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STRONGLY REVERSIBLE CLASSES IN $\mathrm{SL}(n, \mathbb{C})$

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Abstract An element of a group is called *strongly reversible* or *strongly real* if it can be expressed as a product of two involutions. We provide necessary and sufficient conditions for an element of $\mathrm{SL}(n, \mathbb{C})$ to be a product of two involutions. In particular, we classify the strongly reversible conjugacy classes in $\mathrm{SL}(n, \mathbb{C})$.

Keywords: reversibility; strongly reversible elements; bireflectional elements; reversing symmetry group; Jordan canonical form; Weyr canonical form.

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1. Introduction

Let G be a group. An element $g \in G$ is called an involution if $g^2 = e$. It is a question of general interest whether every element of G can be expressed as a product of involutions; see [10, 13, 17–19]. It is fascinating how, even in very large groups, the product of a small number of involutions suffices to express any arbitrary element; see [14, p. 76]. For instance, in the special linear group $\mathrm{SL}(n, \mathbb{C})$, consisting of all $n \times n$ matrices over $\mathbb{C}$ with determinant 1, every element can be written as a product of four involutions; see [12].

A related problem is to understand the product of two involutions in a group; see [2, 14]. This problem is well-understood in the context of general linear groups; see [23, 4, 9, 11]. It follows that for a given field $\mathbb{F}$, an element of the general linear group $\mathrm{GL}(n, \mathbb{F})$

can be written as a product of two involutions if and only if it is conjugate to its inverse in $\mathrm{GL}(n,\mathbb{F})$. However, this does not hold for the special linear group, e.g., the unipotent Jordan block in $\mathrm{SL}(2,\mathbb{C})$.

An element of a group G is called *strongly reversible* if it can be expressed as a product of two involutions. Such elements are also known as *strongly real* or *bireflectional* in the literature. It is easy to see that an element $g \in G$ is strongly reversible if and only if g is conjugate to g^{-1} by an involution in G. When the conjugating element of g is not necessarily an involution, g is known as a *reversible* or *real* element. A conjugacy class in G is called *reversible conjugacy class* if it contains a reversible element, and a *strongly reversible conjugacy class* if it contains a strongly reversible element. Most investigations in the literature have focused on understanding the equivalence between the reversible and strongly reversible conjugacy classes. For an elaborate exposition of this theme, we refer to the monograph [14]. In this article, we use the notion of reversibility to classify the product of two involutions in $\mathrm{SL}(n,\mathbb{C})$. Such classification is known for the finite special linear group $\mathrm{SL}(n,q)$; see [5].

By strong reversibility in a group G, we mean a classification of strongly reversible elements in G. Recall that an element of $\mathrm{GL}(n,\mathbb{C})$ is reversible if and only if it is strongly reversible; see [23, 14]. A complete list of reversible elements in $\mathrm{GL}(n,\mathbb{C})$ can be found in [14, Section 4.2]. Note that if two matrices in $\mathrm{SL}(n,\mathbb{C})$ are conjugate by an element of $\mathrm{GL}(n,\mathbb{C})$, then by a suitable scaling, we can assume that both matrices are conjugate by an element of $\mathrm{SL}(n,\mathbb{C})$. Therefore, the classification of reversible elements in $\mathrm{SL}(n,\mathbb{C})$ follows from the corresponding classification in $\mathrm{GL}(n,\mathbb{C})$ and is given by the following result. We refer to Lemma 2.5 for the Jordan decomposition of matrices over $\mathbb{C}$.

Theorem 1.1 O'Farrell and Short [14, Theorem 4.2] *An element $A \in \mathrm{SL}(n,\mathbb{C})$ is reversible if and only if the Jordan blocks in the Jordan decomposition of A can be partitioned into singletons $\{\mathrm{J}(\mu,k)\}$ or pairs $\{\mathrm{J}(\lambda,m),\mathrm{J}(\lambda^{-1},m)\}$, where $\mu,\lambda \in \mathbb{C}\setminus\{0\}$ such that $\mu \in \{-1,+1\}$ and $\lambda \notin \{-1,+1\}$.*

However, unlike $\mathrm{GL}(n,\mathbb{C})$, there exist reversible elements in $\mathrm{SL}(n,\mathbb{C})$ that are not strongly reversible in $\mathrm{SL}(n,\mathbb{C})$. Classifying strongly reversible elements in $\mathrm{SL}(n,\mathbb{C})$ is a natural problem of interest; see [14, p. 77]. In [22], the authors proved that if $n \not\equiv 2 \pmod 4$, then every reversible element in $\mathrm{SL}(n,\mathbb{C})$ is strongly reversible; see [22, Theorem 3.1.1]. Recently, in [8], the notion of adjoint reality was introduced, and the authors used this concept to classify the strongly reversible unipotent elements in $\mathrm{SL}(n,\mathbb{C})$; see [8, Theorem 4.6]. However, these classifications do not cover all strongly reversible elements in $\mathrm{SL}(n,\mathbb{C})$. We are not aware of any literature that provides a complete list of strongly reversible elements in $\mathrm{SL}(n,\mathbb{C})$. The aim of this paper is to offer a complete classification of the strongly reversible elements in $\mathrm{SL}(n,\mathbb{C})$.

We now state the main result of this paper, which classifies the product of two involutions in $\mathrm{SL}(n,\mathbb{C})$.

