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ON A PARTITION PROBLEM OF FINITE ABELIAN GROUPS

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Abstract

Let G be a finite abelian group and $A \subseteq G$. For $n \in G$, denote by $r_A(n)$ the number of ordered pairs $(a_1, a_2) \in A^2$ such that $a_1 + a_2 = n$. Among other things, we prove that for any odd number $t \geq 3$, it is not possible to partition G into t disjoint sets $A_1, A_2, \ldots, A_t$ with $r_{A_1} = r_{A_2} = \cdots = r_{A_t}$.

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1. Introduction

We use $\mathbb{N}$ to denote the set of nonnegative integers. Let G be an abelian semigroup with an arbitrary total ordering. For any subset $A \subseteq G$ and $n \in G$, let

$$r_A(n) = \#\{(a_1, a_2) \in A^2 : a_1 + a_2 = n\},$$
$$r_A^+(n) = \#\{(a_1, a_2) \in A^2 : a_1 + a_2 = n,\ a_1 \leq a_2\}$$

and

$$r_A^-(n) = \#\{(a_1, a_2) \in A^2 : a_1 + a_2 = n,\ a_1 < a_2\},$$

respectively. These representation functions have been studied by many authors (see, for example, the survey paper [7] for a picture of results in this area). An important problem is the inverse problem for representation functions, which seeks to understand sets $A, B \subseteq G$ with the same representation function.

Nathanson [4] determined all pairs of sets $A, B \subseteq \mathbb{N}$ such that r_A and r_B eventually coincide. Kiss *et al.* [2] extended Nathanson's result to 3-fold representation functions. In [3, 6, 8], the authors classified all subsets $A \subseteq \mathbb{N}$ such that r_A^+ and $r_{\mathbb{N}\setminus A}^+$ (respectively r_A^- and $r_{\mathbb{N}\setminus A}^-$) eventually coincide. Nathanson [5] posed the following problem, which, to the best of our knowledge, is still open.

Problem 1.1. Let $t \geq 3$. Does there exist a partition of the nonnegative integers into disjoint sets $A_1, A_2, \ldots, A_t$ whose representation functions $r_{A_1}^+, r_{A_2}^+, \ldots, r_{A_t}^+$ eventually coincide? Characterise all such partitions if they exist. The same problem can be posed for r_A^-.

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Analogously, for a finite abelian group G, one may ask the following question.

PROBLEM 1.2. Let $t \geq 2$. Does there exist a partition of G into disjoint sets $A_1, A_2, \ldots, A_t$ whose representation functions $r_{A_1}, r_{A_2}, \ldots, r_{A_t}$ coincide? Characterise all such partitions if they exist. The same problem can be posed for r_A^+ and r_A^-.

For $t = 2$, we have a complete classification.

THEOREM 1.3. *Let G be a finite abelian group with $|G| \geq 2$ and $A \subseteq G$. Denote the 2-torsion subgroup of G by $G_2 := \{g \in G : 2g = 0\}$. Then:*

- $r_A = r_{G\setminus A}$ *if and only if* $|G|$ *is even and* $|A| = |G|/2$*;*
- $r_A^+ = r_{G\setminus A}^+$ *(respectively* $r_A^- = r_{G\setminus A}^-$*) if and only if* $|G|$ *is even, and* $|A \cap H| = |H|/2$ *for every coset* H *of* G_2.

We make some progress toward Problem 1.2 for r_A and $t \geq 3$. Our main result is the following theorem.

THEOREM 1.4. *Let G be a finite abelian group and $t \geq 3$ an odd number. Then it is not possible to partition G into t disjoint sets $A_1, A_2, \ldots, A_t$ with $r_{A_1} = r_{A_2} = \cdots = r_{A_t}$.*

We also pose the following conjecture.

CONJECTURE 1.5. Let G be a finite abelian group and $t \geq 2$. Suppose that $A_1, A_2, \ldots, A_t$ form a partition of G with $r_{A_1} = r_{A_2} = \cdots = r_{A_t}$; then t divides $|G_2|$.

