order 4 and graph B does not. However, what if there is no such egregious distinction? The author suggests examining the adjacency matrices, which might look different, but then you might be able to relabel the vertices and they turn out to be the same. There were several places in this chapter where I looked at the solutions and wondered about their rigour. One example asks the student to determine whether a graph is planar, fully justifying their answer, and the solution states that it is not, using Kuratowski's theorem, since it does not contain a subgraph which is a *subdivision* (formed when you insert extra vertices into one for more edges) of K_5. How do you know that? Have you looked at all possible subdivisions? This rather worries me, wearing my Olympiad hat, as a sort of 'Trust Me' argument.

However, this sort of issue strikes me as generic for many of the subject areas covered by the mantle of Discrete Mathematics, which probably has as many undetermined problems as number theory, and one can hardly censure the author for it. So, despite this reservation, I am sure that his book will turn out to be very useful to those who are teaching the discipline for the first time, as well as whose who aren't, and, of course, their students.

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Bounded gaps between primes by Kevin Broughan, pp. 590, £39.99 (paper), ISBN 978-1-108-79920-1, Cambridge University Press (2021).

Let p_n denote the nth prime, and g_n the nth gap between successive primes. The twin-primes conjecture states that $g_n = 2$ for infinitely many n. Being essentially an additive problem concerned with primes which are defined, ostensibly, in terms of multiplication only, such a basic problem is bound to be difficult. Indeed there was no method to tackle the problem until G. H. Hardy and J. E. Littlewood created their powerful circle method, which is particularly suitable for the investigation of such additive problems. The method shows that knowledge of the distribution of primes in arithmetic progressions, and exponential sums, will be central to the investigation of the conjecture. If the argument can be completed successfully then not only is the conjecture true, but the counting function for twin-primes has the asymptotic value $C_2 x/(\log x)^2$, where the twin-primes constant $C_2 \approx 1.32$ is an explicitly defined number. Computer counts of prime-twins up to various x of modest size show that there is good agreement with the formula.

The prime number theorem shows that, for large n, the gap g_n has the average value $\log p_n$, so that $c = \liminf \left(g_n / \log p_n\right) \leqslant 1$. From their circle method, together with the assumption of the Generalised Riemann Hypothesis (GRH), which implies that the distribution of primes in arithmetic progressions is uniform with respect to the moduli of the progressions, Hardy and Littlewood established in 1923 that $c < 1$. The dependence on GRH was removed by P. Erdős in 1940 and, for the remaining part of the twentieth century, the upper estimate for c was duly brought down, but 'only' to slightly better than $c < \frac{1}{4}$. We say 'only' because of the spectacular achievements in the present century, and we put it in quotes because such achievements are based on the many important ideas and results from distinguished mathematicians in their approaches to the reduction of the bound. Thus, besides the development in sieve methods, there is the important Bombieri-Vinogradov theorem which shows that, in the distribution of primes up to x in arithmetic progressions, there is uniformity with respect to the modulus varying up to nearly $\sqrt{x}$, at least in 'almost all' progressions; the theorem is often used to establish unconditional results in prime number theory which were dependent on GRH previously. However, the '$\sqrt{x}$ barrier' limits the application of the theorem to many of the more interesting problems. During 1985–1990 important results which go beyond the barrier, and indeed beyond what GRH implies, were established by E. Bombieri, J. Friedlander and H. Iwaniec in three papers concerned with primes in arithmetic progressions to large moduli, but the method was neither powerful enough nor general enough to be applied to the prime gap problem. Meanwhile, P. Elliott and H. Halberstam had put forward their conjecture (EH) which amounts to saying that there is still uniformity in the distribution of primes in arithmetic progressions for the moduli to vary significantly beyond $\sqrt{x}$. Also relevant to prime gaps is H. Maier's matrix method, which shows that there are unexpected deviations in the distribution of primes, from which he deduced the then best bound $c < 0.2485$ in 1988.

The astonishing news came in 2006 that $c = 0$, which was soon followed by the announcement in 2009 of the even better result that g_n is infinitely often not substantially bigger than $\sqrt{\log p_n}$. Then, in 2013, Yitang Zhang succeeded in establishing the truly sensational result that $C = \liminf g_n < \infty$; in fact he showed that $C \leqslant 7 \times 10^7$ in the proof of his theorem. The book being reviewed tells the story behind the achievements toward the twin-primes conjecture, particularly those in the early part of the present century, and including the work of a team of mathematicians trying to bring the estimate of C down to something tangible—the goal being $C = 2$, of course.

