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DETERMINATION OF $[n\theta]$ BY ITS SEQUENCE OF DIFFERENCES

BY

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ABSTRACT. For any real number θ, let $f_\theta(n) = [(n+1)\theta]-[n\theta]-[\theta]$ $(n = 1, 2, \ldots)$ where $[x]$ denotes the greatest integer not exceeding x. A method is given for computing f_θ from its first few terms. A similar method is given for computing the characteristic function $g_\theta(n)$ of $[n\theta]$. The given methods converge rapidly, and generalize previous results of Bernoulli, Markoff, and Stolarsky. Note that either of the sequences f_θ and g_θ determines the sequence $[n\theta]$ $(n = 1, 2, \ldots)$.

1. **Introduction.** Johann Bernoulli and later André Markoff determined the values of a sequence of the form $[n\theta]$ $(n = 1, 2, \ldots;$ θ any irrational) by means of the differences $f(n) = f_\theta(n) = [(n+1)\theta]-[n\theta]-[\theta]$, where $[x]$ is the integer part of x. Thus $f_\theta(n) \in \{0, 1\}$ for all n. They did this by using the partial quotients $a_1, a_2, \ldots$ of the simple continued fraction expansion of $\theta = [0, a_1, a_2 \ldots]$, when $0<\theta<1$. See [22].

In this note we shall compute these differences for any real number θ by using the denominators q_i of the convergents p_i/q_i of θ. Since the q_i grow exponentially faster than the partial quotients, convergence is that much faster. A related result is obtained for the characteristic function

$$g(n) = g_\theta(n) = \begin{cases} 1 & \text{if there exists an integer } m \text{ satisfying } n = [m\theta] \\ 0 & \text{otherwise.} \end{cases}$$

2. **Main results.** If $C = (c_1, c_2, \ldots)$ is an infinite sequence, we denote its first t elements by $(C, t) = (c_1, \ldots, c_t)$. By $(C, t)^\infty$ we denote the infinite concatenation of (C, t) with itself, namely $(C, t)^\infty = (c_1, \ldots, c_t, c_1, \ldots, c_t, \ldots)$. Let $C^j = (c_1^j, c_2^j, \ldots)$ be a family of sequences $(j = 1, 2, \ldots)$. We say that $\lim_{j\to\infty} C^j = C$, if for every i there exists $j(i)$ such that $c_i^j = c_i$ for all $j > j(i)$.

Let l be a positive integer, and $T = \{t_l, t_{l+1}, \ldots\}$ a finite or infinite sequence of positive integers which is strictly increasing. Define $T_n(C)$ recursively by $T_1(C) = (C, t_l)^\infty$, $T_{n-l+1}(C) = (T_{n-l}(C), t_n)^\infty$ $(n > l)$. If T is infinite, let $T_\infty(C) = \lim_{n\to\infty} T_n(C)$. The limit obviously exists.

Let $\theta = [a_0, a_1, \ldots] > 0$, where it is understood that a_N is the last partial quotient if $\theta = p_N/q_N$ is rational. Whenever we refer in the sequel to N, a_N, p_N or q_N, it is understood that the corresponding θ is rational, namely, $\theta = p_N/q_N$.

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Since a_N can be replaced by $a_N - 1, 1$ if $a_N > 1$ and $a_{N-1}, 1$ can be replaced by $a_{N-1}+1$ if $a_N = 1$ ($\theta \neq 1$), we can choose N even or odd at will. Also $f_\theta(n+q_N) = f_\theta(n)$, $g_\theta(n+p_N) = g_\theta(n)$ for every positive integer n, that is, f_θ, g_θ are periodic with periods q_N, p_N respectively (θ rational). Moreover, $f_\theta = f_{\theta+k}$ for every integer k (θ real). In particular, $f_\theta = f_{\theta-[\theta]}$. If θ is an integer, then clearly: (i) $f_\theta(n) = 0$ for all n; (ii) $g_\theta(n) = 1$ if and only if $n \equiv 0 \pmod{\theta}$. Finally, $g_\theta(n) = 1$ for all n if $\theta \leq 1$. So f_θ is non-trivial only if θ is nonintegral and g_θ only if $\theta > 1$ and θ is non-integral. Hence we may assume without loss of generality that $q_N > 1$ for computing f_θ, and p_N, $q_N > 1$ for computing g_θ. (Though we could assume $p_N > q_N > 1$ in the latter case, this will not be necessary.)