Problem 1.2 was also asked in [1] for h-fold representation functions and Theorem 1.4 gives a partial solution. Theorem 1.3 was also proved in [1]. We provide a new proof here, since the ingredients in the proof are also needed for proving Theorem 1.4.

2. Proof of results

Throughout this section, G is a finite abelian group. Our main tool is the generating function in the group algebra $\mathbb{C}[G]$ associated to a set $A \subseteq G$. Recall that the elements of $\mathbb{C}[G]$ are of the form

$$f(x) = \sum_{g \in G} a_g x^g,$$

where a_g is a complex number for every $g \in G$. The multiplication in $\mathbb{C}[G]$ is given by

$$\Big(\sum_{g \in G} a_g x^g\Big)\Big(\sum_{g \in G} b_g x^g\Big) = \sum_{g_1, g_2 \in G} a_{g_1} b_{g_2} x^{g_1+g_2} = \sum_{g \in G}\Big(\sum_{\substack{g_1, g_2 \in G \\ g_1+g_2=g}} a_{g_1} b_{g_2}\Big) x^g.$$

For any subset $A \subseteq G$, write

$$f_A(x) = \sum_{a \in A} x^a \in \mathbb{C}[G].$$

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Then

$$f_A(x)^2 = \sum_{n\in G}\Big(\sum_{\substack{a_1,a_2\in A\\ a_1+a_2=n}} 1\Big)x^n = \sum_{n\in G} r_A(n)x^n, \tag{2.1}$$

$$f_A(x)^2 + f_A(x^2) = \sum_{n\in G}\Big(r_A(n) + \sum_{\substack{a\in G\\ 2a=n}} 1\Big)x^n = \sum_{n\in G} 2r_A^+(n)x^n \tag{2.2}$$

and

$$f_A(x)^2 - f_A(x^2) = \sum_{n\in G}\Big(r_A(n) - \sum_{\substack{a\in G\\ 2a=n}} 1\Big)x^n = \sum_{n\in G} 2r_A^-(n)x^n. \tag{2.3}$$

We use χ_A to denote the characteristic function of A, that is,

$$\chi_A(x) = \begin{cases} 1, & x\in A,\\ 0, & x\in G\backslash A.\end{cases}$$

For any function $f : G \to \mathbb{Z}$ and map $\varphi : G \to G'$, let $f^\varphi : G' \to \mathbb{Z}$ be defined as

$$f^\varphi(n) = \sum_{m\in\varphi^{-1}(n)} f(m), \quad n \in G'.$$

For any group homomorphism $\varphi : G \to G'$, we have a natural induced homomorphism of group algebras $\varphi_* : \mathbb{C}[G] \to \mathbb{C}[G']$, namely

$$\varphi_*\Big(\sum_{g\in G} a_g x^g\Big) = \sum_{g\in G} a_g x^{\varphi(g)} = \sum_{g\in G'}\Big(\sum_{n\in\varphi^{-1}(g)} a_n\Big)x^g.$$

Proof of Theorem 1.3. Let $A \subseteq G$ and write $B = G\backslash A$. If $r_A = r_B$, then

$$|A|^2 = \sum_{n\in G} r_A(n) = \sum_{n\in G} r_B(n) = |B|^2 \tag{2.4}$$

and hence $|A| = |B| = |G|/2$. Now suppose that $|G|$ is even and $|A| = |G|/2$. It follows from (2.1) that $r_A = r_B$ if and only if

$$f_A(x)^2 = f_B(x)^2$$

or, equivalently,

$$(f_A(x) - f_B(x))(f_A(x) + f_B(x)) = 0. \tag{2.5}$$

To see that (2.5) holds, decompose G into a direct sum of cyclic groups, say

$$G \cong \bigoplus_{i=1}^{k} \mathbb{Z}_{m_i}.$$

Fixing a generator g_i of $\mathbb{Z}_{m_i}$ for every i and setting $x^{g_i} = x_i$, we thus obtain an isomorphism

$$\mathbb{C}[G] \cong \mathbb{C}[x_1, x_2, \dots, x_k]/(x_1^{m_1} - 1, x_2^{m_2} - 1, \dots, x_k^{m_k} - 1).$$

