But, again, Equation (58) determines the leading normal pressure term, showing

that there is no balance without cavitation. The case where $\nu\beta_0$ is equal to one allows a balance only if U_s is zero.

In conclusion, the solutions constructed for the Coulomb law, assuming no cavitation, are invalid for all ranges of bed-line inclination α.

4. Velocity power law

The law represented by Equation (2) can be rewritten as

$$\Sigma_s = E\frac{\lambda}{2\mu}U_s{}^{(1/m)-1}\left(\frac{V_s}{U_s}\right)^{1/m}, \tag{62}$$

but we lack data for the physical constant E. However, if we suppose that t_s is less than or equal to $\rho gh \sin \alpha$ (a condition compatible with the gravity driving force) then

$$\Sigma_s \leqslant O(\kappa\lambda/h)$$
$$\leqslant O(\lambda/h),$$

since κ is of the order of one for U_b greater than zero. This is satisfied by the equation

$$\Sigma_s = e\left(\frac{\lambda}{h}\right)\kappa^{1-(1/m)}\left(\frac{V_s}{U_s}\right)^{1/m}, \tag{63}$$

where e is less than or equal to the order of one provided that κ is of order one. If any solution with e of order one predicts a smaller κ, and hence a V_s of the order of U_b and U_s, then larger values of t_s occur, presumably with compensating bed tensions as with the Coulomb law, so that cavitation occurs. However, the form of Equation (63) includes all possible valid situations. By construction, the equation

$$e = E/\{(2\mu)^{1/m}(\rho g \sin \alpha)^{1-(1/m)}h^{1-(2/m)}\}, \tag{64}$$

is independent of the solution variable U_s, but depends explicitly on α and on h if m is not equal to 2. The perfect slip law $\Sigma_s = 0$ is given by the limit as e tends to zero.

Now V_s is of order U_b, and if

$$U_b/U_s = 1 - \kappa + U_0$$
$$\approx 0,$$

then Equation (62) approaches the perfect slip law. These conditions do not, in general, follow Morland (1976), so we assume that

$$\frac{U_b}{U_s} = 1 + \kappa + U_0 = O(1), \tag{65}$$

and hence from Equation (33)

$$\left(\frac{V_s}{U_s}\right)^{1/m} = (1 - \kappa + U_0)^{1/m} + O(\epsilon). \tag{66}$$

With the restriction of Equation (34) the ϵ^0 terms of Equation (63) give

$$\gamma_0 = 0, \qquad U_0 = 0, \tag{67}$$

and then, the ϵ^0 terms of Equations (23) to (25) are identical with the perfect slip balance implying that

$$\left.\begin{array}{l} V_0 = 0, \\ \Theta_0 = 0, \\ \Omega_0 = 0. \end{array}\right\} \tag{68}$$

Next, the ϵ terms of Equation (63) give

$$U_1 = 0: \qquad \gamma_1\kappa\left\{1 - e\left(\frac{1-\kappa}{\kappa}\right)^{1/m}\right\} = 0. \tag{69}$$

One solution to these equations is

$$\gamma_1 \neq 0, \qquad \left(\frac{\kappa}{1-\kappa}\right)^{1/m} = e, \tag{70}$$

thus determining κ, U_b, and U_s independent of $F(X)$. Again, the ϵ balance of Equations (23) to (25) is identical with the perfect slip solution, so

$$V_1'' - \bar{\omega}^2 V_1 = -(1-\kappa)\,\bar{\omega}^2\{F - AH[F']\}, \tag{71}$$

and, for the sinusoidal bed (Equation (51)),

$$V_1 = \frac{(1-\kappa)\,\bar{\omega}^2}{1+\bar{\omega}^2}(\cos X + A \sin X). \tag{72}$$

In the limit as e tends to zero, Equation (69) implies that $\gamma_1 = 0$ as in the perfect slip solution.
  The alternative solution of Equation (69) is simply

$$\gamma_1 = 0, \tag{73}$$

and the perfect-slip results (Equations (71) and (72)) again follow. Now, the ϵ^2 balance in Equation (63) requires that

$$\gamma_2 + H[U_2'] - FH[V_1''] = e\gamma_2\left(\frac{1-\kappa}{\kappa}\right)^{1/m}. \tag{74}$$