For a given $\theta > 0$, let

$$s = \min\{i : q_i > 1\}, \qquad r = \min\{i : p_i > 1\}.$$

Note that s is well-defined if $q_N > 1$ and r is well-defined if $p_N > 1$. Of course both s and r are well-defined if θ is irrational. Since

$$\begin{aligned} p_{-1} &= 1, & p_0 &= a_0, & p_n &= a_n p_{n-1} + p_{n-2} & (n \geq 1) \\ q_{-1} &= 0, & q_0 &= 1, & q_n &= a_n q_{n-1} + q_{n-2} & (n \geq 1), \end{aligned}$$

we have (for $\theta > 0$) $s = 1$ or 2, $r = 0, 1, 2$ or 3 ($r = 0$ or 1 for $\theta > 1$). We prove:

THEOREM 1. *Let $T = (q_s, q_{s+1}, \ldots)$ be a partial sequence of the denominators of the convergents of $\theta > 0$. If θ is irrational, then $T_\infty(f_\theta) = f_\theta$. If $\theta = p_N/q_N$ is rational ($q_N > 1$) and N is even, then $T_{N-s+1}(f_\theta) = f_\theta$.*

THEOREM 2. *Let $T = (p_r, p_{r+1}, \ldots)$ be a partial sequence of the numerators of the convergents of $\theta > 0$. If θ is irrational, then $T_\infty(g_\theta) = g_\theta$. If $\theta = p_N/q_N$ is rational ($p_N, q_N > 1$) and N is even, then $T_{n-r+1}(g_\theta) = g_\theta$.*

EXAMPLE. Let $\theta = [1, 2, 3, 1, 2] = 36/25$. Then

$$\begin{aligned} &p_0 = 1, \quad p_1 = 3, \quad p_2 = 10, \quad p_3 = 13, \quad p_4 = 36 \\ &q_0 = 1, \quad q_1 = 2, \quad q_2 = 7, \quad q_3 = 9, \quad q_4 = 25, \quad s = r = 1, \quad N = 4. \end{aligned}$$

n	$[n\theta]$	$f_\theta(n)$	$g_\theta(n)$
1	1	0	1
2	2	1	1
3	4	⋮	0

For f_θ we have $T = (2, 7, 9, 25)$, and therefore $T_1(f) = (01)^\infty$, $T_2(f) = (0101010)^\alpha$, $T_3(f) = (010101001)^\infty$, $T_4(f) = T_{N-s+1}(f) = (0101010010101010010101010)^\infty = f$. Similarly, for g_θ we have $T = (3, 10, 13, 36)$, and so $T_1(g) = (110)^\infty$, $T_2(g) = (1101101101)^\infty$, $T_3(g) = (1101101101110)^\infty$, $T_4(g) = T_{N-r+1}(g) = (110110110111011011011011101101101101)^\infty = g$.

Theorem 1 was proved by K. B. Stolarsky [21] for the subset of irrational

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algebraic numbers of the form $\theta=[1, a, a, \ldots]=\frac{1}{2}(2-a+\sqrt{a^2+4})$ (a any positive integer), and Theorem 2 for the number $\theta=[1,1,1,\ldots]=\frac{1}{2}(1+\sqrt{5})$. Both results are extended here to all real numbers and short proofs are given. We remark that Stolarsky [21] gave two different proofs of his results and an extensive and useful bibliography on complementary sequences, especially those related to sequences of the form $[n\alpha+\gamma]$. In the meantime we came across a few additional references. A subset of these are listed at the end at the suggestion of Stolarsky.

3. **Proofs.** The proof of Theorem 1 is based on the fact that $f_\theta(q+q_n)=f_\theta(q)$ if q is not too large (see Corollary below). In fact for all q, $f_{p_n/q_n}(q+q_n)=f_{p_n/q_n}(q)$. This suggests use of approximation properties of the convergents of θ.

LEMMA 1. *Let $p_n/q_n (n>0)$ be the n-th convergent of a real number θ ($n\le N$ if θ is rational). Then $0<q<q_n$ implies $|q_{n-1}\theta-p_{n-1}|\le|q\theta-p|$ for every integer p.*

This is a standard result in the theory of continued fractions. See e.g. [14, Theorem 7.13] (where it is stated for irrational θ).