Using this isomorphism,

$$f_A(x) + f_B(x) = \sum_{n\in G} x^n = \prod_{i=1}^{k}(1 + x_i + \cdots + x_i^{m_i-1}). \tag{2.6}$$

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Let $\overline{f}_A, \overline{f}_B \in \mathbb{C}[x_1, x_2, \ldots, x_k]$ be an inverse image of f_A, f_B respectively under the projection map

$$\pi : \mathbb{C}[x_1, x_2, \ldots, x_k] \to \mathbb{C}[x_1, \ldots, x_k]/(x_1^{m_1} - 1, x_2^{m_2} - 1, \ldots, x_k^{m_k} - 1).$$

The value of $\overline{f}_A(1, 1, \ldots, 1)$ does not depend on the choice of $\overline{f}_A$, since the difference of two choices is a polynomial in the ideal $(x_1^{m_1} - 1, x_2^{m_2} - 1, \ldots, x_k^{m_k} - 1)$, which vanishes at $(1, 1, \ldots, 1)$. Thus, we see that

$$\overline{f}_A(1, 1, \ldots, 1) = \sum_{n \in A} 1 = |A|$$

and similarly

$$\overline{f}_B(1, 1, \ldots, 1) = |B|.$$

It follows that

$$\overline{f}_A(1, 1, \ldots, 1) - \overline{f}_B(1, 1, \ldots, 1) = |A| - |B| = 0. \tag{2.7}$$

Hilbert's Nullstellensatz states that if $P \in \mathbb{C}[x_1, x_2, \ldots, x_k]$ and P vanishes at some $(a_1, a_2, \ldots, a_k) \in \mathbb{C}^k$, then P is in the maximal ideal $(x_1 - a_1, x_2 - a_2, \ldots, x_k - a_k)$. By (2.7) and Hilbert's Nullstellensatz, $\overline{f}_A - \overline{f}_B \in (x_1 - 1, x_2 - 1, \ldots, x_k - 1)$; in other words,

$$\overline{f}_A - \overline{f}_B = \sum_{i=1}^{k} (x_i - 1) h_i \tag{2.8}$$

for some $h_1, h_2, \ldots, h_k \in \mathbb{C}[x_1, x_2, \ldots, x_k]$. Applying the projection map π to (2.8) and multiplying by (2.6),

$$\begin{aligned}(f_A - f_B)(f_A + f_B) &= \left(\sum_{i=1}^{k} (x_i - 1)\pi(h_i)\right) \prod_{j=1}^{k} (1 + x_j + \cdots + x_j^{m_j - 1}) \\ &= \sum_{i=1}^{k} \left((x_i^{m_i} - 1)\pi(h_i) \prod_{\substack{1 \le j \le k \\ j \ne i}} (1 + x_j + \cdots + x_j^{m_j - 1}) \right) \\ &= 0 \in \mathbb{C}[x_1, \ldots, x_k]/(x_1^{m_1} - 1, x_2^{m_2} - 1, \ldots, x_k^{m_k} - 1).\end{aligned}$$

Hence, (2.5) holds.

If $r_A^+ = r_B^+$ (respectively $r_A^- = r_B^-$), then

$$\binom{|A| + 1}{2} = \sum_{n \in G} r_A^+(n) = \sum_{n \in G} r_B^+(n) = \binom{|B| + 1}{2}$$

or, respectively,

$$\binom{|A|}{2} = \sum_{n \in G} r_A^-(n) = \sum_{n \in G} r_B^-(n) = \binom{|B|}{2}$$

and again we have $|A| = |B|$. Now suppose that $|G|$ is even and $|A| = |G|/2$. Noting that we have already proved that $f_A^2 = f_B^2$, it follows from (2.2) and (2.3) that $r_A^+ = r_B^+$