The product $FH[V_1'']$ is in general the sum of a constant Γ and a periodic term $W(X)$, as in the perfect slip solution, so

$$U_2' = -H[W], \tag{75a}$$

$$\kappa\left\{1 - e\left(\frac{1-\kappa}{\kappa}\right)^{1/m}\right\} = \frac{\Gamma h\epsilon^2}{\lambda}, \tag{75b}$$

where a non-zero Γ implies that γ_2 is not equal to zero. As

$$\gamma_2 \rightarrow O(1/\epsilon),$$
$$\frac{h\epsilon^2}{\kappa\lambda} \rightarrow O(\epsilon),$$

(which is equivalent to γ_1 being non-zero), Equation (75) approaches the result (Equation (70)). Thus, the first solution is given by taking small values for $h\epsilon^2$ in Equation (75b). For the sinusoidal bed (by equation (M88))

$$FH[V_1''] = \frac{(1-\kappa)\,\bar{\omega}^2}{2(1+\bar{\omega}^2)}\{1 - \cos 2X - A \sin 2X\}, \tag{76}$$

and U_2 and V_2 are given by the perfect slip solutions (equations (M90) and (M94)). Thus the velocity perturbation is changed only by the scale factor U_s, which changes with κ (Equation (8)). Now Equation (75b) becomes

$$\frac{\kappa}{1-\kappa} - e\left(\frac{\kappa}{1-\kappa}\right)^{1-(1/m)} - \frac{h\epsilon^2\bar{\omega}}{2\lambda_*(1+\bar{\omega}^2)} = 0. \tag{77}$$

When $e = 0$, Equation (77) reduces to the perfect slip result (equation (M89)).
  The basal-sliding velocity is given by

$$U_b = \frac{1-\kappa}{\kappa}\frac{\rho g \sin \alpha\, h^2}{2\mu}, \tag{78}$$

and, for fixed values of h and $\bar{\omega}$, as e increases $\kappa/(1-\kappa)$ increases and so U_b decreases. As expected, friction decreases the sliding velocity. Calculations have been made for $m = 2$. Figures 2 and 3 show the variation of U_b with $\bar{\omega}$ for values for $h\epsilon^2$ of 0.1, 0.4, 1, and 4, for $e = 0.5$ (Fig. 2) and $e = 1$ (Fig. 3), compared with U_b of the perfect-slip solution ($e = 0$).

In the perfect-slip case the ratio $U_b\epsilon^2/h$ is independent of $h\epsilon^2$. For $\epsilon \approx 0.1$, this range of $h\epsilon^2$ covers values of h running from 10–400 m. If e, ϵ and $\bar{\omega}$ are fixed, we see that U_b decreases as h decreases. Also, for fixed e, the variation of U_b with $\bar{\omega}$ (the length scale of the undulation) decreases as h decreases, and, in particular, U_b is very insensitive to changes in $\bar{\omega}$ provided h is small. (As the graph ordinates are proportional to U_b/h (Figs 2 and 3), so these effects are more significant than the Figures appear to suggest. Figure 4 shows the variation of the ratio U_b/U_s with $\bar{\omega}$ for the values of $h\epsilon^2$ used in the previous Figures. This ratio decreases with increase of e for each value of $h\epsilon^2$ and $\bar{\omega}$, including the case where $e = 0$ (figure 3, Morland (1976)). At fixed e, U_b/U_s becomes less sensitive to a change in $\bar{\omega}$ as $h\epsilon^2$ decreases. At both fixed e and $\bar{\omega}$, the ratio increases as $h\epsilon^2$ decreases.

