CENTRALISER DIMENSION OF FREE PARTIALLY COMMUTATIVE NILPOTENT GROUPS OF CLASS 2

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Abstract. In an effort to extend the theory of algebraic geometry over groups beyond free groups, Duncan, Kazatchkov and Remeslennikov have studied the notion of centraliser dimension for free partially commutative groups. In this paper we consider the centraliser dimension of free partially commutative nilpotent groups of class 2, showing that a free partially commutative nilpotent group of class 2 with non-commutation graph Γ has the same centraliser dimension as the free partially commutative group represented by the non-commutation graph Γ.

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1. Introduction. There has been much interest in recent years in the development of algebraic geometry over groups (see [**1**, **10**, **12**]), inspired by various successes in the study of equations over groups, such as Makanin's algorithm for deciding whether a system of equations over a free group has a solution, and the solution of Tarski's problem concerning the elementary theory of free groups by Kharlampovich and Myasnikov [**9**], and, independently, Sela [**16**]. In [**3**], Chiswell and Remeslennikov showed that if V is an irreducible algebraic set defined by a one-variable system of equations over a non-abelian free group, F, then either $V = F$, or V is a point, or there exist elements $f, g, h \in F$ such that $V = f C_F(g)h$, where $C_F(g)$ is the centraliser in F of g. It is therefore believed that the investigation of the structure of the centraliser lattice of a group is a first step in the construction of algebraic geometry over that group. The centraliser dimension of a group is defined to be the height of its centraliser lattice, a concept which has been studied in numerous papers, for example [**15**], [**11**], [**2**], [**13**] and [**4**]. To extend the theory of algebraic geometry over groups beyond free groups, Duncan, Kazatchkov and Remeslennikov [**5**] have studied the notion of centraliser dimension for free partially commutative groups, also known as graph groups, trace groups or right-angle Artin groups.

Given an arbitrary variety of groups, $\mathcal{V}$, we may define the class of free partially commutative $\mathcal{V}$ groups in an obvious fashion. In this paper we consider the centraliser dimension of free partially commutative nilpotent groups of class 2. We begin, in Section 3, by considering the form of powers and roots of group elements. We then investigate the structure of centralisers in free partially commutative nilpotent groups of class 2 in Section 4. In Section 5 we show that to calculate centraliser dimension in these groups we only need to consider chains of centralisers of generating elements. This result is then used to show that a free partially commutative nilpotent group of class 2 with non-commutation graph Γ has the same centraliser dimension as the free partially commutative group represented by non-commutation graph Γ.

2. Preliminaries.   In this section, we introduce some preliminary definitions which we will use throughout the paper. For further information on nilpotent groups the reader is referred to the texts by Hall [**7**] and Neumann [**14**].

Let G be a group. The *commutator* of two elements $g_1, g_2 \in G$ is denoted by

$$[g_1, g_2] = g_1^{-1} g_2^{-1} g_1 g_2.$$

For H, K subgroups of G, we let $[H, K]$ denote the subgroup of G generated by all commutators of the form $[h, k]$ for some $h \in H, k \in K$.

A group G is *nilpotent of class 2* if it has a central series of length 2, and no central series of G has shorter length. In particular, the lower central series of G is finite of length 2:

$$G > [G, G] > [[G, G], G] = 1.$$

Therefore a group G is nilpotent of class 2 if and only if $[[G, G], G] = 1$ and $[G, G] \neq 1$. In particular we may note that $[g, h]$ is in the centre of G for all elements $g, h \in G$. This definition may be further refined to show that if G has presentation $G = \langle X | R \rangle$, then G is nilpotent of class 2 if and only if G is non-abelian and $[[x, y], z] = 1$ for all $x, y, z \in X$. Thus a free nilpotent group of class 2 has presentation

$$G = \langle X | \{[[x, y], z] : x, y, z \in X\} \rangle.$$

A group G is said to be *(free) partially commutative* if it has a presentation $G = \langle X | R \rangle$ such that

$$R \subseteq \{[x, y] : x, y \in X\}.$$

Further information on this class of groups may be found in [**4**], [**5**] and [**6**]. We say G is a *(free) partially commutative nilpotent group of class 2* if $G = \langle X | R \rangle$ where

$$R = \{[[x, y], z] : x, y, z \in X\} \cup R',$$
$$R' \subsetneqq \{[x, y] : x, y \in X\}.$$

In both of these cases, we define the *non-commutation graph*, $\Gamma(G)$, of G to be the graph with vertex set X, and with an edge connecting vertices x_i and x_j if and only if $[x_i, x_j] \notin R$.

3. Powers and Roots of Group Elements.   Every element g of the free nilpotent group of class 2 and rank n,

$$\langle x_1, x_2, \ldots, x_n | \{[[x, y], z] : x, y, z \in \{x_1, x_2, \ldots, x_n\}\} \rangle,$$

may be written uniquely in the normal form

$$
\begin{aligned}
g &= x_1^{r_1} \ldots x_n^{r_n} [x_1, x_2]^{s_{1,2}} \ldots [x_1, x_n]^{s_{1,n}} [x_2, x_3]^{s_{2,3}} \ldots [x_2, x_n]^{s_{2,n}} \ldots [x_{n-1}, x_n]^{s_{n-1,n}}, \\
&= x_1^{r_1} x_2^{r_2} \ldots x_n^{r_n} \prod_{1 \leqslant i < j \leqslant n} [x_i, x_j]^{s_{i,j}},
\end{aligned}
\tag{1}
$$

for some $r_i, s_{i,j} \in \mathbb{Z}$ (see Hall [**8**] for further details).

Let

$$G = \langle X | R \rangle,$$
$$= \langle x_1, x_2, \ldots, x_n | \{[[x_i, x_j], x_k] : i, j, k \in \{1, 2, \ldots, n\}\}, [x_{i_1}, x_{j_1}], \ldots, [x_{i_m}, x_{j_m}]\rangle,$$

be a partially commutative nilpotent group of class 2.

LEMMA. *Every element $g \in G$ may be written uniquely in the normal form*

$$g = x_1^{r_1} x_2^{r_2} \ldots x_n^{r_n} \prod_{\substack{1 \leqslant i < j \leqslant n \\ [x_i, x_j] \neq 1}} [x_i, x_j]^{s_{i,j}}$$

for some $r_i, s_{i,j} \in \mathbb{Z}$.
    If

$$h = x_1^{r_1'} x_2^{r_2'} \ldots x_n^{r_n'} \prod_{\substack{1 \leqslant i < j \leqslant n \\ [x_i, x_j] \neq 1}} [x_i, x_j]^{s_{i,j}'},$$

then gh has normal form

$$gh = x_1^{r_1 + r_1'} x_2^{r_2 + r_2'} \ldots x_n^{r_n + r_n'} \prod_{\substack{1 \leqslant i < j \leqslant n \\ [x_i, x_j] \neq 1}} [x_i, x_j]^{s_{i,j} + s_{i,j}' - r_i' r_j}.$$

For $k \in \mathbb{Z}$, the kth power of g has normal form

$$g^k = x_1^{kr_1} x_2^{kr_2} \ldots x_n^{kr_n} \prod_{\substack{1 \leqslant i < j \leqslant n \\ [x_i, x_j] \neq 1}} [x_i, x_j]^{ks_{i,j} - \frac{1}{2}(k-1)kr_i r_j}.$$

Proof. Let $G' = \langle X | R_{G'} \rangle$ be the free nilpotent group of class 2 generated by the set X, and let

$$R' = \{[x_{i_1}, x_{j_1}], [x_{i_2}, x_{j_2}], \ldots, [x_{i_m}, x_{j_m}]\},$$

so that $G = \langle X | R_{G'}, R' \rangle$. Note that R' is a subset of the centre of G', and therefore the subgroup, N, of G' generated by R' is a normal subgroup of G'. By Von Dyck's Theorem, there exists an epimorphism, $\phi : G' \to G$ such that ϕ fixes every $x \in X$, and $\ker(\phi) = N$, the normal closure of R'.