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Theorem 1.2 *Let A be a reversible element in $\mathrm{SL}(n,\mathbb{C})$. Let s denote the number of Jordan blocks of the form $\{\mathrm{J}(\mu,4k+2)\}$, $\mu\in\{-1,+1\}$, and t denote the number of pairs of the form $\{\mathrm{J}(\lambda,2m+1),\mathrm{J}(\lambda^{-1},2m+1)\}$, $\lambda\notin\{-1,+1\}$, in the Jordan decomposition of A. Then A is strongly reversible in $\mathrm{SL}(n,\mathbb{C})$ if and only if at least one of the following conditions holds.*

(1) *There is a Jordan block $\mathrm{J}(\mu,2r+1)$, $\mu\in\{-1,+1\}$, of odd size in the Jordan decomposition of A.*
(2) $s+t\equiv 0 \pmod 2$.

A key ingredient of our approach used in this paper is a description of the reversers. For a group G, the *centraliser and reverser* of an element g of G are, respectively, defined as

$$\mathcal{Z}_G(g):=\{f\in G\mid fg=gf\}\ \text{ and }\ \mathcal{R}_G(g):=\{h\in G\mid hgh^{-1}=g^{-1}\}.$$

The set $\mathcal{R}_G(g)$ of reversers for a reversible element $g\in G$ is a right coset of the centraliser $\mathcal{Z}_G(g)$. Thus, the *reversing symmetry group* or *extended centraliser* $\mathcal{E}_G(g):=\mathcal{Z}_G(g)\cup\mathcal{R}_G(g)$ is a subgroup of G in which $\mathcal{Z}_G(g)$ has an index of at most 2. Therefore, to find the reversing symmetry group $\mathcal{E}_G(g)$ of a reversible element $g\in G$, it is sufficient to specify one reverser of g that is not in the centraliser; see [1], [14, Section 2.1.4] for more details.

The centraliser of each element in the group $\mathrm{GL}(n,\mathbb{C})$ has been well studied in the literature; see [15, Proposition 3.1.2], [6, Theorem 9.1.1]. It follows that to find the reversing symmetry group $\mathcal{E}_{\mathrm{GL}(n,\mathbb{C})}(A)$ of an arbitrary reversible element $A\in\mathrm{GL}(n,\mathbb{C})$, it is sufficient to find a reverser for the Jordan form of A; see § 3. In this paper, using some combinatorial identities, we have described a reverser for certain types of Jordan forms in $\mathrm{GL}(n,\mathbb{C})$, which are summarised in Table 1. We refer to Definition 2.4 for the notation of the Jordan block used in Table 1. Moreover, the following notation is required to state such an explicit description of reversers.

Definition 1.3. *For a non-zero $\lambda\in\mathbb{C}$, define $\Omega(\lambda,n):=[x_{i,j}]_{n\times n}\in\mathrm{GL}(n,\mathbb{C})$ as follows*

(1) $x_{i,j}=0$ for all $1\leq i,j\leq n$ such that $j<i$,
(2) $x_{n,n}=1$ and $x_{i,n}=0$ for all $1\leq i\leq n-1$,
(3) For all $1\leq i\leq j\leq n-1$, define

$$x_{i,j}=-\lambda^{-1}x_{i+1,j}-\lambda^{-2}x_{i+1,j+1}. \tag{1.1}$$

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Table 1. Involutory reversing symmetries for Jordan forms in $\mathrm{GL}(n,\mathbb{C})$.

Sr No.	Jordan form	Reversing involution
1	$\mathrm{J}(\mu, n)$, $\mu \in \{-1,+1\}$	$\Omega(\mu,n)$
2	$\begin{pmatrix} \mathrm{J}(\lambda,n) & \\ & \mathrm{J}(\lambda^{-1},n) \end{pmatrix}$, $\lambda \in \mathbb{C}\setminus\{-1,0,+1\}$	$\begin{pmatrix} & \Omega(\lambda,n) \\ \Big(\Omega(\lambda,n)\Big)^{-1} & \end{pmatrix}$

In view of Theorem 1.1 and Lemma 2.5, we can use Table 1 to construct a suitable reverser for each reversible element of $\mathrm{GL}(n,\mathbb{C})$, which is also an involution in $\mathrm{GL}(n,\mathbb{C})$. It follows that every reversible element in $\mathrm{GL}(n,\mathbb{C})$ is strongly reversible. This constructive proof of strong reversibility in $\mathrm{GL}(n,\mathbb{C})$ may be of independent interest.

Another key tool used in this paper is the notion of the *Weyr canonical form*; see [15, 20]. The centraliser (and hence the reverser) of a reversible element written in Weyr canonical form is a block upper triangular matrix, making it more suitable for our purposes than the corresponding Jordan canonical form; see § 2.4 and Remark 5.2 for more details.

Although we restrict ourselves to the field of complex numbers, the methods used in this paper may extend to classify strongly reversible elements in the special linear group over an algebraically closed field of characteristic not equal to 2. It is worth mentioning that in [16], the authors proved that an element A of the general linear group $\mathrm{GL}(n,\mathbb{F})$ over a field $\mathbb{F}$ is conjugate to $-A^{-1}$ if and only if A can be expressed as a product of an involution and a skew-involution (i.e., an element σ such that $\sigma^2 = -\mathrm{I}_n$); see [16, Theorem 5].