LEMMA 2. *Suppose that $n>0$ ($n\le N$ with N even if θ is rational), and $0<q<q_n$. Then $[(q+q_{n-1})\theta]=p_{n-1}+[q\theta]$.*

Proof. Consider the difference $\delta=(q+q_{n-1})\theta-(p_{n-1}+[q\theta])$. It suffices to show that $0\le\delta<1$, because then $[\delta]=0$, which is what the lemma claims.

If n is even, Lemma 1 implies $0<p_{n-1}-q_{n-1}\theta\le q\theta-[q\theta]<1$. Thus $0\le\delta<1-(p_{n-1}-q_{n-1}\theta)<1$. If n is odd, then $0<q_{n-1}\theta-p_{n-1}\le 1+[q\theta]-q\theta$, which implies $0\le\delta\le1$. If θ is irrational, then $\delta<1$. If θ is rational, then $n\le N-1$ since N is even. Hence $q+q_{n-1}<q_n+q_{n-1}\le q_{N-1}+q_{N-2}\le q_N$. Now

$$\delta=1\Rightarrow\theta=\frac{p_{n-1}+[q\theta]+1}{q+q_{n-1}}=\frac{p_N}{q_N}.$$

Since $(p_N, q_N)=1$, this implies $q+q_{n-1}\ge q_N$, a contradiction. Hence $\delta<1$.

COROLLARY. *Suppose that $n>0$ ($n\le N$ with N even if θ is rational), and $0<q<q_n-1$. Then $f_\theta(q+q_{n-1})=f_\theta(q)$.*

Proof. By Lemma 2, $f_\theta(q+q_{n-1})=[(q+q_{n-1}+1)\theta]-[(q+q_{n-1})\theta]-[\theta]=p_{n-1}+[(q+1)\theta]-p_{n-1}-[q\theta]-[\theta]=f_\theta(q)$.

Proof of Theorem 1. By definition, $T_1(f)=(f, q_s)^\infty$. Suppose that we showed already $T_{i-s+1}(f)=(f, q_i)^\infty$ for some $i\ge s$, where, by definition, $T_{i-s+1}(f)=(T_{i-s}(f), q_i)^\infty$ ($i>s$). By the Corollary, $f_\theta(q)$ is periodic with period q_i for $1\le q\le q_i+q_{i+1}-2$, where $0\le i$ ($<N\equiv0 \pmod 2$ if θ is rational). This implies $(f, q_i+q_{i+1}-2)=(T_{i-s+1}(f), q_i+q_{i+1}-2)$. Since $q_i\ge q_s\ge2$, we have, in particular, $(f, q_{i+1})^\infty=(T_{i-s+1}(f), q_{i+1})^\infty=T_{i-s+2}(f)$. So by induction, $T_{i-s+1}(f)=(f, q_i)^\infty$ for all $i\ge s$.

If θ is irrational, $T_\infty(f_\theta)=\lim_{i\to\infty} T_{i-s+1}(f_\theta)=\lim_{i\to\infty}(f_\theta, q_i)^\infty=f_\theta$.

If $\theta=p_N/q_N$ is rational, then f_θ is periodic with period q_N, and so $T_{N-s+1}(f_\theta)=(f_\theta, q_N)^\infty=f_\theta$.

LEMMA 3. *Suppose that* $n>0$ ($n\le N$ *with* N *even if* θ *is rational*), *and* $0<p<p_n-1$. *Then* $g_\theta(p+p_{n-1})=g_\theta(p)$.

Proof. It is well-known from the theory of continued fractions [14] that $q_n\theta=p_n+(-1)^n/q'_{n+1}$, where $0<n$ ($<N$ if θ is rational), $q'_{n+1}=a'_{n+1}q_n+q_{n-1}$, $a'_{n+1}=[a_{n+1}, a_{n+2}, \ldots,]$. Hence

$$[q_n\theta]=\begin{cases} p_n & \text{if } n \text{ is even}\\ p_n-1 & \text{otherwise,}\end{cases}$$

for $0<n$ ($\le N\equiv 0 \pmod 2$ if θ is rational).