Since G is a nilpotent group of class 2, every element of G may be written in the form (1). In addition, for each commutator $[x_i, x_j] \in R'$ we may set $s_{i,j} = 0$, since $[x_i, x_j] = 1$. Therefore every element g in G may be written in the form

$$g = x_1^{r_1} x_2^{r_2} \ldots x_n^{r_n} \prod_{\substack{1 \leqslant i < j \leqslant n \\ [x_i, x_j] \neq 1}} [x_i, x_j]^{s_{i,j}}, \tag{2}$$

for some $r_i, s_{i,j} \in \mathbb{Z}$. It remains to show that this is a unique form for elements of G.
    Let

$$g = x_1^{r_1} \ldots x_n^{r_n} \prod_{\substack{1 \leqslant i < j \leqslant n \\ [x_i, x_j] \neq 1}} [x_i, x_j]^{s_{i,j}}, \qquad h = x_1^{r_1'} \ldots x_n^{r_n'} \prod_{\substack{1 \leqslant i < j \leqslant n \\ [x_i, x_j] \neq 1}} [x_i, x_j]^{s_{i,j}'},$$

be elements of G written in the form (2). We wish to write the element gh in the form (2). First we note that for any group H, and for any two elements a and b of H, we have the relation

$$ba = abb^{-1}a^{-1}ba = ab[b, a] = ab[a, b]^{-1}.$$

Now suppose that a and b are elements of G. Since G is a nilpotent group of class 2, we may use the above relation to see that

$$
\begin{aligned}
b^s a^r &= b^{s-1}aba^{r-1}[a, b]^{-1}, \\
&= ab^s a^{r-1}[a, b]^{-s}, \\
&= a^r b^s [a, b]^{-rs},
\end{aligned}
$$

for all $r, s \in \mathbb{N}$. Further note that

$$[a, b^{-1}] = b^{-1}[a, b^{-1}]b = b^{-1}a^{-1}bab^{-1}b = (a^{-1}b^{-1}ab)^{-1} = [a, b]^{-1},$$

and similarly that $[a^{-1}, b] = [a, b]^{-1}$ and $[a^{-1}, b^{-1}] = [a, b]$. Therefore, for all $r, s \in \mathbb{N}$ and all $\varepsilon, \delta \in \{-1, 1\}$, the relation

$$
\begin{aligned}
b^{\delta s} a^{\varepsilon r} &= a^{\varepsilon r} b^{\delta s} [a^\varepsilon, b^\delta]^{-rs}, \\
&= a^{\varepsilon r} b^{\delta s} [a, b]^{-(\varepsilon r)(\delta s)},
\end{aligned}
$$

holds in G.

We now use this relation to rearrange elements of our group G into the form (2). For all $i \in \{1, 2, \dots, n-1\}$ and all $j \in \{i+1, i+2, \dots, n\}$, we see that

$$x_j^{r_j} x_i^{r_i'} = x_i^{r_i'} x_j^{r_j} [x_i, x_j]^{-r_i' r_j}.$$

Therefore

$$
\begin{aligned}
&x_1^{r_1} x_2^{r_2} \dots x_n^{r_n} x_1^{r_1'} x_2^{r_2'} \dots x_n^{r_n'} \\
&= x_1^{r_1} x_2^{r_2} \dots x_{n-1}^{r_{n-1}} x_1^{r_1'} x_n^{r_n} x_2^{r_2'} \dots x_n^{r_n'} [x_1, x_n]^{-r_1' r_n}, \\
&= x_1^{r_1+r_1'} x_2^{r_2} x_3^{r_3} \dots x_n^{r_n} x_2^{r_2'} x_3^{r_3'} \dots x_n^{r_n'} \prod_{1 < j \leqslant n} [x_1, x_j]^{-r_1' r_j}, \\
&= x_1^{r_1+r_1'} x_2^{r_2+r_2'} \dots x_n^{r_n+r_n'} \prod_{1 \leqslant i < j \leqslant n} [x_i, x_j]^{-r_i' r_j},
\end{aligned}
$$

which implies that

$$gh = x_1^{r_1+r_1'} x_2^{r_2+r_2'} \dots x_n^{r_n+r_n'} \prod_{\substack{1 \leqslant i < j \leqslant n \\ [x_i, x_j] \neq 1}} [x_i, x_j]^{s_{i,j} + s_{i,j}' - r_i' r_j}.$$

Note that this is in form (2).

Now let $k \in \mathbb{Z}$ and consider the element g^k. We wish to show that

$$g^k = x_1^{kr_1} x_2^{kr_2} \dots x_n^{kr_n} \prod_{\substack{1 \leqslant i < j \leqslant n \\ [x_i, x_j] \neq 1}} [x_i, x_j]^{ks_{i,j} - \frac{1}{2}(k-1)kr_i r_j},$$

for all integers k. We begin with the case where k is a positive integer, and assume that g^k may be written in the above form, which is clearly true for $k = 1$. Noting that $g^{k+1} = g^k g$, we may apply our earlier result for products to see that,

$$g^{k+1} = x_1^{kr_1+r_1} x_2^{kr_2+r_2} \ldots x_n^{kr_n+r_n} \prod_{\substack{1 \leqslant i < j \leqslant n \\ [x_i,x_j] \neq 1}} [x_i, x_j]^{s_{i,j}+ks_{i,j}-\frac{1}{2}(k-1)kr_ir_j-kr_ir_j},$$

$$= x_1^{(k+1)r_1} x_2^{(k+1)r_2} \ldots x_n^{(k+1)r_n} \prod_{\substack{1 \leqslant i < j \leqslant n \\ [x_i,x_j] \neq 1}} [x_i, x_j]^{(k+1)s_{i,j}-\frac{1}{2}k(k+1)r_ir_j}.$$

Therefore, by induction, g^k is of the required form for all positive integers k. The result also clearly holds for $k = 0$. Now let $k = -1$. Using our earlier technique for rearranging group elements, we see that

$$g^{-1} = x_n^{-r_n} x_{n-1}^{-r_{n-1}} \ldots x_1^{-r_1} \prod_{\substack{1 \leqslant i < j \leqslant n \\ [x_i,x_j] \neq 1}} [x_i, x_j]^{-s_{i,j}},$$

$$= x_1^{-r_1} x_2^{-r_2} \ldots x_n^{-r_n} \prod_{\substack{1 \leqslant i < j \leqslant n \\ [x_i,x_j] \neq 1}} [x_i, x_j]^{-s_{i,j}-r_ir_j},$$

and therefore the result holds for $k = -1$. Finally note that for any positive integer k, we have

$$g^{-k} = (g^k)^{-1},$$
$$= x_1^{-kr_1} x_2^{-kr_2} \ldots x_n^{-kr_n} \prod_{\substack{1 \leqslant i < j \leqslant n \\ [x_i,x_j] \neq 1}} [x_i, x_j]^{-(ks_{i,j}-\frac{1}{2}(k-1)kr_ir_j)-k^2r_ir_j},$$

$$= x_1^{-kr_1} x_2^{-kr_2} \ldots x_n^{-kr_n} \prod_{\substack{1 \leqslant i < j \leqslant n \\ [x_i,x_j] \neq 1}} [x_i, x_j]^{-ks_{i,j}-\frac{1}{2}(-k-1)(-k)r_ir_j},$$

and therefore g^k is of the required form for all integers k.

Now suppose that $g = h$ in G, so that

$$gh^{-1} = x_1^{r_1-r_1'} x_2^{r_2-r_2'} \ldots x_n^{r_n-r_n'} \prod_{\substack{1 \leqslant i < j \leqslant n \\ [x_i,x_j] \neq 1}} [x_i, x_j]^{s_{i,j}-s_{i,j}'-r_i'r_j'+r_i'r_j},$$

$$= 1.$$

Recall that there exists an epimorphism $\phi : G' \to G : x_i \mapsto x_i$ such that $\ker(\phi) = N$, where N is the subgroup of G generated by R'. This implies that the inverse image of gh^{-1} under ϕ is

$$\phi^{-1}(gh^{-1}) = \left\{ \prod_{[x_i,x_j] \in R'} [x_i, x_j]^{t_{i,j}} \,\middle|\, t_{i,j} \in \mathbb{Z} \right\}.$$

Since $\phi(gh^{-1}) = gh^{-1} = 1$, this implies that the relation

$$x_1^{r_1-r_1'} x_2^{r_2-r_2'} \dots x_n^{r_n-r_n'} \prod_{\substack{1 \leqslant i < j \leqslant n \\ [x_i, x_j] \notin R'}} [x_i, x_j]^{s_{i,j}-s_{i,j}'-r_i'r_j'+r_i'r_j} = \prod_{\substack{1 \leqslant i < j \leqslant n \\ [x_i, x_j] \in R'}} [x_i, x_j]^{t_{i,j}},$$

holds in the group G' for some $t_{i,j} \in \mathbb{Z}$. By the uniqueness of the normal form for G', this implies

(i) $r_i - r_i' = 0$ for all $i \in \{1, 2, \dots, n\}$; and

(ii) $s_{i,j} - s_{i,j}' - r_i'r_j' + r_i'r_j = 0$ for all $1 \leqslant i < j \leqslant n$ such that $[x_i, x_j] \notin R'$.

Therefore $r_i = r_i'$ for all $i \in \{1, 2, \dots, n\}$ and $s_{i,j} = s_{i,j}'$ for all $1 \leqslant i < j \leqslant n$ such that $[x_i, x_j] \notin R'$. This implies that (2) is a unique normal form for elements of a partially commutative nilpotent group of class 2. $\square$

Let g, h be elements of G. If g and h are both (positive) powers of some common element, then clearly there exist positive integers k_1, k_2 such that $g^{k_1} = h^{k_2}$. We now show that the converse is also true.