Finally, consider the subgroup $\mathrm{SL}(n,\mathbb{C}) \ltimes \mathbb{C}^n$ of the affine group $\mathrm{GL}(n,\mathbb{C}) \ltimes \mathbb{C}^n$. An element $g = (A, v) \in \mathrm{GL}(n,\mathbb{C}) \ltimes \mathbb{C}^n$ acts on $\mathbb{C}^n$ as an affine transformation: $g(x) = A(x) + v$. In [7], the authors proved that an element $g = (A, v) \in \mathrm{GL}(n,\mathbb{C}) \ltimes \mathbb{C}^n$ is reversible (respectively, strongly reversible) if and only if A is reversible (respectively, strongly reversible) in $\mathrm{GL}(n,\mathbb{C})$; see [7, Theorem 1.1]. However, the situation in $\mathrm{SL}(n,\mathbb{C}) \ltimes \mathbb{C}^n$ is more subtle than in $\mathrm{GL}(n,\mathbb{C}) \ltimes \mathbb{C}^n$. There are elements in $\mathrm{SL}(n,\mathbb{C}) \ltimes \mathbb{C}^n$ that are not reversible in $\mathrm{SL}(n,\mathbb{C}) \ltimes \mathbb{C}^n$, even though their linear part is reversible in $\mathrm{SL}(n,\mathbb{C})$. For example, consider the affine transformation: $x \mapsto x + 1$ for all $x \in \mathbb{C}^n$. Reversibility in $\mathrm{SL}(n,\mathbb{C}) \ltimes \mathbb{C}^n$ is intricately related to reversibility in $\mathrm{SL}(n,\mathbb{C})$; see [7, Lemmas 3.2–3.4]. The results of this article, together with those in [7], can be used to classify the reversible and strongly reversible elements in $\mathrm{SL}(n,\mathbb{C}) \ltimes \mathbb{C}^n$. We will investigate this in a subsequent work.

Structure of the paper. In § 2, we fix some notation and recall background related to the Jordan and Weyr canonical forms. We explore the reversing symmetry groups of certain Jordan forms in $\mathrm{SL}(n,\mathbb{C})$ in § 3. In § 4, we classify the strong reversibility of certain Jordan forms by analysing the structure of the corresponding reversing symmetry groups. We also classify strongly reversible semisimple elements in § 4. We classify strongly reversible unipotent elements using the notion of the Weyr canonical form in § 5. Finally, in § 6, we prove Theorem 1.2.

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2. Preliminaries

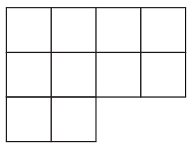
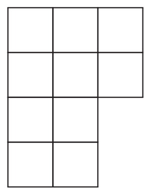
In this section, we will fix some notations and introduce the notion of the Jordan and Weyr canonical forms in M$(n, \mathbb{C})$, the algebra of $n \times n$ complex matrices.

2.1. Notation for partition of n

In this section, we will recall some notation for partitioning a positive integer n.

Definition 2.1. cf. [15] *A partition of a positive integer n is a finite sequence $(n_1, n_2, \dots, n_r)$ of positive integers such that $n_1 + n_2 + \cdots + n_r = n$ and $n_1 \geq n_2 \geq \cdots \geq n_r \geq 1$. Moreover, the conjugate partition (or dual partition) of the partition $(n_1, n_2, \dots, n_r)$ of n is the partition $(m_1, m_2, \dots, m_{n_1})$ such that $m_j = |\{i \mid n_i \geq j\}|$.*

For every positive integer n, we can represent each of its partitions using a diagram known as a Young diagram. The Young diagram of a specific partition $(n_1, n_2, \dots, n_r)$ of n consists of n boxes arranged into r rows, where the length of the i-th row is n_i. We can obtain the Young diagram corresponding to the conjugate partition of a given partition of n by flipping the Young diagram of the given partition over its main diagonal from upper left to lower right. For example, the Young diagrams corresponding to the partition $(4, 4, 2)$ and its conjugate partition $(3, 3, 2, 2)$ are given as follows:

 and 

We recall an alternative notation for partitioning a positive integer n, as introduced in [8, Section 3.3].

Definition 2.2. *A partition of a positive integer n is a list of the form*

$$\mathbf{d}(n) := [d_1^{t_{d_1}}, \dots, d_s^{t_{d_s}}],$$

where $t_{d_i}, d_i \in \mathbb{N}, 1 \leq i \leq s$, such that $\sum_{i=1}^{s} t_{d_i} d_i = n, t_{d_i} \geq 1$ and $d_1 > \cdots > d_s > 0$. Moreover, for a partition $\mathbf{d}(n) = [d_1^{t_{d_1}}, \dots, d_s^{t_{d_s}}]$ of n, define $\mathbb{N}_{\mathbf{d}(n)} := \{d_i \mid 1 \leq i \leq s\}, \mathbb{E}_{\mathbf{d}(n)} := \mathbb{N}_{\mathbf{d}(n)} \cap (2\mathbb{N})$, $\mathbb{O}_{\mathbf{d}(n)} := \mathbb{N}_{\mathbf{d}(n)} \setminus \mathbb{E}_{\mathbf{d}(n)}$, and $\mathbb{E}^2_{\mathbf{d}(n)} := \{\eta \in \mathbb{E}_{\mathbf{d}(n)} \mid \eta \equiv 2 \pmod 4\}$. Furthermore, note that $|\mathbb{E}^2_{\mathbf{d}(n)}| = \sum_{\eta \in \mathbb{E}^2_{\mathbf{d}(n)}} t_\eta$.