If there exists $0<p<p_n-1$ satisfying $g_\theta(p)=1$, let $p=[q\theta]$. Then $q>0$. Moreover, $[q\theta]=p<p_n-1\le[q_n\theta]\Rightarrow q<q_n$. By Lemma 2, $[(q+q_{n-1})\theta]=p+p_{n-1}$. and so $g_\theta(p)=1\Rightarrow g_\theta(p+p_{n-1})=g_\theta(p)$.

Now let $0<x<p_n-1$ satisfy $g_\theta(x)=0$. Then $x=p+i$ for some $p=[q\theta]$, where $0<i<f_\theta(q)+[\theta]$, so $1\le i\le[\theta]$. Also,

$$(1) \qquad [\theta]\le f_\theta(n)+[\theta]=[(n+1)\theta]-[n\theta]\le[\theta]+1$$

for all integers n. Since $p<p+i<p_n-1$, we have $q+1\le q_n$ by the first part of the proof.

It suffices to show that

$$(2) \qquad i<f_\theta(q+q_{n-1})+[\theta],$$

because then $[(q+q_{n-1})\theta]<[(q+q_{n-1})\theta]+i=p+p_{n-1}+i$ (by Lemma 2) $<$ $[(q+q_{n-1}+1)\theta]$ (by (2)), and thus also $g_\theta(p+i)=0\Rightarrow g_\theta(p+i+p_{n-1})=g_\theta(p+i)$.

If $q+1<q_n$, then $f_\theta(q+q_{n-1})=f_\theta(q)$ by the Corollary. Since $0<i<f_\theta(q)+[\theta]$, this establishes (2). Also if $i<[\theta]$ ($\le f_\theta(q+q_{n-1})+[\theta]$), then (2) is clear. So it suffices to show that $q+1<q_n$ for $i=[\theta]$. If n is even, this is immediate from the right-hand side of (1): $[(q+1)\theta]\le p+i+1<p_n=[q_n\theta]\Rightarrow q+1<q_n$. For $0<n$ ($<N$ if θ is rational) we have $[(q_n+q_{n-1})\theta]=p_n+p_{n-1}+[y]$, where $y=1/q'_n-1/q'_{n+1}$ if n is odd, as we may assume. If $n<N$, then

$$q'_n=a'_nq_{n-1}+q_{n-2}\le(a_n+1)q_{n-1}+q_{n-2}=q_n+q_{n-1}\le q_{n+1}$$
$$q'_{n+1}=a'_{n+1}q_n+q_{n-1}\ge a_{n+1}q_n+q_{n-1}=q_{n+1},$$

so $y\ge 0$. Thus

$$(3) \qquad [(q_n+q_{n-1})\theta]\ge p_n+p_{n-1} \qquad (n \text{ odd}).$$

Now $i=[\theta]<f_\theta(q)+[\theta]\Rightarrow f_\theta(q)=1$. Using Lemma 2 we write this in the form $p_{n-1}+[(q+1)\theta]-[(q+q_{n-1})\theta]=[\theta]+1$. Since $f_\theta(q+q_{n-1})\le 1$, we

have $[(q+q_{n-1}+1)\theta]-[(q+q_{n-1})\theta]\le[\theta]+1$. Subtracting, $[(q+q_{n-1}+1)\theta]\le p_{n-1}+[(q+1)\theta]\le p_{n-1}+[q_n\theta]$ (since $q+1\le q_n$) $=p_{n-1}+p_n-1$ (since n is odd) $<[(q_n+q_{n-1})\theta]$ (by (3)). Hence $q+1<q_n$ also for n odd.

Proof of Theorem 2. We follow the proof of Theorem 1 verbatim, with Corollary, f, q, q_i, s replaced by Lemma 3, g, p, p_i, r respectively.

NOTE. Had we confined ourselves to irrational $\theta>0$, the proof could have been shortened considerably. For example, instead of Lemma 3, it would have been advantageous to prove: $\theta>1$ irrational $\Rightarrow g_\theta=f_{1/\theta}$, which has a one-line proof. This result, which is of independent interest, implies Theorem 2 immediately. If θ is rational, $g_\theta(n)=f_{1/\theta}(n)$ holds, except that $n\equiv 0 \pmod{p_N} \Rightarrow g_\theta(n)=1$, $f_{1/\theta}(n)=0$, and $n\equiv -1 \pmod{p_N} \Rightarrow g_\theta(n)=0$, $f_{1/\theta}(n)=1$. This leads to a "proof by cases" of Theorem 2 and hence a somewhat different route was preferred.

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