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(respectively $r_A^- = r_B^-$) if and only if $f_A(x^2) = f_B(x^2)$. Consider the homomorphism $\varphi : G \to 2G$ given by $\varphi(x) = 2x$ for $x \in G$, where $2G := \{2x : x \in G\}$. The kernel of φ is $\ker \varphi = G_2 = \{g \in G : 2g = 0\}$. Since

$$f_A(x^2) = \sum_{n\in G} \chi_A(n)x^{2n} = \sum_{m\in 2G}\Big(\sum_{n\in\varphi^{-1}(m)} \chi_A(n)\Big)x^m = \sum_{m\in 2G} \chi_A^{\varphi}(m)x^m,$$

and similarly

$$f_B(x^2) = \sum_{n\in G} \chi_B(n)x^{2n} = \sum_{m\in 2G} \chi_B^{\varphi}(m)x^m,$$

it follows that $f_A(x^2) = f_B(x^2)$ if and only if

$$\chi_A^{\varphi}(m) = \chi_B^{\varphi}(m)$$

for every $m \in 2G$. Note that

$$\chi_A^{\varphi}(m) = \sum_{n\in\varphi^{-1}(m)} \chi_A(n) = |A \cap \varphi^{-1}(m)|$$

and similarly $\chi_B^{\varphi}(m) = |B \cap \varphi^{-1}(m)|$. Thus, $f_A(x^2) = f_B(x^2)$ if and only if

$$|A \cap H| = |B \cap H| = |H|/2$$

for every coset $H = \varphi^{-1}(m)$ of G_2. This completes the proof of Theorem 1.3. □

We now proceed to prove Theorem 1.4. Our strategy is to study f_A under projections of G onto various cyclic groups.

Lemma 2.1. *Let $t \geq 3$ be an odd integer. Suppose that $A_1, A_2, \ldots, A_t$ form a partition of G with $r_{A_1} = r_{A_2} = \cdots = r_{A_t}$. Then for any cyclic quotient map $\varphi : G \to \mathbb{Z}_q$ with q a prime power, $\chi_{A_i}^{\varphi}$ is a constant function on $\mathbb{Z}_q$ for every $i = 1, 2, \ldots, t$.*

Proof. Write $q = p^k$ with p prime and $k > 0$. With the same argument as in (2.4), we first conclude that $|A_1| = |A_2| = \cdots = |A_t|$. The lemma is proved by induction on k.

For $k = 1$, let

$$g_{A_i} := \varphi_*(f_{A_i}) = \sum_{g\in G} \chi_{A_i}(g)x^{\varphi(g)} = \sum_{n\in\mathbb{Z}_p}\Big(\sum_{m\in\varphi^{-1}(n)} \chi_{A_i}(m)\Big)x^n = \sum_{n\in\mathbb{Z}_p} \chi_{A_i}^{\varphi}(n)x^n$$

in $\mathbb{C}[\mathbb{Z}_p] \cong \mathbb{C}[x]/(x^p - 1)$. In treating the divisibility of polynomials, we can consider g_{A_i} as a polynomial in $\mathbb{C}[x]$ by taking an inverse image in $\mathbb{C}[x]$. Since $f_{A_i}^2 = f_{A_j}^2$ in $\mathbb{C}[G]$, we have $g_{A_i}^2 = g_{A_j}^2$ in $\mathbb{C}[\mathbb{Z}_p]$, that is, $x^p - 1 \mid g_{A_i}^2 - g_{A_j}^2$. In particular, $\Phi_p(x) \mid g_{A_i}^2 - g_{A_j}^2$, where $\Phi_m(x)$ denotes the mth cyclotomic polynomial. Note that $\Phi_p(x)$ is irreducible over $\mathbb{Z}$, and g_{A_i} also has integral coefficients; therefore, either $\Phi_p(x) \mid g_{A_i} - g_{A_j}$ or $\Phi_p(x) \mid g_{A_i} + g_{A_j}$.