LEMMA. *Let g, h be elements of G. For any $k_1, k_2 \in \mathbb{N}$, if $g^{k_1} = h^{k_2}$ then g and h are both powers of a common element. Further, if, in addition, $k_1 = k_2$ then $g = h$.*

Proof. Let

$$g = x_1^{r_1} \dots x_n^{r_n} \prod_{\substack{1 \leqslant i < j \leqslant n \\ [x_i, x_j] \neq 1}} [x_i, x_j]^{s_{i,j}}, \qquad h = x_1^{r_1'} \dots x_n^{r_n'} \prod_{\substack{1 \leqslant i < j \leqslant n \\ [x_i, x_j] \neq 1}} [x_i, x_j]^{s_{i,j}'},$$

be normal forms for elements $g, h \in G$. Suppose that $g^{k_1} = h^{k_2}$ for some $k_1, k_2 \in \mathbb{N}$. This implies that

(i) $k_1 r_i = k_2 r_i'$ for all $i \in \{1, 2, \dots, n\}$; and,

(ii) for all $1 \leqslant i < j \leqslant n$ such that $[x_i, x_j] \neq 1$,

$$k_1 s_{i,j} - \frac{1}{2}(k_1 - 1)k_1 r_i r_j = k_2 s_{i,j}' - \frac{1}{2}(k_2 - 1)k_2 r_i' r_j'. \tag{3}$$

We may assume without loss of generality that $k_1 \leqslant k_2$. Note that if $k_1 = k_2$ then $r_i = r_i'$ for all $i \in \{1, 2, \dots, n\}$, and further $s_{i,j} = s_{i,j}'$ for $1 \leqslant i < j \leqslant n$ such that $[x_i, x_j] \neq 1$. Therefore, if $k_1 = k_2$ then $g = h$. Further note that if k_1 and k_2 are not coprime, say $(k_1, k_2) = t > 1$, then $k_1 = \alpha_1 t$ and $k_2 = \alpha_2 t$ for some $\alpha_1, \alpha_2 \in \mathbb{N}$ with $(\alpha_1, \alpha_2) = 1$. This implies that $(g^{\alpha_1})^t = (h^{\alpha_2})^t$, so that $g^{\alpha_1} = h^{\alpha_2}$. It remains to consider the possibility that $(k_1, k_2) = 1$. In this case, k_1 divides r_i' and k_2 divides r_i for all $i \in \{1, 2, \dots, n\}$.

Suppose k_2 is odd. For some $t_i \in \mathbb{Z}$ we have

$$g = x_1^{k_2 t_1} x_2^{k_2 t_2} \dots x_n^{k_2 t_n} \prod_{\substack{1 \leqslant i < j \leqslant n \\ [x_i, x_j] \neq 1}} [x_i, x_j]^{s_{i,j}}.$$

Now for

$$g' = x_1^{t_1} x_2^{t_2} \dots x_n^{t_n} \prod_{\substack{1 \leqslant i < j \leqslant n \\ [x_i, x_j] \neq 1}} [x_i, x_j]^{u_{i,j}},$$

we have

$$(g')^{k_2} = x_1^{k_2 t_1} x_2^{k_2 t_2} \ldots x_n^{k_2 t_n} \prod_{\substack{1 \leqslant i < j \leqslant n \\ [x_i, x_j] \neq 1}} [x_i, x_j]^{k_2 u_{i,j} - \frac{1}{2}(k_2 - 1)k_2 t_i t_j}.$$

Therefore, $g = (g')^{k_2}$ for some element g' of G if and only if there exist integers $u_{i,j}$ such that

$$s_{i,j} = k_2 u_{i,j} - \frac{1}{2}(k_2 - 1)k_2 t_i t_j,$$

for all $1 \leqslant i < j \leqslant n$ such that $[x_i, x_j] \neq 1$.

We may rearrange equation (3) to give

$$k_1 s_{i,j} = k_2 s'_{i,j} - \frac{1}{2}(k_2 - 1)k_2 r'_i r'_j + \frac{1}{2}(k_1 - 1)k_1 r_i r_j,$$

for all $1 \leqslant i < j \leqslant n$ such that $[x_i, x_j] \neq 1$. Since k_2 is odd, $k_2 - 1$ is even, which together with the fact k_2 divides r_i for all $i \in \{1, 2, \ldots, n\}$, implies that k_2 divides $k_1 s_{i,j}$. Since k_1 and k_2 are coprime, this implies that k_2 divides $s_{i,j}$. Therefore

$$u_{i,j} = \frac{s_{i,j}}{k_2} + \frac{1}{2}(k_2 - 1)t_i t_j,$$

is an integer for all $1 \leqslant i < j \leqslant n$ such that $[x_i, x_j] \neq 1$.

So k_2 odd implies that $g = (g')^{k_2}$ for some $g' \in G$, which in turn implies that $h = (g')^{k_1}$ since $g^{k_1} = h^{k_2}$.

Similarly, if k_1 is odd then $h = (h')^{k_1}$ for some $h' \in G$, so that $g = (h')^{k_2}$. Since $(k_1, k_2) = 1$ implies that at least one of k_1, k_2 is odd, if k_1 and k_2 are coprime, g, h must both be powers of a common element.

We have thus established that for any $k_1, k_2 \in \mathbb{N}$, if $g^{k_1} = h^{k_2}$ then g and h are both powers of a common element. $\square$

We say h is an *(nth) root* of g if there exists a positive integer n such that $h^n = g$. We call h a *least root* of g if $g = h^n$ for some $n \in \mathbb{N}$, and for every $w \in G$ such that $w^m = g$ for some $m \in \mathbb{N}$ we have $m | n$ and $w = h^{\frac{n}{m}}$.

LEMMA. *Every element $g \in G$ has a unique least root.*

Proof. Let

$$g = x_1^{r_1} \ldots x_n^{r_n} \prod_{\substack{1 \leqslant i < j \leqslant n \\ [x_i, x_j] \neq 1}} [x_i, x_j]^{s_{i,j}}, \qquad h = x_1^{r'_1} \ldots x_n^{r'_n} \prod_{\substack{1 \leqslant i < j \leqslant n \\ [x_i, x_j] \neq 1}} [x_i, x_j]^{s'_{i,j}},$$

be normal forms for elements $g, h \in G$. Notice that if g has a least root, it must be unique, since we have already shown that if $u^k = v^k$ for elements u, v of G then $u = v$. We now show that every element g of G has a least root.

First, we note that $h^m = g$ implies that:
  (i) $r_i = m r'_i$ for all $i \in \{1, 2, \ldots, n\}$; and,
  (ii) for all $1 \leqslant i < j \leqslant n$ such that $[x_i, x_j] \neq 1$, we have

$$s_{i,j} = m s'_{i,j} - \frac{1}{2}(m - 1)m r'_i r'_j.$$

Let the greatest common divisor of $r_1, r_2, \ldots, r_n$ be k. If $k = 1$, then g is its own least root. If k is greater than one, let d be the greatest divisor of k such that there exist integers $u_{i,j}$ satisfying

$$s_{i,j} = du_{i,j} - \frac{d-1}{2d} r_i r_j,$$

for all $i, j \in \{1, 2, \ldots, n\}$ such that $[x_i, x_j] \neq 1$. In other words, let d be the greatest divisor of k such that

$$\frac{s_{i,j}}{d} + \frac{d-1}{2d^2} r_i r_j$$

is an integer for all $i, j \in \{1, 2, \ldots, n\}$ such that $[x_i, x_j] \neq 1$. If $d = 1$, then g is its own least root. For $d > 1$, g has a dth root

$$h = x_1^{\frac{r_1}{d}} x_2^{\frac{r_2}{d}} \ldots x_n^{\frac{r_n}{d}} \prod_{\substack{1 \leqslant i < j \leqslant n \\ [x_i, x_j] \neq 1}} [x_i, x_j]^{\frac{s_{i,j}}{d} + \frac{d-1}{2d^2} r_i r_j},$$

and h is its own least root by definition of d. Now, for any root w of g, we have $g = w^r = h^d$ for some $r \in \mathbb{Z}$, so that w and h are both powers of some common element. Since h is its own least root, this means that h must be a root of w, say $w = h^s$. Now $w^r = h^{sr} = h^d$ implies that $d = rs$, so $r \mid d$ and $w = h^{\frac{d}{r}}$. Therefore h is a least root of g. $\square$

4. Centralisers. Let S be a subset of a group G. The *centraliser* of S in G is denoted by

$$C_G(S) = \{x \in G \mid sx = xs \ \forall s \in S\}.$$

The centraliser in G of an element $g \in G$ is

$$C_G(g) = \{h \in G \mid gh = hg\}.$$

The centre of G is denoted by $Z(G)$.