We have introduced two notations $(n_1, n_2, \dots, n_r)$ and $\mathbf{d}(n)$ for the partition of a positive integer n; see Definition 2.1 and Definition 2.2. The following lemma provides the relationship between the partition $\mathbf{d}(n)$ and its conjugate partition $\overline{\mathbf{d}}(n)$. We omit the proof as it is straightforward.

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Lemma 2.3. *Let $\mathbf{d}(n) = [d_1^{t_{d_1}}, \ldots, d_s^{t_{d_s}}]$ be a partition of a positive integer n, as defined in Definition 2.2. Then the conjugate partition $\overline{\mathbf{d}}(n)$ of $\mathbf{d}(n)$ has the following form*

$$\overline{\mathbf{d}}(n) = \Big[(t_{d_1}+t_{d_2}+\cdots+t_{d_s})^{d_s}, (t_{d_1}+t_{d_2}+\cdots+t_{d_{s-1}})^{d_{s-1}-d_s}, \ldots, (t_{d_1}+t_{d_2})^{d_2-d_3}, (t_{d_1})^{d_1-d_2}\Big].$$

2.2. Block Matrices

We can partition a matrix $A \in \mathrm{M}(n, \mathbb{C})$ by choosing a horizontal partitioning of the rows and an independent vertical partitioning of the columns. When the same partitioning is used for both the rows and columns, we refer to the resulting partitioned matrix as a *block matrix* or a *blocked matrix*. For example:

$$A = \left(\begin{array}{ccc|cc} 2 & 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 & 0 \\ \hline 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{array}\right) = \left(\begin{array}{c|c} A_{1,1} & A_{1,2} \\ \hline A_{2,1} & A_{2,2} \end{array}\right) = (A_{i,j})_{1 \le i,j \le 2} \in \mathrm{M}(5, \mathbb{C}).$$

Note that the diagonal blocks $A_{i,i}$ in the block matrix $A = (A_{i,j})_{1 \le i,j \le m}$ are all square sub-matrices. Moreover, when specifying the block structure of matrix A, it is sufficient to specify only the sizes of the diagonal blocks $A_{i,i}$ since the (i, j)-th block $A_{i,j}$ must be a $n_i \times n_j$ matrix, where n_i and n_j are the sizes of the diagonal blocks $A_{i,i}$ and $A_{j,j}$, respectively. Therefore, if the diagonal blocks of A have decreasing size, we can uniquely specify the whole block structure of A by a partition $(n_1, n_2, \ldots, n_r)$ of n such that $n_1 + n_2 + \cdots + n_r = n$ and $n_1 \ge n_2 \ge \cdots \ge n_r \ge 1$. We refer to block matrix A as *block upper triangular* if $A_{i,j} = 0$ for all $i > j$. If all the non-diagonal blocks of a block matrix A are zero matrices, then A is called a *block diagonal* matrix; see [15, Section 1.2] for more details.

The symbol I_r represents the $r \times r$ identity matrix. When $s > r$, we use the notation $\mathrm{I}_{s,r}$ to denote an $s \times r$ matrix, where the first r rows form the identity matrix I_r, and the remaining $(s - r)$ rows consist entirely of zeros. For example, $\mathrm{I}_{3,2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$.

2.3. Jordan canonical form

In this section, we recall the notion of the Jordan canonical form in $\mathrm{M}(n, \mathbb{C})$; see [15, p. 39], [6, Section 2.2] for more details.

Definition 2.4. Gohberg et al. [6, p. 52] *A Jordan block $\mathrm{J}(\lambda, m)$ is an $m \times m$ $(m > 1)$ matrix with $\lambda \in \mathbb{C}$ on the diagonal entries, 1 on all of the super-diagonal entries and 0 elsewhere. For $m = 1$, $\mathrm{J}(\lambda, 1) := (\lambda)$. We will refer to a block diagonal matrix where each diagonal block is a Jordan block as Jordan form.*

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Lemma 2.5. Jordan form in $\mathrm{M}(n,\mathbb{C})$ **[6, Theorem 2.2.1]** *For every* $A \in \mathrm{M}(n,\mathbb{C})$*, there is an invertible matrix* $S \in \mathrm{GL}(n,\mathbb{C})$ *such that*

$$SAS^{-1} = \mathrm{J}(\lambda_1,\, m_1) \oplus \cdots \oplus \mathrm{J}(\lambda_k,\, m_k), \tag{2.1}$$

where $\lambda_1, \ldots, \lambda_k$ *are complex numbers (not necessarily distinct). The form* (2.1) *is uniquely determined by* A *up to a permutation of Jordan blocks.*

The Jordan structure, which lists the sizes of the diagonal blocks in the Jordan form of a matrix, is defined as follows.

Definition 2.6. O'Meara et al. [15, p. 39] *Suppose* $A \in \mathrm{M}(n,\mathbb{C})$ *is similar to the Jordan form* $\mathrm{J}(\lambda, n_1) \oplus \cdots \oplus \mathrm{J}(\lambda, n_r)$ *such that* $n_1 + n_2 + \cdots + n_r = n$ *and* $n_1 \geq n_2 \geq \cdots \geq n_r \geq 1$*. Then the partition* $(n_1, n_2, \ldots, n_r)$ *of* n *is called the Jordan structure of* A*. The Jordan structure is also known as the Segre characteristic in the literature; see* [21].