Note that

$$g_{A_i} \pm g_{A_j} \equiv \sum_{n=0}^{p-1}(\chi_{A_i}^{\varphi}(n) \pm \chi_{A_j}^{\varphi}(n))x^n \pmod{x^p - 1}$$

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and $\Phi_p(x) = 1 + x + \cdots + x^{p-1}$. Thus, $\Phi_p(x) \mid g_{A_i} \pm g_{A_j}$ if and only if $\chi^{\varphi}_{A_i} \pm \chi^{\varphi}_{A_j}$ is a constant function on $\mathbb{Z}_p$. Since

$$\sum_{n\in\mathbb{Z}_p} \chi^{\varphi}_{A_i}(n) = \sum_{n\in\mathbb{Z}_p} \chi^{\varphi}_{A_j}(n) = |A_i| = |A_j| = |G|/t,$$

$\chi^{\varphi}_{A_i} - \chi^{\varphi}_{A_j}$ is constant if and only if $\chi^{\varphi}_{A_i} = \chi^{\varphi}_{A_j}$, and $\chi^{\varphi}_{A_i} + \chi^{\varphi}_{A_j}$ is constant if and only if

$$\chi^{\varphi}_{A_i} + \chi^{\varphi}_{A_j} = \frac{2|G|}{tp},$$

that is,

$$\chi^{\varphi}_{A_j} = \frac{2|G|}{tp} - \chi^{\varphi}_{A_i}. \tag{2.9}$$

Suppose on the contrary that $\chi^{\varphi}_{A_i}$ is not a constant function. Assume that there are a sets A_j among $A_1, A_2, \ldots, A_t$ satisfying $\chi^{\varphi}_{A_j} = \chi^{\varphi}_{A_i}$, and the remaining A_j satisfy (2.9); then

$$\chi^{\varphi}_G = \sum_{j=1}^{t} \chi^{\varphi}_{A_j} = a\chi^{\varphi}_{A_i} + (t-a)\Big(\frac{2|G|}{tp} - \chi^{\varphi}_{A_i}\Big) = \frac{2(t-a)|G|}{tp} + (2a-t)\chi^{\varphi}_{A_i}.$$

Since t is odd, $2a - t \neq 0$; we conclude that χ^{φ}_G is not a constant function, which is clearly a contradiction.

For $k > 1$, we assume that the assertion holds for $k-1$. Let $\alpha : \mathbb{Z}_{p^k} \to \mathbb{Z}_{p^{k-1}}$ be the canonical projection and $\beta = \alpha \circ \varphi$. By the inductive hypothesis, $\chi^{\beta}_{A_i}$ is a constant function for $i = 1, 2, \ldots, t$; thus,

$$1 + x + x^2 + \cdots + x^{p^{k-1}-1} \mid \beta_*(f_{A_i})$$

and therefore

$$1 + x + x^2 + \cdots + x^{p^{k-1}-1} \mid g_{A_i}, \tag{2.10}$$

where $g_{A_i} = \varphi_*(f_{A_i})$. Since $g^2_{A_i} = g^2_{A_j}$ in $\mathbb{C}[\mathbb{Z}_q]$, we have $x^q - 1 \mid g^2_{A_i} - g^2_{A_j}$. It follows that either $\Phi_q \mid g_{A_i} - g_{A_j}$ or $\Phi_q \mid g_{A_i} + g_{A_j}$. By (2.10),

$$1 + x + x^2 + \cdots + x^{p^{k-1}-1} \mid g_{A_i} \pm g_{A_j},$$

and $x - 1 \mid g_{A_i} - g_{A_j}$, since $g_{A_i}(1) - g_{A_j}(1) = |A_i| - |A_j| = 0$.

If $\Phi_q(x) \mid g_{A_i} - g_{A_j}$, then $x^q - 1 \mid g_{A_i} - g_{A_j}$. Since

$$g_{A_i} \pm g_{A_j} \equiv \sum_{n=0}^{q-1} (\chi^{\varphi}_{A_i}(n) \pm \chi^{\varphi}_{A_j}(n))x^n \pmod{x^q - 1}, \tag{2.11}$$

$\chi^{\varphi}_{A_i} = \chi^{\varphi}_{A_j}$. If $\Phi_q \mid g_{A_i} + g_{A_j}$, then

$$1 + x + \cdots + x^{q-1} \mid g_{A_i} + g_{A_j}.$$

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Again, by (2.11), $\chi^{\varphi}_{A_i} + \chi^{\varphi}_{A_j}$ is a constant function and consequently

$$\chi^{\varphi}_{A_j} = \frac{2|G|}{tq} - \chi^{\varphi}_{A_i}.$$

With the same argument as in the case $k = 1$, we see that $\chi^{\varphi}_{A_i}$ is a constant function for $i = 1, 2, \ldots, t$. This completes the proof of the lemma. □

Lemma 2.2. *Let G be a finite abelian group, $|G| = p^k$ with p prime and $f : G \to \mathbb{Z}$. Assume that for any cyclic quotient map $\varphi : G \to \mathbb{Z}_q$, f^{φ} is a constant function. Then f is a constant function.*

Proof. We use induction on k. For $k = 1$, G is cyclic and the result follows from the assumptions.