Let G be a partially commutative nilpotent group of class 2,

$$G = \langle x_1, x_2, \ldots, x_n \mid \{[[x_i, x_j], x_k] : i, j, k \in \{1, 2, \ldots, n\}\}, [x_{i_1}, x_{j_1}], \ldots, [x_{i_m}, x_{j_m}] \rangle,$$

and let g, h be elements of G:

$$g = x_1^{r_1} x_2^{r_2} \ldots x_n^{r_n} \prod_{\substack{1 \leqslant i < j \leqslant n \\ [x_i, x_j] \neq 1}} [x_i, x_j]^{s_{i,j}}, \qquad h = x_1^{r'_1} x_2^{r'_2} \ldots x_n^{r'_n} \prod_{\substack{1 \leqslant i < j \leqslant n \\ [x_i, x_j] \neq 1}} [x_i, x_j]^{s'_{i,j}}.$$

Note that

$$gh = x_1^{r_1 + r'_1} x_2^{r_2 + r'_2} \ldots x_n^{r_n + r'_n} \prod_{\substack{1 \leqslant i < j \leqslant n \\ [x_i, x_j] \neq 1}} [x_i, x_j]^{s_{i,j} + s'_{i,j} - r'_i r_j}$$

and

$$hg = x_1^{r_1'+r_1} x_2^{r_2'+r_2} \ldots x_n^{r_n'+r_n} \prod_{\substack{1 \leqslant i < j \leqslant n \\ [x_i, x_j] \neq 1}} [x_i, x_j]^{s_{i,j}'+s_{i,j}-r_i r_j'}.$$

Therefore g commutes with h precisely when $r_i' r_j = r_i r_j'$ for all i, j such that $1 \leqslant i < j \leqslant n$ and $[x_i, x_j] \neq 1$, and the centraliser of g in G is

$$C_G(g) = \{h \in G | r_i' r_j = r_i r_j' \text{ for } 1 \leqslant i < j \leqslant n \text{ with } [x_i, x_j] \neq 1\}.$$

We may note that $r_i' r_j = r_i r_j'$ if and only if:

  (i) $r_i = 0, r_j \neq 0 \Rightarrow r_i' = 0$;
  (ii) $r_i \neq 0, r_j = 0 \Rightarrow r_j' = 0$; and,
 (iii) $r_i \neq 0, r_j \neq 0 \Rightarrow \frac{r_i'}{r_i} = \frac{r_j'}{r_j}$.

In the case where g is a generator, say $g = x_k$ for some $k \in \{1, 2, \ldots, n\}$, we have

$$C_G(x_k) = \{h \in G | r_i' = 0 \text{ for } 1 \leqslant i \leqslant n \text{ with } [x_i, x_k] \neq 1\}.$$

Returning to the general case, we define the set $A_g = \{x_i | r_i \neq 0\}$, and split A_g into distinct subsets $A_{g,1}, A_{g,2}, \ldots, A_{g,q}$, such that two distinct elements $x_i, x_j \in A_g$ are in the same subset if and only if either:

  (i) $[x_i, x_j] \neq 1$; or,
  (ii) there exist $x_{\alpha_1}, x_{\alpha_2}, \ldots, x_{\alpha_t} \in A_g$ such that $[x_i, x_{\alpha_1}] \neq 1$, $[x_{\alpha_t}, x_j] \neq 1$, and $[x_{\alpha_k}, x_{\alpha_{k+1}}] \neq 1$ for all $k \in \{1, 2, \ldots, t-1\}$.

Recall that the non-commutation graph of a partially commutative nilpotent group $H = \langle X | R \rangle$ is the graph with vertex set X, and with an edge connecting vertices x_i and x_j if and only if $[x_i, x_j] \notin R$. Therefore the $A_{g,i}$ correspond to the vertex sets of the connected components of non-commutation graph $\Gamma(\langle A_g \rangle)$.

Note that h is an element of the centraliser $C_G(g)$ if and only if:

  (i) if $x_i \notin A_g$ and $x_j \in A_g$ with $[x_i, x_j] \neq 1$, then $r_i' = 0$; and
  (ii) if $A_{g,\gamma} = \{x_{i_1}, x_{i_2}, \ldots, x_{i_p}\}$, then

$$\frac{r_{i_1}'}{r_{i_1}} = \frac{r_{i_2}'}{r_{i_2}} = \cdots = \frac{r_{i_p}'}{r_{i_p}},$$

and therefore there exists $k \in \mathbb{Q}$ such that $r_{i_j}' = k r_{i_j}$ for all $j \in \{1, 2, \ldots, p\}$.

We may now write $C_G(g)$ in the following fashion:

$$C_G(g) = \Big\{ x_1^{k_1 r_1 + m_1} x_2^{k_2 r_2 + m_2} \ldots x_n^{k_n r_n + m_n} \prod [x_i, x_j]^{s_{i,j}'} \Big| $$
$$((k_i r_i), m_i, s_{i,j}' \in \mathbb{Z} \, \forall i, j), (x_i, x_j \in A_{g,\gamma} \Rightarrow k_i = k_j),$$
$$(m_i \neq 0 \Rightarrow (x_i \notin A_g, \, [x_i, x_j] = 1 \, \forall x_j \in A_g)) \Big\}.$$

In the case where G is a free nilpotent group of class 2, this simplifies to

$$C_G(g) = \Big\{ x_1^{k r_1 + m_1} x_2^{k r_2 + m_2} \ldots x_n^{k r_n + m_n} \prod [x_i, x_j]^{s_{i,j}'} \Big|$$
$$(k r_i, m_i, s_{i,j}' \in \mathbb{Z} \, \forall i, j), (A_g \neq \emptyset \Rightarrow m_j = 0 \, \forall j) \Big\}.$$

Returning to the general case, notice that, by definition, the distinct $A_{g,i}$ commute pairwise. This allows us to rewrite g in the form,

$$g = u_1 u_2 \ldots u_q \prod_{\substack{1 \leqslant i < j \leqslant n \\ [x_i, x_j] \neq 1}} [x_i, x_j]^{s_{i,j}},$$

where $u_i = x_1^{t_{i,1}} x_2^{t_{i,2}} \ldots x_n^{t_{i,n}}$ with

$$t_{i,j} = \begin{cases} r_j & \text{if } x_j \in A_{g,i}, \\ 0 & \text{otherwise.} \end{cases}$$

Let $p \in \{1, 2, \ldots, q\}$, and let d be the greatest common divisor of $t_{p,1}, t_{p,2}, \ldots, t_{p,n}$. For each $j \in \{1, 2, \ldots, n\}$, let $c_{p,j} \in \mathbb{Z}$ be such that $t_{p,j} = dc_{p,j}$. Then,

$$\begin{aligned} u_p &= x_1^{dc_{p,1}} x_2^{dc_{p,2}} \ldots x_n^{dc_{p,n}}, \\ &= (x_1^{c_{p,1}} x_2^{c_{p,2}} \ldots x_n^{c_{p,n}})^d \prod_{1 \leqslant i < j \leqslant n} [x_i, x_j]^{\frac{1}{2}(d-1)dc_i c_j}. \end{aligned}$$

Let $v_p = x_1^{c_{p,1}} x_2^{c_{p,2}} \ldots x_n^{c_{p,n}}$, and note that $C_G(u_p) = C_G(v_p)$, so that, in particular $[v_i, v_j] = 1$ for all i, j. Further note that we may rewrite the centraliser $C_G(g)$ in the following fashion

$$\begin{aligned} C_G(g) = \{ & v_1^{k_1} v_2^{k_2} \ldots v_q^{k_q} x_1^{m_1} x_2^{m_2} \ldots x_n^{m_n} \prod [x_i, x_j]^{s'_{i,j}} | k_i, m_i, s'_{i,j} \in \mathbb{Z}, \\ & m_i \neq 0 \Rightarrow (x_i \notin A_g, [x_i, x_j] = 1 \; \forall x_j \in A_g) \}. \end{aligned}$$

From this, we can see that $C_G(g) \subseteq C_G(v_i) = C_G(u_i)$ for all $i \in \{1, 2, \ldots, q\}$, so that

$$C_G(g) \subseteq C_G(u_1) \cap C_G(u_2) \cap \cdots \cap C_G(u_q).$$

Conversely, if h commutes with u_i for all $i \in \{1, 2, \ldots, q\}$, then h commutes with $g = u_1 u_2 \ldots u_q \prod [x_i, x_j]^{s_{i,j}}$. Therefore

$$\begin{aligned} C_G(g) &= C_G(u_1) \cap C_G(u_2) \cap \cdots \cap C_G(u_q), \\ &= C_G(v_1) \cap C_G(v_2) \cap \cdots \cap C_G(v_q). \end{aligned}$$

Note in particular that this implies $C_G(g^m) = C_G(g)$ for all non-zero integers m.

We now wish to consider centralisers of subsets of G. For each $\alpha \in \{1, 2, \ldots, p\}$, let

$$g_\alpha = x_1^{r_{\alpha,1}} x_2^{r_{\alpha,2}} \ldots x_n^{r_{\alpha,n}} \prod_{\substack{1 \leqslant i < j \leqslant n \\ [x_i, x_j] \neq 1}} [x_i, x_j]^{s_{\alpha,i,j}}.$$

Then

$$\begin{aligned} C_G(g_1, g_2, \ldots, g_p) &= C_G(g_1) \cap C_G(g_2) \cap \cdots \cap C_G(g_p) \\ &= \Big\{ x_1^{r_1} x_2^{r_2} \ldots x_n^{r_n} \prod [x_i, x_j]^{s_{i,j}} | r_{\alpha,i} r_j = r_i r_{\alpha,j} \text{ for } 1 \leqslant \alpha \leqslant p, \\ & \quad 1 \leqslant i < j \leqslant n \text{ with } [x_i, x_j] \neq 1 \Big\}. \end{aligned}$$

Let $A = A_{g_1} \cup A_{g_2} \cup \cdots \cup A_{g_p}$.