Finally, the relation for normal bed pressure (Equation (30)) has leading terms (recall Equation (40))

$$
\left.
\begin{aligned}
&\alpha = O(\epsilon): \quad -\Sigma_n \sim \beta_1\gamma_1 + \epsilon(\beta_1\gamma_2 - H[V_1']), \\
&\alpha = O(1): \quad -\Sigma_n \sim \epsilon(\beta_0\gamma_1 - H[V_1']).
\end{aligned}
\right\} \tag{79}
$$

Now if λ/h is of order ϵ, Equation (77) reduces to Equation (70) with $\gamma_1 \neq 0$, $\kappa = O(1)$, and, for such relatively thin glaciers, Equation (79) shows that there is no cavitation if α is of order ϵ, or if α is of order one and

$$
\beta_0\gamma_1 \geqslant (H[V_1'])_{\max}.
$$

For the sinusoidal bed the latter condition becomes

$$
\alpha = O(1): \quad \tan\alpha \leqslant \frac{\lambda(1+\bar{\omega}^2)e^m}{h\epsilon\bar{\omega}^2}\left(1 + \frac{p_a}{\rho g h \cos\alpha}\right). \tag{80}
$$
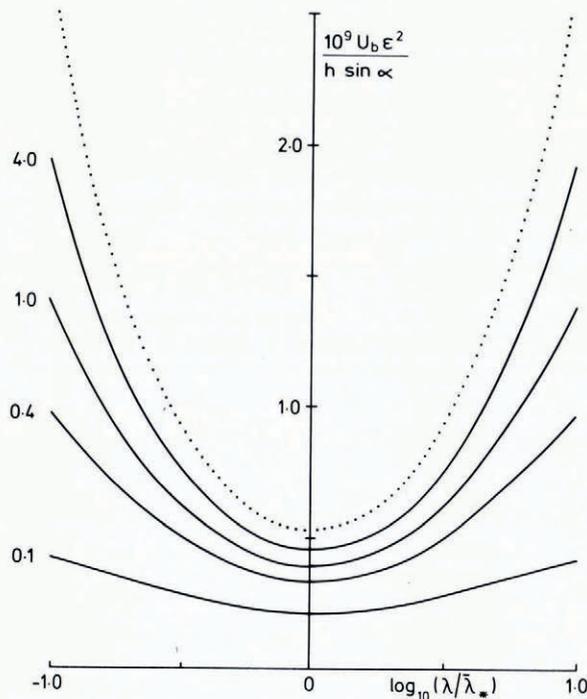
Fig. 2. *Variation of basal sliding velocity U_b with undulation length λ (SI-units) for $e = 0.5$. The perfect-slip solution $(e = 0)$ is shown by the dotted line.*
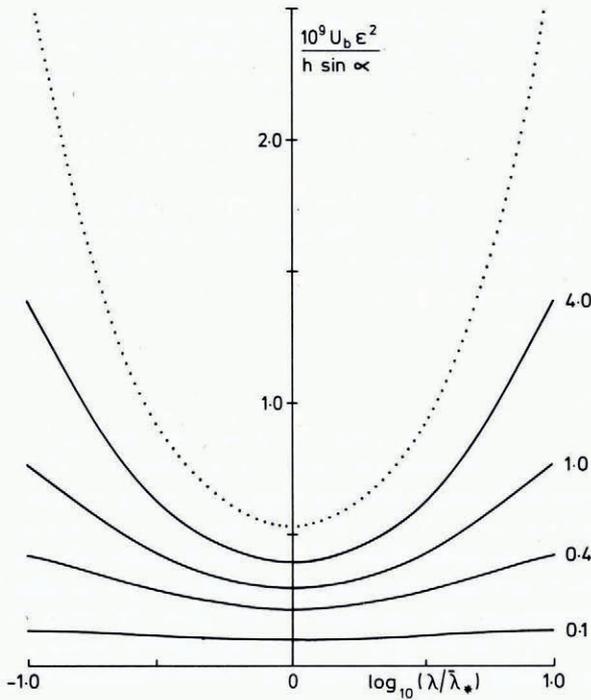
Fig. 3. *Variation of basal sliding velocity U_b with undulation length λ (SI-units) for $e = 1.0$. The perfect-slip solution ($e = 0$) is shown by the dotted line.*
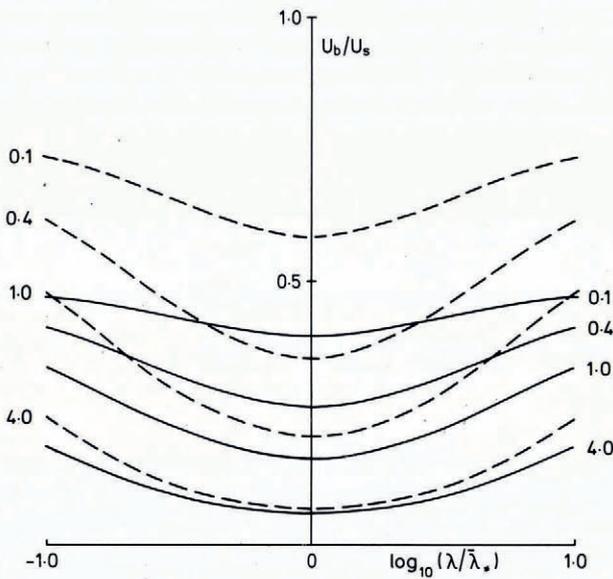
Fig. 4. *Variation of the ratio of basal-sliding velocity to surface velocity with undulation length λ for different values of $h\epsilon^2$ (SI-units). These values are shown against the curves. The dashed lines are for $e = 0.5$, the full lines are for $e = 1.0$.*