2.4. Weyr canonical form

In this section, we recall the notion of the Weyr canonical form introduced by the Czech mathematician Eduard Weyr in 1885. The Weyr form is preferred over the Jordan form when dealing with problems concerning matrix centralisers. We refer to [15, 20, 21] for a detailed discussion on the theory of Weyr canonical forms.

Definition 2.7. O'Meara et al. [15, Definition 2.1.1] *A* basic Weyr matrix *with eigenvalue* λ *is a matrix* $W \in \mathrm{M}(n,\mathbb{C})$ *of the following form: There exists a partition* $(n_1, n_2, \ldots, n_r)$ *of* n *such that, when* W *is viewed as an* $r \times r$ *blocked matrix* $(W_{ij})_{1 \leq i,j \leq r}$*, where the* (i, j)*-th block* W_{ij} *is an* $n_i \times n_j$ *matrix, the following three features hold.*

(1) The main diagonal blocks $W_{i,i}$ are the $n_i \times n_i$ scalar matrices $\lambda \mathrm{I}_{n_i}$ for all $1 \leq i \leq r$.
(2) The first super-diagonal blocks $W_{i,i+1}$ are the $n_i \times n_{i+1}$ matrices in reduced row-echelon of rank n_{i+1} (i.e., $W_{i,i+1} = \mathrm{I}_{n_i, n_{i+1}}$) for all $1 \leq i \leq r$.
(3) All other blocks of W are zero matrices (i.e., $W_{ij} = 0$ when $j \neq i, i+1$).

In this case, we say that W has the *Weyr structure* $(n_1, n_2, \ldots, n_r)$.

In other words, a basic Weyr matrix is a block upper triangular matrix where the diagonal blocks are scalar matrices (i.e., scalar multiples of identity matrices). The super-diagonal blocks consist of identity matrices augmented by rows of zeros, and all the other blocks are zero matrices. An $n \times n$ scalar matrix can be viewed as a basic Weyr matrix with the trivial Weyr structure (n). On the other hand, Jordan blocks are basic Weyr matrices with the Weyr structure $(1, 1, 1, \ldots, 1)$.

Definition 2.8. O'Meara et al. [15, Definition 2.1.5] *Let* $W \in \mathrm{M}(n,\mathbb{C})$*, and let* $\lambda_1, \lambda_2, \ldots, \lambda_k$ *be the distinct eigenvalues of* W*. We say that* W *is in Weyr form (or is a Weyr matrix) if it is a direct sum of basic Weyr matrices, one for each distinct*

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eigenvalue. In other words, W has the following form:

$$W = W_1 \oplus W_2 \oplus \cdots \oplus W_k,$$

where W_i is a basic Weyr matrix with eigenvalue λ_{i} for all $1 \leq i \leq k$.

Theorem 2.9 O'Meara et al. [15, Theorem 2.2.4] *Up to a permutation of the basic Weyr blocks, each square matrix $A \in \mathrm{M}(n, \mathbb{C})$ is similar to a unique Weyr matrix W. The matrix W is called the Weyr (canonical) form of A.*

In the Jordan form, a matrix with a single eigenvalue can be expressed as a direct sum of several Jordan blocks, each corresponding to that eigenvalue. In contrast, the Weyr form of a matrix with a single eigenvalue is simply the corresponding basic Weyr matrix for that eigenvalue. This highlights a significant difference between the Jordan and Weyr forms of a matrix. In this article, we will use the notation A_W to denote the Weyr form corresponding to $A \in \mathrm{M}(n, \mathbb{C})$. The following result recalls the centraliser of a basic Weyr matrix in $\mathrm{M}(n, \mathbb{C})$.

Proposition 2.10. O'Meara et al. [15, Proposition 2.3.3] *Let $W \in \mathrm{M}(n, \mathbb{C})$ be an $n \times n$ basic Weyr matrix with the Weyr structure $(n_1, \ldots, n_r)$, where $r \geq 2$. Let K be an $n \times n$ matrix, blocked according to the partition $(n_1, \ldots, n_r)$, and let $K_{i,j}$ denote its (i, j)-th block of size $n_i \times n_j$ for all $1 \leq i, j \leq r$. Then W and K commute if and only if K is a block upper triangular matrix such that*

$$K_{i,j} = \begin{pmatrix} K_{i+1,j+1} & * \\ 0 & * \end{pmatrix} \text{ for all } 1 \leq i \leq j \leq r-1.$$

Here, $K_{i,j}$ is written as a block matrix where the zero denotes the zero matrix of size $(n_i - n_{i+1}) \times n_{j+1}$. The asterisk entries $(*)$ indicate no restrictions on the entries in that part of the matrix. The column of asterisks disappears if $n_j = n_{j+1}$, and the row $\begin{pmatrix} 0 & * \end{pmatrix}$ disappears if $n_i = n_{i+1}$.

2.5. Duality between the Jordan and Weyr Forms

In this section, we will recall the duality between the Jordan and Weyr canonical forms. Each partition $(n_1, n_2, \ldots, n_r)$ of n determines a Young diagram. The Weyr structure $(m_1, m_2, \ldots, m_{n_1})$ is the conjugate partition of the Jordan structure $(n_1, n_2, \ldots, n_r)$. Therefore, by transposing the Young diagram (writing its columns as rows) of partition $(n_1, n_2, \ldots, n_r)$, we get a Young diagram that corresponds to the conjugate partition $(m_1, m_2, \ldots, m_{n_1})$ of $(n_1, n_2, \ldots, n_r)$. More precisely, m_j is the number of n_i's greater than or equal to j; see Definition 2.1. Moreover, it is worth noting that if $\mathbf{d}(n)$ is a partition corresponding to the Jordan structure $(n_1, n_2, \ldots, n_r)$, then the corresponding Weyr structure $(m_1, m_2, \ldots, m_{n_1})$ can also be represented by the conjugate partition $\overline{\mathbf{d}}(n)$; see Lemma 2.3.