Then

$$C_G(g_1, g_2, \ldots, g_p) = \left\{ x_1^{r_1} x_2^{r_2} \ldots x_n^{r_n} \prod [x_i, x_j]^{s_{i,j}} \right|$$
$$(x_i, x_j \in A_{g_\alpha, \gamma} \Rightarrow r_i = k_{\alpha, \gamma} r_{\alpha, i}, r_j = k_{\alpha, \gamma} r_{\alpha, j}),$$
$$\left. (x_i \notin A, x_j \in A \text{ with } [x_i, x_j] \neq 1 \Rightarrow r_i = 0) \right\}.$$

5. Centraliser Dimension. Let G be a group. The *centraliser dimension*, $\mathrm{cdim}(G)$, of G is the maximum length of a strict chain of centralisers of subsets of G. Where no finite maximum exists, G is said to have infinite centraliser dimension. Thus if G had a strict centraliser chain of length n,

$$G = C_G(1) \supsetneq C_G(S_1) \supsetneq C_G(S_2) \supsetneq \cdots \supsetneq C_G(S_n) = Z(G),$$

then $\mathrm{cdim}(G) \geqslant n$. If $\mathrm{cdim}(G) = n$ then there exist elements $g_1, g_2, \ldots, g_n \in G$ such that

$$G = C_G(1) \supsetneq C_G(g_1) \supsetneq C_G(g_1, g_2) \supsetneq \cdots \supsetneq C_G(g_1, g_2, \ldots, g_n) = Z(G).$$

Suppose G is free nilpotent of class 2;

$$G = \langle x_1, x_2, \ldots, x_n | \{ [[x_i, x_j], x_k] : i, j, k \in \{1, 2, \ldots, n\} \} \rangle.$$

Then the centre of G is

$$Z(G) = \left\{ \prod_{1 \leqslant i < j \leqslant n} [x_i, x_j]^{s_{i,j}} \,\middle|\, s_{i,j} \in \mathbb{Z} \right\}.$$

Let g_1, g_2 be elements of G, where each g_α has normal form

$$g_\alpha = x_1^{r_{\alpha,1}} x_2^{r_{\alpha,2}} \ldots x_n^{r_{\alpha,n}} \prod_{1 \leqslant i < j \leqslant n} [x_i, x_j]^{s_{\alpha,i,j}}.$$

The centraliser of g_1 in G is

$$C_G(g_1) = \left\{ x_1^{kr_{1,1}+m_1} x_2^{kr_{1,2}+m_2} \ldots x_n^{kr_{1,n}+m_n} \prod [x_i, x_j]^{s_{i,j}} | s_{i,j}, kr_{1,i}, m_i \in \mathbb{Z}, \right.$$
$$\left. (r_{1,i} \neq 0 \Rightarrow m_j = 0 \,\forall j) \right\},$$
$$= \begin{cases} \{ x_1^{kr_{1,1}} x_2^{kr_{1,2}} \ldots x_n^{kr_{1,n}} \prod [x_i, x_j]^{s_{i,j}} | s_{i,j}, (kr_{1,i}) \in \mathbb{Z} \} & \text{for } g_1 \notin Z(G), \\ G & \text{for } g_1 \in Z(G), \end{cases}$$

and if $C_G(g_1) \neq G$ then

$$C_G(g_1, g_2) = \begin{cases} C_G(g_1) & \text{if } \exists k \in \mathbb{Q} \text{ such that } kr_{1,i} = r_{2,i} \,\forall i, \\ Z(G) & \text{otherwise.} \end{cases}$$

Thus the length of a centraliser chain can be no greater than two. For any $g \in G$ which is not an element of the centre of G we have a maximal chain

$$G = C_G(1) \supsetneq C_G(g) \supsetneq Z(G)$$

and the centraliser dimension of a free nilpotent group, G, of class 2 is two.

In [**5**] it is shown that the centraliser dimension of a partially commutative group $G = \langle X | R \rangle$ is equal to the maximum possible length of a strictly descending chain of centralisers of subsets of X. We will now show that this is also the case for partially commutative nilpotent groups of class 2.

PROPOSITION. *Let $G = \langle X | R \rangle$ be a partially commutative nilpotent group of class 2. Then the centraliser dimension of G is finite. Further, if $\mathrm{cdim}(G) = d$ then there exist elements $x_{i_1}, x_{i_2}, \ldots, x_{i_d}$ of X such that*

$$G \supsetneqq C_G(x_{i_1}) \supsetneqq C_G(x_{i_1}, x_{i_2}) \supsetneqq \cdots \supsetneqq C_G(x_{i_1}, x_{i_2}, \ldots, x_{i_d}) = Z(G)$$

is a centraliser chain in G.

Proof. Let

$$G = C_G(1) \supsetneqq C_G(g_1) \supsetneqq C_G(g_1, g_2) \supsetneqq \ldots \supsetneqq C_G(g_1, g_2, \ldots, g_d) \qquad (4)$$

be a centraliser chain in G. We wish to show that this implies there exist elements $x_{i_1}, x_{i_2}, \ldots, x_{i_d}$ in X such that

$$G \supsetneqq C_G(x_{i_1}) \supsetneqq C_G(x_{i_1}, x_{i_2}) \supsetneqq \cdots \supsetneqq C_G(x_{i_1}, x_{i_2}, \ldots, x_{i_d})$$

is a centraliser chain in G. Let $X = \{x_1, x_2, \ldots, x_n\}$, and for each $\alpha \in \{1, 2, \ldots, d\}$, let g_α have normal form

$$g_\alpha = x_1^{r_{\alpha,1}} x_2^{r_{\alpha,2}} \ldots x_n^{r_{\alpha,n}} \prod_{1 \leqslant i < j \leqslant n} [x_i, x_j]^{s_{\alpha,i,j}}.$$

We now rewrite each g_α in the form

$$g_\alpha = u_{\alpha,1} u_{\alpha,2} \ldots u_{\alpha,q_\alpha} \prod_{1 \leqslant i < j \leqslant n} [x_i, x_j]^{s_{\alpha,i,j}},$$

where $u_{\alpha,i} = x_1^{t_{\alpha,i,1}} x_2^{t_{\alpha,i,2}} \ldots x_n^{t_{\alpha,i,n}}$ with

$$t_{\alpha,i,j} = \begin{cases} r_{\alpha,j} & \text{if } x_j \in A_{g_\alpha,i}, \\ 0 & \text{otherwise.} \end{cases}$$

Further, for each $i \in \{1, 2, \ldots, q_\alpha\}$, we let $c_{\alpha,i}$ be the greatest common divisor of $t_{\alpha,i,1}, t_{\alpha,i,2}, \ldots, t_{\alpha,i,n}$, and let

$$v_{\alpha,i} = x_1^{t'_{\alpha,i,1}} x_2^{t'_{\alpha,i,2}} \ldots x_n^{t'_{\alpha,i,n}},$$

where $t_{\alpha,i,j} = t'_{\alpha,i,j} c_{\alpha,i}$ for all $j \in \{1, 2, \ldots, n\}$. Recall that for elements written in this form,

$$C_G(g_\alpha) = C_G(v_{\alpha,1}) \cap C_G(v_{\alpha,2}) \cap \cdots \cap C_G(v_{\alpha,q_\alpha}).$$

By (4),

$$C_G(g_1, g_2, \ldots, g_{\alpha-1}) \neq C_G(g_1, g_2, \ldots, g_\alpha) = C_G(g_1, g_2, \ldots, g_{\alpha-1}) \cap C_G(g_\alpha),$$

which implies that

$$C_G(g_1, g_2, \ldots, g_{\alpha-1}) \nsubseteq C_G(g_\alpha).$$

Therefore there exists some $k \in \{1, 2, \ldots, q_\alpha\}$ such that

$$C_G(g_1, g_2, \ldots, g_{\alpha-1}) \nsubseteq C_G(v_{\alpha,k}).$$

Reorder the $v_{\alpha,i}$ as necessary so that $k = 1$. Noting that $C_G(g_\alpha) \subseteq C_G(v_{\alpha,1})$, we may deduce that

$$C_G(g_1, \ldots, g_{\alpha-1}) \supsetneq C_G(g_1, \ldots, g_{\alpha-1}, v_{\alpha,1}) \supseteq C_G(g_1, \ldots, g_{\alpha-1}, g_\alpha).$$

Further note that,

$$C_G(g_1, g_2, \ldots, g_i) = C_G(g_1, \ldots, g_{\alpha-1}, v_{\alpha,1}, g_{\alpha+1}, \ldots, g_i) \cap C_G(v_{\alpha,2}, v_{\alpha,3}, \ldots, v_{\alpha,q_\alpha}),$$

for all $i \in \{\alpha, \alpha + 1, \ldots, d\}$, and therefore the chain (4) implies that

$$C_G(g_1, g_2, \ldots, g_{\alpha-1}, v_{\alpha,1}) \supsetneq C_G(g_1, g_2, \ldots, g_{\alpha-1}, v_{\alpha,1}, g_{\alpha+1}, g_{\alpha+2}, \ldots, g_j)$$
$$\supsetneq C_G(g_1, g_2, \ldots, g_{\alpha-1}, v_{\alpha,1}, g_{\alpha+1}, g_{\alpha+2}, \ldots, g_{j+1}),$$

for all $j \in \{\alpha + 1, \alpha + 2, \ldots, d - 1\}$. This means that we can replace g_α by $v_{\alpha,1}$ in the centraliser chain (4) and still have a strictly descending chain of length d. For each $\alpha \in \{1, 2, \ldots, d\}$ we therefore assume that $q_\alpha = 1$, that $s_{\alpha,i,j} = 0$ for all $i, j \in \{1, 2, \ldots, n\}$, and that $r_{\alpha,1}, r_{\alpha,2}, \ldots, r_{\alpha,n}$ are coprime.