There are nine chapters in the book. The first chapter gives the development of ideas and methods applied to the conjecture, particularly the significance of the Bombieri-Vinogradov theorem. The second chapter is on Brun's sieve, which includes the proof of his famous theorem that the reciprocal sum over twin-primes is finite, and Selberg's sieve, the preferred tool for the investigation because of its relative simplicity in applications. Such sieves are very useful tools, but there is an inherent deficiency, called the 'parity obstacle': the inability to discern the parity of the number of prime divisors of the unsifted numbers. Chapter 3 is mainly concerned with the methods and achievements of mathematicians in the twentieth century mentioned in the first two paragraphs above.

Chapter 4 is on the breakthrough in 2006 of Dan Goldston, Yoichi Motohashi, János Pintz, Cem Yildirim (GMPY) which delivers $c = 0$, and the even better result, by GPY, mentioned earlier; besides an outline of the method, a fair amount of technical detail is given. "The work of GMPY was celebrated throughout the number theory world … Their work is seminal, in that many of the new ideas they introduced … have found their way into other developments and applications." There is also a proof of $C \leqslant 16$ under the assumption of EH, and it is pointed out that Motohashi and Pintz had shown in 2008 that, for the GPY method to work, the relevant moduli in Bombieri-Vinogradov may be restricted to squarefree numbers with no large prime factors—indeed a specific feature of the later contribution of Zhang was to break the $\sqrt{x}$ barrier with respect to such moduli.

Chapters 5 and 6 are on the astonishing result of Zhang, and James Maynard's impressive work giving a radical simplification of the argument which delivers $C \leqslant 600$; the independent contribution from Terence Tao is also mentioned. Maynard also proved that $\liminf_{n \to \infty} (p_{n+m} - p_n) \leqslant Km^3 e^{4m}$, where K does not depend on m. Chapter 7 is on the refinement of Maynard's results by Polymath8b which gives successive improvements on the estimate of C, with the current best being $C \leqslant 246$. The titles of the last two chapters are 'Variations on Bombieri-Vinogradov' and

'Further work and the epilogue', in which the achievements of PMGY, Zhang, Maynard, Tao and the Polymath8 teams are summarised. The chapters also include various statements on topics such as the Siegel-Walfisz theorem, Vaughan's identity, speculations on the generalisation of the Elliott-Halberstam conjecture, gaps between almost primes, and consecutive primes in arithmetic progressions with a fixed common difference. There are also brief mentions of topics not directly related to prime gaps: the large sieve, Kloostermann sums, extension of Artin's primitive root conjecture, modular forms and elliptic curves.

There are nine appendices which include the following topics: Bessel functions of the first kind, the Brun-Titchmarsh inequality for multiplicative functions, exponential sums, and the dispersion method of Linnik. There is also a PGpack mini-manual for a set of functions written to assist the reader to reproduce, and possibly extend, the calculations mentioned in the book. There are 215 items of references. Although there are only a few misprints, I read with alarm that my friend Roger Baker is referred to as the "late Roger Baker". I can reassure readers that Roger is still very much alive and enjoying rude health.

Mathematical research, be it the creation of a theory or the solution of a difficult problem, is a human endeavour. It is mainly the activity of sole individuals, with perhaps one or two collaborators, but there is now a new phenomenon: the massive collaborations over the internet on specific projects with the aim of finding solutions to various famous problems. There is a section in the book explaining what a Polymath project is, and also specific information on the contributors to Polymath8a/b, but it is not a book just telling us who did what and when in the pursuit of the twin-primes conjecture. The reader who really wants to know how Zhang's spectacular theorem was arrived at will need to absorb a large amount of technical detail, and such an individual may feel that there is a lack of overall coherence in the presentation. The book is thus more useful for graduate students in number theory who are already familiar with much of the material, whereas other readers should perhaps first read the more easily digestible book [1] by Vicky Neale. Nevertheless, perhaps even a casual reader may find it an interesting read, and will appreciate that mathematical research is a human activity well worthwhile pursuing.

Reference

1. Vicky Neale, *Closing the gap*, Oxford University Press (2017), reviewed in *Math. Gaz*. 102 (November 2018) p. 561.

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When least is best: how mathematicians discovered many clever ways to make things as small (or as large) as possible by Paul J. Nahin, pp. 392, £20 (paper), 978-0-69121-876-2, Princeton University Press (2021).

I found this an enjoyable and engaging read. It brings to life the maths of optimisation by portraying its creators and the problems they were trying to solve. Woven into the mix are problems for the reader, with solutions, computer explorations, cultural links to films and books where the ideas are used, and references for further reading. I enjoyed Judith Grabiner's quote that "The derivative was first *used*; it was then *discovered*; it was then *explored and developed*; and it was finally *defined*" and the details of Fermat and Descartes' strained relationship. I was