The following result establishes the duality between Jordan and Weyr structures of complex matrices.

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Theorem 2.11 O'Meara et al. [15, Theorem 2.4.1] *The Weyr and Jordan structures of a nilpotent $n \times n$ matrix A (or a matrix with a single eigenvalue) correspond to partitions of n that are conjugate (or dual) to each other. Furthermore, the Weyr and Jordan forms of a square matrix are conjugate to each other by a permutation matrix.*

The following example illustrates Proposition 2.10 and describes the centraliser of a Weyr matrix.

Example 2.12. Let $A = \left(\begin{array}{c|c|c} \mathrm{J}(1,4) & & \\ \hline & \mathrm{J}(1,4) & \\ \hline & & \mathrm{J}(1,2) \end{array}\right) := \mathrm{J}(1,4) \oplus \mathrm{J}(1,4) \oplus \mathrm{J}(1,2)$ be a unipotent Jordan form in $\mathrm{GL}(10,\mathbb{C})$ with Jordan structure $(4,4,2)$. Then the Weyr form A_W corresponding to the Jordan form A has the Weyr structure $(3,3,2,2)$ and can be given as follows:

$$A_W = \left(\begin{array}{c|c|c|c} \mathrm{I}_3 & \mathrm{I}_3 & & \\ \hline & \mathrm{I}_3 & \mathrm{I}_{3,2} & \\ \hline & & \mathrm{I}_2 & \mathrm{I}_2 \\ \hline & & & \mathrm{I}_2 \end{array}\right).$$

Furthermore, Proposition 2.10 implies that a matrix $B \in \mathrm{GL}(10,\mathbb{C})$ commuting with the basic Weyr matrix A_W has the following form:

$$B = \left(\begin{array}{ccc|ccc|cc|cc} a & b & e & h & i & l & p & q & v & w \\ c & d & f & j & k & m & r & s & x & y \\ 0 & 0 & g & 0 & 0 & n & t & u & z & \alpha \\ \hline & & & a & b & e & h & i & p & q \\ & & & c & d & f & j & k & r & s \\ & & & 0 & 0 & g & 0 & 0 & t & u \\ \hline & & & & & & a & b & h & i \\ & & & & & & c & d & j & k \\ \hline & & & & & & & & a & b \\ & & & & & & & & c & d \end{array}\right). \tag{2.2}$$

Here empty blocks represent blocks with zero matrices. Observe that partitions $(4,4,2)$ and $(3,3,2,2)$ representing the Weyr and Jordan structure of A, respectively, are conjugate (or dual) to each other; see Definition 2.1.

3. Reversing symmetry groups in $\mathrm{GL}(n,\mathbb{C})$

This section explores the structure of the reversing symmetry group for specific types of Jordan forms (listed in Table 1) in $\mathrm{GL}(n,\mathbb{C})$, which may be of independent interest. We will then apply these results to investigate strong reversibility in $\mathrm{SL}(n,\mathbb{C})$.

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Remark 3.1. Recall the notation $\Omega(\lambda, n)$ introduced in Definition 1.3. For more clarity, we can write $\Omega(\lambda, n) = [x_{i,j}]_{1\leq i,j\leq n} \in \mathrm{GL}(n, \mathbb{C})$, where $x_{i,j}$ are as follows

$$x_{i,j} = \begin{cases} 0 & \text{if } j < i \\ 0 & \text{if } j = n, i \neq n \\ (-1)^{n-i}\,\lambda^{-2(n-i)} & \text{if } j = i \\ (-1)^{n-i}\binom{n-i-1}{j-i}\lambda^{-2n+i+j} & \text{if } i < j, j \neq n \end{cases}, \tag{3.1}$$

where $\lambda \in \mathbb{C} \setminus \{0\}$ and $\binom{n-i-1}{j-i}$ denotes the binomial coefficients. Observe that for all $1 \leq i \leq j \leq n-1$, we can also write Equation (1.1) as follows

$$x_{i,j} = -\lambda^{-2}x_{i+1,j+1} + \lambda^{-3}x_{i+2,j+1} - \lambda^{-4}x_{i+3,j+1} + \cdots + (-1)^{(n-i)}\lambda^{-(n-i+1)}x_{n,j+1}. \tag{3.2}$$

The following example gives relationship between $\Omega(\lambda, 4)$ and $\mathrm{J}(\lambda, 4)$ for $\lambda \neq 0$.