Now suppose that $g_\alpha \in C_G(g_1, g_2, \ldots, g_d)$ for some $\alpha \in \{1, 2, \ldots, d\}$. For each $i \in \{1, 2, \ldots, d\}$, note that $g_i \notin Z(G)$, and therefore there exists some $x \in X$ such that $[g_i, x] \neq 1$. This implies that, for some $d' > d$, there exist elements $g_{d+1}, g_{d+2}, \ldots, g_{d'}$ of X such that

$$G \supsetneq C_G(g_1) \supsetneq C_G(g_1, g_2) \supsetneq \ldots \supsetneq C_G(g_1, g_2, \ldots, g_{d'}) = C,$$

and $g_i \notin C$ for all $i \in \{1, 2, \ldots, d'\}$. Note that if we can replace g_i with an element of X for each $i \in \{1, 2, \ldots, d'\}$ so that this chain still holds, then it is certainly true for the shorter chain (4). Therefore we may assume that there exists no $\alpha \in \{1, 2, \ldots, d\}$ such that $g_\alpha \in C_G(g_1, g_2, \ldots, g_d)$.

We now have a centraliser chain

$$G = C_G(1) \supsetneq C_G(g_1) \supsetneq C_G(g_1, g_2) \supsetneq \ldots \supsetneq C_G(g_1, g_2, \ldots, g_d),$$

where, for all $\alpha \in \{1, 2, \ldots, d\}$,
  (1) $g_\alpha = x_1^{r_{\alpha,1}} x_2^{r_{\alpha,2}} \ldots x_n^{r_{\alpha,n}}$, with $r_{\alpha,i}$ and $r_{\alpha,j}$ both non-zero only if either
    (i) $[x_i, x_j] \neq 1$; or,
    (ii) for some $m \neq 0$, there exist $k_1, k_2, \ldots, k_m \in \{1, 2, \ldots, n\}$ such that
      (a) r_{α,k_l} is non-zero for all $l \in \{1, 2, \ldots, m\}$; and,
      (b) $[x_i, x_{k_1}] \neq 1$, $[x_{k_l}, x_{k_{l+1}}] \neq 1$ for all $l \in \{1, 2, \ldots, m-1\}$, and $[x_{k_m}, x_j] \neq 1$;
  (2) $r_{\alpha,1}, r_{\alpha,2}, \ldots, r_{\alpha,n}$ are coprime, and therefore g_α is a least root; and,
  (3) $g_\alpha \notin C_G(g_1, g_2, \ldots, g_d)$.
For each $\alpha \in \{1, 2, \ldots, d\}$, define $\beta_\alpha \in \{1, 2, \ldots, d\}$ to be minimal such that $g_\alpha \notin C_G(g_1, g_2, \ldots, g_{\beta_\alpha})$.

Suppose that for some $\alpha \in \{2, 3, \ldots, d\}$ we have $\beta_\alpha = \beta > \alpha + 1$ and $g_\alpha \notin X^{\pm 1}$. Since g_α is a least root, the condition that $g_\alpha \notin X^{\pm 1}$ implies that there exist $i_1, i_2 \in \{1, 2, \ldots, n\}$ such that r_{α, i_1} and r_{α, i_2} are both non-zero and $[x_{i_1}, x_{i_2}] \neq 1$. This allows us to define an element

$$w_\alpha = x_1^{t_{\alpha, 1}} x_2^{t_{\alpha, 2}} \ldots x_n^{t_{\alpha, n}}$$

of G such that

(i) $t_{\alpha, i} = 0$ if and only if $r_{\alpha, i} = 0$;

(ii) $t_{\alpha, 1}, t_{\alpha, 2}, \ldots, t_{\alpha, n}$ are coprime, and therefore w_α is a least root; and,

(iii) $[g_\alpha, w_\alpha] \neq 1$.

For example, let

$$t'_{\alpha, i} = \begin{cases} 2 r_{\alpha, i_1} & \text{if } i = i_1, \\ r_{\alpha, i} & \text{otherwise,} \end{cases}$$

and let c be the greatest common divisor of $t'_{\alpha, 1}, t'_{\alpha, 2}, \ldots, t'_{\alpha, n}$. If we let $t_{\alpha, i}$ be such that $t'_{\alpha, i} = c t_{\alpha, i}$ for all $i \in \{1, 2, \ldots, n\}$, then w_α satisfies the required conditions. We have chosen our definition of w_α such that

(i) $A_{w_\alpha} = A_{g_\alpha}$; and,

(ii) $\langle g_\alpha \rangle \cap \langle w_\alpha \rangle = 1$.

Further note that g_α is an element of $C_G(g_1, g_2, \ldots, g_{\beta-1})$, but fails to commute with w_α, and therefore

$$C_G(g_1, g_2, \ldots, g_{\beta-1}) \supsetneq C_G(g_1, g_2, \ldots, g_{\beta-1}, w_\alpha).$$

Now note that

$$C_G(g_\alpha) = \langle g_\alpha \rangle \times \left\{ x_1^{m_1} x_2^{m_2} \ldots x_n^{m_n} \prod [x_i, x_j]^{s_{i,j}} \, | \, m_i, s_{i,j} \in \mathbb{Z}, \right.$$
$$\left. m_i \neq 0 \Rightarrow ([x_i, x_j] = 1 \, \forall x_j \in A_{g_\alpha}) \right\},$$

and, since $A_{g_\alpha} = A_{w_\alpha}$,

$$C_G(w_\alpha) = \langle w_\alpha \rangle \times \left\{ x_1^{m_1} x_2^{m_2} \ldots x_n^{m_n} \prod [x_i, x_j]^{s_{i,j}} \, | \, m_i, s_{i,j} \in \mathbb{Z}, \right.$$
$$\left. m_i \neq 0 \Rightarrow ([x_i, x_j] = 1 \, \forall x_j \in A_{g_\alpha}) \right\}.$$

Further $\langle g_\alpha \rangle \cap \langle w_\alpha \rangle = 1$ implies that

$$C_G(g_\alpha) \cap C_G(w_\alpha) = \left\{ x_1^{m_1} x_2^{m_2} \ldots x_n^{m_n} \prod [x_i, x_j]^{s_{i,j}} \, | \, m_i, s_{i,j} \in \mathbb{Z}, \right.$$
$$\left. m_i \neq 0 \Rightarrow ([x_i, x_j] = 1 \, \forall x_j \in A_{g_\alpha}) \right\},$$
$$= C_G(g_\alpha) \cap \left\{ x_1^{r_1} x_2^{r_2} \ldots x_n^{r_n} \prod [x_i, x_j]^{s_{i,j}} \, | \, r_{\alpha, i} \neq 0 \Rightarrow r_i = 0 \right\}.$$

Note, too, that since g_α and g_β do not commute, we must have

$$C_G(g_\alpha) \cap C_G(g_\beta) \subseteq \left\{ x_1^{r_1} x_2^{r_2} \ldots x_n^{r_n} \prod [x_i, x_j]^{s_{i,j}} \, | \, r_{\alpha, i} \neq 0 \Rightarrow r_i = 0 \right\}.$$

We now combine these results to see that

$$
\begin{aligned}
C_G(&g_1, g_2, \ldots, g_{\beta-1}, w_\alpha) \\
&= C_G(g_1, g_2, \ldots, g_\alpha, \ldots, g_{\beta-1}) \cap C_G(w_\alpha) \\
&= C_G(g_1, g_2, \ldots, g_{\beta-1}) \cap (C_G(g_\alpha) \cap C_G(w_\alpha)), \\
&= C_G(g_1, \ldots, g_{\beta-1}) \cap \left(C_G(g_\alpha) \cap \left\{ x_1^{r_1} \ldots x_n^{r_n} \prod [x_i, x_j]^{s_{i,j}} | r_{\alpha,i} \neq 0 \Rightarrow r_i = 0 \right\} \right), \\
&= C_G(g_1, g_2, \ldots, g_\alpha, \ldots, g_{\beta-1}) \cap \left\{ x_1^{r_1} \ldots x_n^{r_n} \prod [x_i, x_j]^{s_{i,j}} | r_{\alpha,i} \neq 0 \Rightarrow r_i = 0 \right\}, \\
&\supseteq C_G(g_1, g_2, \ldots, g_\alpha, \ldots, g_{\beta-1}) \cap (C_G(g_\alpha) \cap C_G(g_\beta)), \\
&= C_G(g_1, g_2, \ldots, g_\beta),
\end{aligned}
$$

and therefore,

$$
C_G(g_1, g_2, \ldots, g_{\beta-1}) \supsetneq C_G(g_1, g_2, \ldots, g_{\beta-1}, w_\alpha) \supseteq C_G(g_1, g_2, \ldots, g_\beta).
$$