For λ/h of order ϵ^2, γ_1 zero and γ_2 non-zero, Equation (79) shows that Σ_n oscillates about zero to the order of ϵ when α is of the order of one, so cavitation always occurs then. But for α of the order of ϵ, cavitation does not occur if $\beta_1\gamma_2$ is greater than or equal to $(H[V_1'])_{\max}$. For the sinusoidal bed, neglecting the small coefficient A, the condition becomes

$$\alpha = O(\epsilon): \quad \tan\alpha \leqslant \frac{\epsilon\lambda(1+\bar{\omega}^2)}{h\epsilon^2\bar{\omega}^2}\left(1+\frac{p_a}{\rho gh\cos\alpha}\right)\frac{\kappa}{1-\kappa}. \tag{81}$$

As with the perfect-slip result of Morland (1976) (equation (M102)), there is a critical limit α_c, of the order ϵ which increased with the friction coefficient e through the factor $\kappa/(1-\kappa)$.

Thus, the law represented by Equation (2) allows non-cavitation solutions for "thin glaciers" for all α less than or equal to α_c (of order one), and for "thick glaciers" for α less than or equal to α_c (of order ϵ).

5. Concluding remarks

The flow solutions which have been established for both friction laws exhibit features which differ from each other and from the perfect-slip solution. With the Coulomb law (Equation (1)), the normal bed traction becomes a tensile stress over finite sections of the profile however small the inclination α; this implies the onset of cavitation and the failure of the solution which assumes contact everywhere. In the perfect-slip solution the bed pressure remains positive everywhere provided that α is less than or equal to α_c where α_c is of the order of ϵ. Since the mean tangential traction must be less than the mean down-plane gravity force, and by Equation (1) t_s is of the same order as the normal pressure $-t_n$ in the mathematical solution, regions of large normal pressure are counterbalanced by regions of negative pressure accompanied by negative values of t_s (the traction driving the glacier). Thus, the Coulomb law is possible only if a significant amount of cavitation takes place. In contrast, the power-velocity law (Equation (2)) gives a bed pressure which is everywhere positive if α is less than or equal to α_c (of order one), provided the glacier depth h is not too large, and if α is less than or equal to α_c (α_c of order ϵ) for deep glaciers. Furthermore, the predicted basal-sliding velocity for a sinusoidal bed is smaller and varies much less with the length scale of the undulation than it does in the perfect-slip solution, both the magnitude and variation decreasing as h decreases. These broad features of the different friction laws may be helpful in the construction of friction models from empirical data.

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