Example 3.2. For a non-zero $\lambda \in \mathbb{C}$, consider $\Omega(\lambda, 4)$ as defined in Definition 1.3. Then

$$\begin{aligned} \Omega(\lambda, 4)\,\mathrm{J}(\lambda^{-1}, 4) &= \begin{pmatrix} -\lambda^{-6} & -2\lambda^{-5} & -\lambda^{-4} & 0 \\ & \lambda^{-4} & \lambda^{-3} & 0 \\ & & -\lambda^{-2} & 0 \\ & & & 1 \end{pmatrix} \begin{pmatrix} \lambda^{-1} & 1 & 0 & 0 \\ & \lambda^{-1} & 1 & 0 \\ & & \lambda^{-1} & 1 \\ & & & \lambda^{-1} \end{pmatrix} \\ &= \begin{pmatrix} -\lambda^{-7} & -3\lambda^{-6} & -3\lambda^{-5} & -\lambda^{-4} \\ & \lambda^{-5} & 2\lambda^{-4} & \lambda^{-3} \\ & & -\lambda^{-3} & -\lambda^{-2} \\ & & & \lambda^{-1} \end{pmatrix} = \begin{pmatrix} \lambda^{-1} & -\lambda^{-2} & \lambda^{-3} & -\lambda^{-4} \\ & \lambda^{-1} & -\lambda^{-2} & \lambda^{-3} \\ & & \lambda^{-1} & -\lambda^{-2} \\ & & & \lambda^{-1} \end{pmatrix} \\ &\quad \times \begin{pmatrix} -\lambda^{-6} & -2\lambda^{-5} & -\lambda^{-4} & 0 \\ & \lambda^{-4} & \lambda^{-3} & 0 \\ & & -\lambda^{-2} & 0 \\ & & & 1 \end{pmatrix} = \Big(\mathrm{J}(\lambda, 4)\Big)^{-1}\,\Omega(\lambda, 4). \end{aligned}$$

Furthermore, if $\lambda \in \{-1, +1\}$, then $\lambda^{-1} = \lambda$ and $\Omega(\lambda, 4)$ is a involution. Therefore, $\mathrm{J}(\lambda, 4)$ is a strongly reversible element in $\mathrm{GL}(4, \mathbb{C})$ for $\lambda \in \{-1, +1\}$.

The following lemma generalises Example 3.2.

Lemma 3.3. *Let $\Omega(\lambda, n) \in \mathrm{GL}(n, \mathbb{C})$ be as defined in Definition 1.3. Then we have*

$$\Omega(\lambda, n)\,\mathrm{J}(\lambda^{-1}, n) = \Big(\mathrm{J}(\lambda, n)\Big)^{-1}\,\Omega(\lambda, n).$$

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Proof. Write $\mathrm{J}(\lambda^{-1},n) = [a_{i,j}]_{n\times n}$ and $\Big(\mathrm{J}(\lambda,n)\Big)^{-1} = [b_{i,j}]_{n\times n}$, where

$$a_{i,j} = \begin{cases} \lambda^{-1} & \text{if } j = i \\ 1 & \text{if } j = i+1 \\ 0 & \text{otherwise} \end{cases} \text{ and } b_{i,j} = \begin{cases} \lambda^{-1} & \text{if } j = i \\ (-1)^k \lambda^{-(k+1)} & \text{if } j = i+k \\ 0 & \text{otherwise} \end{cases}.$$

Let $\Omega(\lambda,n) = [x_{i,j}]_{n\times n}$. Note that for all $1 \le i \le n$, we have

$$\Big(\Omega(\lambda,n)\,\mathrm{J}(\lambda^{-1},n)\Big)_{i,i} = \Big(\Big(\mathrm{J}(\lambda,n)\Big)^{-1}\,\Omega(\lambda,n)\Big)_{i,i} = \lambda^{-1}x_{i,i}.$$

Since matrices under consideration are upper triangular, so it is enough to prove the following

$$\Big(\Omega(\lambda,n)\,\mathrm{J}(\lambda^{-1},n)\Big)_{i,j} = \Big(\Big(\mathrm{J}(\lambda,n)\Big)^{-1}\,\Omega(\lambda,n)\Big)_{i,j} = \lambda^{-1}\,x_{i,j} + x_{i,j-1} \text{ for all } 1 \le i < j \le n.$$

To see this, note that for all $1 \le i < j \le n$, we have

$$\Big(\Omega(\lambda,n)\,\mathrm{J}(\lambda^{-1},n)\Big)_{i,j} = \sum_{r=1}^{n} x_{i,r}\,a_{r,j} = \sum_{r=i}^{j} x_{i,r}\,a_{r,j} = x_{i,j-1} + x_{i,j}\lambda^{-1} = \lambda^{-1}x_{i,j} + x_{i,j-1}.$$

Furthermore, for all $1 \le i < j \le n$, we have

$$\begin{aligned}\Big(\Big(\mathrm{J}(\lambda,n)\Big)^{-1}\,\Omega(\lambda,n)\Big)_{i,j} &= \sum_{r=1}^{n} b_{i,r}x_{r,j} = \sum_{r=i}^{j} b_{i,r}x_{r,j} = b_{i,i}x_{i,j} + b_{i,i+1}x_{i+1,j} + \cdots + b_{i,j}x_{j,j} \\ &= \lambda^{-1}x_{i,j} + (-\lambda^{-2}x_{i+1,j} + \lambda^{-3}x_{i+2,j} + \cdots + (-1)^{(j-i)}\lambda^{-(j-i+1)}x_{j,j}).\end{aligned}$$

Using Equations (1.1) and (3.2), we get

$$\Big(\Big(\mathrm{J}(\lambda,n)\Big)^{-1}\,\Omega(\lambda,n)\Big)_{i,j} = \lambda^{-1}\,x_{i,j} + x_{i,j-1} \text{ for all } 1 \le i < j \le n.$$

Hence, the proof follows. □

Next, we want to find a relationship between $\Omega(\lambda,n)$ and $\Omega(\lambda^{-1},n)$, which will be used for constructing reversers that are involutions for strongly reversible elements in $\mathrm{SL}(n,\mathbb{C})$. For this, we will use some well-known combinatorial identities. We refer to [3, Section 1.2] for the basic notions related to the binomial coefficients. Recall the following well-known binomial identities concerning binomial coefficients.