Note that this implies that

$$
C_G(g_1, g_2, \ldots, g_i) \subseteq C_G(g_1, \ldots, g_{\beta-1}, w_\alpha, g_{\beta+1}, \ldots, g_i)
$$

for all $i \in \{\beta + 1, \beta + 2, \ldots, d\}$, and further the strictness of chain (4) implies that

$$
C_G(g_1, g_2, \ldots, g_i) \nsubseteq C_G(g_{i+1})
$$

for all $i \in \{1, 2, \ldots, d-1\}$. This implies that

$$
C_G(g_1, \ldots, g_{\beta-1}, w_\alpha, g_{\beta+1}, \ldots, g_i) \nsubseteq C_G(g_{i+1}),
$$

and therefore

$$
C_G(g_1, \ldots, g_{\beta-1}, w_\alpha, g_{\beta+1}, \ldots, g_i) \supsetneq C_G(g_1, \ldots, g_{\beta-1}, w_\alpha, g_{\beta+1}, \ldots, g_{i+1}),
$$

for all $i \in \{\beta, \beta + 1, \ldots, d-1\}$. We may therefore replace g_β with w_α in the chain (4).

For $p \in \{\alpha, \alpha + 1, \ldots, \beta - 1\}$, note that

$$
C_G(g_1, g_2, \ldots, g_p) = C_G(g_1, g_2, \ldots, g_p, w_\alpha) \times \langle g_\alpha \rangle.
$$

Since (4) is a strict centraliser chain, so is the chain

$$
C_G(g_1, \ldots, g_\alpha, w_\alpha) \supsetneq C_G(g_1, \ldots, g_{\alpha+1}, w_\alpha) \supsetneq \cdots \supsetneq C_G(g_1, \ldots, g_{\beta-1}, w_\alpha).
$$

Therefore

$$
\begin{aligned}
G \supsetneq C_G(g_1) &\supsetneq \cdots \supsetneq C_G(g_1, \ldots, g_\alpha) \supsetneq C_G(g_1, \ldots, g_\alpha, w_\alpha) \supsetneq \ldots \\
&\supsetneq C_G(g_1, \ldots, g_{\beta-1}, w_\alpha) \supsetneq \cdots \supsetneq C_G(g_1, \ldots, g_{\beta-1}, g_{\beta+1}, \ldots, g_d, w_\alpha)
\end{aligned}
$$

is a strict centraliser chain of length d, and further $\beta_\alpha = \alpha + 1$.

We apply this process to each such α, in numerical order, to obtain a chain (4) with the property that, for all $\alpha \in \{1, 2, \ldots, d\}$, either $g_\alpha \in X^{\pm 1}$ or $\beta_\alpha \leqslant \alpha + 1$. Note that if $g_\alpha \in X^{-1}$ then we may replace g_α with $g_\alpha^{-1} \in X$ since $C_G(g) = C_G(g^{-1})$ for all $g \in G$. Therefore we may further assume that our chain (4) has the property that, for all $\alpha \in \{1, 2, \ldots, d\}$, either $g_\alpha \in X$ or $\beta_\alpha \leqslant \alpha + 1$.

Suppose $g_\alpha \notin X$, with $g_j \in X$ for all $j < \alpha$. We have two possibilities for the value of β_α:

(1) $\beta_\alpha < \alpha + 1$

Note that since $C_G(g_1, g_2, \ldots, g_{\alpha-1})$ and $C_G(g_1, g_2, \ldots, g_\alpha)$ are distinct, there must exist some $u \in G$ such that $[u, g_i] = 1$ for all $i \in \{1, 2, \ldots, \alpha - 1\}$ and $[u, g_\alpha] \neq 1$. Further, $[u, g_\alpha] \neq 1$ implies that there exists some $k \in \{1, 2, \ldots, n\}$ such that $r_{\alpha,k} \neq 0$ and $[u, x_k] \neq 1$. Therefore

$$C_G(g_1, g_2, \ldots, g_{\alpha-1}) \supsetneq C_G(g_1, g_2, \ldots, g_{\alpha-1}, x_k).$$

Now

$$C_G(x_k) \cap \{x_1^{r_1} x_2^{r_2} \ldots x_k^{r_k} \ldots x_n^{r_n} \prod [x_i, x_j]^{s_{i,j}} | r_k = 0\}$$
$$\supseteq C_G(g_\alpha) \cap \{x_1^{r_1} x_2^{r_2} \ldots x_n^{r_n} \prod [x_i, x_j]^{s_{i,j}} | r_{\alpha,i} \neq 0 \Rightarrow r_i = 0\}.$$

Therefore, since $\beta_\alpha < \alpha + 1$ implies that g_α is not an element of the centraliser $C_G(g_1, g_2, \ldots, g_\alpha)$, we have

$$C_G(g_1, g_2, \ldots, g_{\alpha-1}, x_k)$$
$$\supseteq C_G(g_1, g_2, \ldots, g_{\alpha-1}) \cap C_G(x_k)$$
$$\cap \left\{ x_1^{r_1} x_2^{r_2} \ldots x_n^{r_n} \prod [x_i, x_j]^{s_{i,j}} | r_k = 0 \right\},$$
$$\supseteq C_G(g_1, g_2, \ldots, g_{\alpha-1}) \cap C_G(g_\alpha)$$
$$\cap \left\{ x_1^{r_1} x_2^{r_2} \ldots x_n^{r_n} \prod [x_i, x_j]^{s_{i,j}} | r_{\alpha,i} \neq 0 \Rightarrow r_i = 0 \right\},$$
$$= C_G(g_1, g_2, \ldots, g_{\alpha-1}, g_\alpha).$$

We have now shown that

$$C_G(g_1, g_2, \ldots, g_{\alpha-1}) \supsetneq C_G(g_1, g_2, \ldots, g_{\alpha-1}, x_k) \supseteq C_G(g_1, g_2, \ldots, g_\alpha).$$

Note that if

$$C_G(g_1, \ldots, g_{\alpha-1}, x_k, g_{\alpha+1}, \ldots, g_i) = C_G(g_1, \ldots, g_{\alpha-1}, x_k, g_{\alpha+1}, \ldots, g_{i+1})$$

for some $i \in \{\alpha, \alpha + 1, \ldots, d - 1\}$ then

$$C_G(g_1, g_2, \ldots, g_i) \subseteq C_G(g_1, \ldots, g_{\alpha-1}, x_k, g_{\alpha+1}, \ldots, g_i) \subseteq C_G(g_{i+1})$$

contradicting the strictness of chain (4). We may therefore replace g_α by the generator x_k, so that (4) is a strict centraliser chain of length d such that g_i is an element of X for all $i \in \{1, 2, \ldots, \alpha\}$.

(2) $\beta = \beta_\alpha = \alpha + 1$

Choose $k, l \in \{1, 2, \ldots, n\}$ such that $r_{\alpha,k} \neq 0$, $r_{\alpha,l} \neq 0$ and $[x_k, x_l] \neq 1$. Note that $g_\alpha \in C_G(g_1, g_2, \ldots, g_{\alpha-1})$ implies that $x_i \in C_G(g_1, g_2, \ldots, g_{\alpha-1})$ for all $i \in \{1, 2, \ldots, n\}$ such that $r_{\alpha,i} \neq 0$, since $g_j \in X$ for all $j \in \{1, 2, \ldots, \alpha - 1\}$. In particular $x_k, x_l \in C_G(g_1, g_2, \ldots, g_{\alpha-1})$. Therefore $x_l \notin C_G(x_k)$ implies that

$$C_G(g_1, g_2, \ldots, g_{\alpha-1}) \supsetneq C_G(g_1, g_2, \ldots, g_{\alpha-1}, x_k),$$

and $x_k \notin C_G(x_l)$ implies that

$$C_G(g_1, g_2, \ldots, g_{\alpha-1}, x_k) \supsetneq C_G(g_1, g_2, \ldots, g_{\alpha-1}, x_k, x_l).$$