Now let $k > 1$ and assume that the assertion holds for all smaller cases. We may assume that G is not cyclic, otherwise the result again follows by assumption. For any subgroup $0 \neq H < G$, consider the quotient map $\varphi : G \to G/H$. Applying the inductive hypothesis to G/H and f^{φ}, we conclude that f^{φ} is a constant function. Thus, for any $x, y \in G$,

$$\sum_{m\in(x+H)} f(m) = \sum_{m\in(y+H)} f(m). \tag{2.12}$$

Let $H_1, H_2, \ldots, H_r$ be all subgroups of G of order p. Since G is not cyclic, G has at least two direct summands; thus, $r \geq 2$.

It is clear that $H_i \cap H_j = \{0\}$ for all $1 \leq i < j \leq r$. Let $G_p < G$ be the p-torsion subgroup. Every nonzero element of G_p belongs to exactly one H_i, while 0 belongs to every H_i. Let $x, y \in G$ be such that $x - y \in G_p$. Summing over all cosets of H_i containing x,

$$\sum_{i=1}^{r} \sum_{m\in(x+H_i)} f(m) = (r-1)f(x) + \sum_{m\in(x+G_p)} f(m) \tag{2.13}$$

and similarly

$$\sum_{i=1}^{r} \sum_{m\in(y+H_i)} f(m) = (r-1)f(y) + \sum_{m\in(y+G_p)} f(m). \tag{2.14}$$

Applying (2.12) with $H = H_i$ for $i = 1, 2, \ldots, r$ and summing,

$$\sum_{i=1}^{r} \sum_{m\in(x+H_i)} f(m) = \sum_{i=1}^{r} \sum_{m\in(y+H_i)} f(m). \tag{2.15}$$

Noting that $x + G_p = y + G_p$, it follows from (2.13)–(2.15) that $f(x) = f(y)$, that is, f is constant on each coset of G_p. For any $x, y \in G$, applying (2.12) with $H = G_p$ yields

$$f(x) = \frac{1}{|G_p|} \sum_{m\in(x+G_p)} f(m) = \frac{1}{|G_p|} \sum_{m\in(y+G_p)} f(m) = f(y).$$

This completes the proof of the lemma. □

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We are now ready to prove Theorem 1.4.

PROOF OF THEOREM 1.4. Suppose on the contrary that there exists a partition of G into disjoint sets $A_1, A_2, \ldots, A_t$ such that $r_{A_1} = r_{A_2} = \cdots = r_{A_t}$. It is clear that $|A_i| = |G|/t$ for all $1 \le i \le t$. Let p be a prime divisor of t and

$$H := \{g \in G : p^k \cdot g = 0 \text{ for some } k > 0\}.$$

Since H is a direct summand of G, let $\varphi : G \to H$ be the projection map. By Lemma 2.1, $(\chi_{A_i}^{\varphi})^{\psi} = \chi_{A_i}^{\psi\circ\varphi}$ is a constant function for any cyclic quotient map $\psi : H \to \mathbb{Z}_q$. By Lemma 2.2, we conclude that $\chi_{A_i}^{\varphi} = c \in \mathbb{Z}$ is a constant function. Thus,

$$|H| \cdot c = \sum_{n\in H} \chi_{A_i}^{\varphi}(n) = \sum_{m\in G} \chi_{A_i}(m) = |A_i| = \frac{|G|}{t}. \tag{2.16}$$

However, $|G|/|H|$ is not divisible by p by definition of H, and $p \mid t$; hence, (2.16) cannot hold. This completes the proof of Theorem 1.4. □

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