Note that

$$C_G(x_k, x_l) \cap \left\{ x_1^{r_1} x_2^{r_2} \ldots x_n^{r_n} \prod [x_i, x_j]^{s_{i,j}} | r_k = r_l = 0 \right\}$$
$$\supseteq C_G(g_\alpha) \cap \left\{ x_1^{r_1} x_2^{r_2} \ldots x_n^{r_n} \prod [x_i, x_j]^{s_{i,j}} | r_{\alpha,i} \neq 0 \Rightarrow r_i = 0 \right\},$$

so that

$$C_G(g_1, g_2, \ldots, g_{\alpha-1}, x_k, x_l)$$
$$\supseteq C_G(g_1, g_2, \ldots, g_{\alpha-1}) \cap C_G(x_k, x_l)$$
$$\cap \left\{ x_1^{r_1} x_2^{r_2} \ldots x_n^{r_n} \prod [x_i, x_j]^{s_{i,j}} | r_k = r_l = 0 \right\},$$
$$\supseteq C_G(g_1, g_2, \ldots, g_{\alpha-1}) \cap C_G(g_\alpha)$$
$$\cap \left\{ x_1^{r_1} x_2^{r_2} \ldots x_n^{r_n} \prod [x_i, x_j]^{s_{i,j}} | r_{\alpha,i} \neq 0 \Rightarrow r_i = 0 \right\}.$$

Note further that

$$C_G(g_1, g_2, \ldots, g_{\alpha+1}) \subsetneq C_G(g_1, g_2, \ldots, g_\alpha),$$

and, since $g_\alpha \notin C_G(g_1, g_2, \ldots, g_\alpha, g_{\alpha+1})$, that

$$C_G(g_1, g_2, \ldots, g_{\alpha+1}) \subseteq \left\{ x_1^{r_1} x_2^{r_2} \ldots x_n^{r_n} \prod [x_i, x_j]^{s_{i,j}} | r_{\alpha,i} \neq 0 \Rightarrow r_i = 0 \right\}.$$

Therefore

$$C_G(g_1, g_2, \ldots, g_{\alpha-1}, x_k, x_l) \supseteq C_G(g_1, g_2, \ldots, g_{\alpha+1}).$$

We have now shown that

$$C_G(g_1, g_2, \ldots, g_{\alpha-1}) \supsetneq C_G(g_1, g_2, \ldots, g_{\alpha-1}, x_k)$$
$$\supsetneq C_G(g_1, g_2, \ldots, g_{\alpha-1}, x_k, x_l) \supseteq C_G(g_1, g_2, \ldots, g_{\alpha-1}, g_\alpha, g_{\alpha+1}).$$

Note, in a similar fashion to case (1), that the strictness of (4) implies that

$$C_G(g_1, g_2, \ldots, g_{\alpha-1}, x_k, x_l, g_{\alpha+2}, \ldots, g_i)$$
$$\supsetneq C_G(g_1, g_2, \ldots, g_{\alpha-1}, x_k, x_l, g_{\alpha+2}, \ldots, g_{i+1})$$

for all $i \in \{\alpha + 1, \alpha + 2, \ldots, d - 1\}$. We may therefore replace the existing g_α and $g_{\alpha+1}$ by x_k and x_l respectively to give a chain (4) for which $g_i \in X$ for all $i \in \{1, 2, \ldots, \alpha\}$.

Repeat for each α such that $g_\alpha \notin X$.

We have now shown that for any strict centraliser chain in G of finite length, d, there exist generators $x_{i_1}, x_{i_2}, \ldots, x_{i_d}$ of X such that

$$G \supsetneq C_G(x_{i_1}) \supsetneq C_G(x_{i_1}, x_{i_2}) \supsetneq \cdots \supsetneq C_G(x_{i_1}, x_{i_2}, \ldots, x_{i_d}).$$

This implies that the centraliser dimension of G is bounded above by $|X|$, and is therefore finite. Further there exist $\mathrm{cdim}(G)$ generators in X which give a maximal chain. $\qquad\square$

Recall that the non-commutation graph of a partially commutative or partially commutative nilpotent group $G = \langle X|R \rangle$ is the graph with vertex set X, and with an edge connecting x_i and x_j if and only if $[x_i, x_j] \notin R$.

THEOREM. *Let $G = \langle x_1, x_2, \ldots, x_n|R \rangle$ be a partially commutative nilpotent group of class 2, with non-commutation graph Γ. Let $G' = \langle x_1, x_2, \ldots, x_n|R' \rangle$ be the partially commutative group with non-commutation graph Γ. Then $\mathrm{cdim}(G) = \mathrm{cdim}(G')$.*

Proof.
(1) Let

$$C_{G'}(x_{i_1}, x_{i_2}, \ldots, x_{i_m}) \supsetneq C_{G'}(x_{i_1}, x_{i_2}, \ldots, x_{i_{m+1}})$$

for some elements $x_{i_j} \in \{x_1, x_2, \ldots, x_n\}$ and some $m \geqslant 0$. This implies that there exists some $g \in G'$ such that
 (i) $[g, x_{i_j}]_{G'} = 1$ for all $1 \leqslant j \leqslant m$; and,
 (ii) $[g, x_{i_{m+1}}]_{G'} \neq 1$.
In [6], it is shown that if H is a free partially commutative group and $x, h \in H$ with x a generator, then we have $[x, h]_H = 1$ if and only if $[x, x']_H = 1$ for all generators x' which occur in h when written as a product of generators of minimal length. Therefore, there exists some generator $y \in \{x_1, x_2, \ldots, x_n\}$ such that
 (i) $[y, x_{i_j}]_{G'} = 1$ for all $1 \leqslant j \leqslant m$; and,
 (ii) $[y, x_{i_{m+1}}]_{G'} \neq 1$.
Since G and G' have the same non-commutation graph, we also have
 (i) $[y, x_{i_j}]_G = 1$ for all $1 \leqslant j \leqslant m$, and,
 (ii) $[y, x_{i_{m+1}}]_G \neq 1$.
Therefore $y \in C_G(x_{i_1}, x_{i_2}, \ldots, x_{i_m})$ and $y \notin C_G(x_{i_1}, x_{i_2}, \ldots, x_{i_{m+1}})$, so that

$$C_G(x_{i_1}, x_{i_2}, \ldots, x_{i_m}) \supsetneq C_G(x_{i_1}, x_{i_2}, \ldots, x_{i_{m+1}}).$$

Now let $\mathrm{cdim}(G') = d$. This implies that there exists a maximal centraliser chain in G' of the form,

$$G' \supsetneq C_{G'}(x_{i_1}) \supsetneq C_{G'}(x_{i_1}, x_{i_2}) \supsetneq \ldots \supsetneq C_{G'}(x_{i_1}, x_{i_2}, \ldots, x_{i_d})$$

for some elements $x_{i_j} \in \{x_1, x_2, \ldots, x_n\}$. Therefore

$$G \supsetneq C_G(x_{i_1}) \supsetneq C_G(x_{i_1}, x_{i_2}) \supsetneq \cdots \supsetneq C_G(x_{i_1}, x_{i_2}, \ldots, x_{i_d}),$$

is a centraliser chain in G, and $\mathrm{cdim}(G) \geqslant d = \mathrm{cdim}(G')$.
(2) Let

$$C_G(x_{i_1}, x_{i_2}, \ldots, x_{i_m}) \supsetneq C_G(x_{i_1}, x_{i_2}, \ldots, x_{i_{m+1}})$$

for some elements $x_{i_j} \in \{x_1, x_2, \ldots, x_n\}$ and some $m \geqslant 0$. Then there exists some element g of G such that
 (i) $[g, x_{i_j}]_G = 1$ for all $1 \leqslant j \leqslant m$; and,
 (ii) $[g, x_{i_{m+1}}]_G \neq 1$.

Recall that for $x \in \{x_1, x_2, \ldots, x_n\}$, we have

$$C_G(x) = \left\{ x_1^{r_1} x_2^{r_2} \ldots x_n^{r_n} \prod [x_i, x_j]_G^{s_{i,j}} \,|\, r_i, s_{i,j} \in \mathbb{Z}, r_i = 0 \text{ if } [x, x_i]_G \neq 1 \right\}.$$

Therefore, if g has normal form $g = x_1^{r_1} x_2^{r_2} \ldots x_n^{r_n} \prod [x_i, x_j]_G^{s_{i,j}}$ then

(i) $r_i = 0$ for all $i \in \{1, 2, \ldots, n\}$ such that $[x_i, x_{i_k}]_G \neq 1$ for some $k \in \{1, 2, \ldots, m\}$; and,

(ii) $r_l \neq 0$ for some $l \in \{1, 2, \ldots, n\}$ such that $[x_l, x_{i_{m+1}}]_G \neq 1$.

Note that $r_l \neq 0$ implies $[x_l, x_{i_j}]_G = 1$ for all $j \in \{1, 2, \ldots, m\}$. Since G and G' share the same non-commutation graph, we have now shown that $x_l \in C_{G'}(x_{i_1}, x_{i_2}, \ldots, x_{i_m})$ but $x_l \notin C_{G'}(x_{i_1}, x_{i_2}, \ldots, x_{i_{m+1}})$. Therefore

$$C_{G'}(x_{i_1}, x_{i_2}, \ldots, x_{i_m}) \supsetneq C_{G'}(x_{i_1}, x_{i_2}, \ldots, x_{i_{m+1}}).$$

Now suppose that $\text{cdim}(G) = d$. By the same method used in part (1) of the proof, we must have $\text{cdim}(G') \geqslant d = \text{cdim}(G)$.

We thus obtain the required result, $\text{cdim}(G) = \text{cdim}(G')$. $\qquad\square$

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