(1) *Pascal's rule:* $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$ for all $1 \le k \le n$.
(2) *Newton's identity:* $\binom{n}{k}\binom{k}{r} = \binom{n}{r}\binom{n-r}{k-r}$ for all $0 \le r \le k \le n$.

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(3) For $n \geq 1$, $\sum_{k=0}^{n}(-1)^k\binom{n}{k} = 0$.

In the following lemma, we compute inverse of $\Omega(\lambda, n)$ in $\mathrm{GL}(n, \mathbb{C})$.

Lemma 3.4. *Let $\Omega(\lambda, n) \in \mathrm{GL}(n, \mathbb{C})$ be as defined in Definition 1.3. Then we have*

$$\Big(\Omega(\lambda, n)\Big)^{-1} = \Omega(\lambda^{-1}, n).$$

Proof. Let $\Omega(\lambda, n) = [x_{i,j}]_{n\times n}$ and $\Omega(\lambda^{-1}, n) = [y_{i,j}]_{n\times n}$. Then for all $1 \leq i, j \leq n$, observe that $x_{i,j}$ are given by Equation (3.1) and

$$y_{i,j} = \begin{cases} 0 & \text{if } j < i \\ 0 & \text{if } j = n, i \neq n \\ (-1)^{n-i}\,\lambda^{2(n-i)} & \text{if } j = i \\ (-1)^{n-i}\binom{n-i-1}{j-i}\,\lambda^{-(-2n+i+j)} & \text{if } i < j, j \neq n. \end{cases}$$

Here, condition (3) of Definition 1.3 can be checked using Pascal's rule (1).

Let $g = [g_{i,j}]_{n\times n} := \Omega(\lambda, n)\,\Omega(\lambda^{-1}, n)$. Then g is an upper triangular matrix with diagonal entries equal to 1, and $g_{i,j} = \sum_{k=1}^{n} x_{i,k}\, y_{k,j} = \sum_{k=i}^{j} x_{i,k}\, y_{k,j}$ for all $1 \leq i < j \leq n$. This implies that for all $1 \leq i < j \leq n$, we have

$$g_{i,j} = \sum_{k=i}^{j}(-1)^{n-i}\binom{n-i-1}{k-i}\lambda^{-2n+i+k}\,(-1)^{n-k}\binom{n-k-1}{j-k}\lambda^{-(-2n+k+j)}.$$

Therefore, for all $1 \leq i < j \leq n$, we have

$$g_{i,j} = \lambda^{i-j}\sum_{k=i}^{j}(-1)^{(-i-k)}\binom{n-i-1}{k-i}\binom{n-k-1}{j-k}. \tag{3.3}$$

By substituting $r = k - i$ in Equation (3.3), we get

$$g_{i,j} = \lambda^{i-j}\sum_{r=0}^{j-i}(-1)^{(-2i-r)}\binom{n-i-1}{r}\binom{(n-i-1)-r}{(j-i)-r}.$$

In view of the Newton's identity (2) and identity (3), we get

$$g_{i,j} = \lambda^{i-j}\binom{n-i-1}{j-i}\sum_{r=0}^{j-i}(-1)^r\binom{j-i}{r} = 0 \text{ for all } 1 \leq i < j \leq n.$$

Hence, $g = \mathrm{I}_n$ in $\mathrm{GL}(n, \mathbb{C})$. This completes the proof. □

In the following result, we show that $\Omega(\mu, n)$ is an involution for $\mu \in \{-1, +1\}$.

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Lemma 3.5. *Let* $\Omega(\mu,n)\in \mathrm{GL}(n,\mathbb{C})$ *be as defined in Definition 1.3. If* $\mu\in\{-1,+1\}$*, then* $\Omega(\mu,n)$ *is an involution in* $\mathrm{GL}(n,\mathbb{C})$*.*

Proof. In view of Lemma 3.4, we have

$$\Big(\Omega(\mu,n)\Big)^{-1}=\Omega(\mu^{-1},n).$$

If $\mu\in\{-1,+1\}$, then $\mu^{-1}=\mu$. Hence, the proof follows. □

Remark 3.6. The proof of Table 1 follows from Lemma 3.3 and Lemma 3.4.

3.1. Reverser set of certain Jordan forms

In this section, we will compute the reverser set $\mathcal{R}_{\mathrm{GL}(n,\mathbb{C})}(A)$ for certain Jordan forms in $\mathrm{GL}(n,\mathbb{C})$. We introduce a notation for upper triangular Toeplitz matrices and some definitions to formulate our results.

Definition 3.7. Gohberg et al. [6, p. 297] *For* $\mathbf{x}=(x_1,x_2,\dots,x_n)\in\mathbb{C}^n$*, we define* $\mathrm{Toep}_n(\mathbf{x})\in \mathrm{M}(n,\mathbb{C})$ *as*

$$\mathrm{Toep}_n(\mathbf{x}):=[x_{i,j}]_{n\times n}=\begin{cases}0 & \text{if } i>j\\ x_{j-i+1} & \text{if } i\le j\end{cases},\quad \text{where } 1\le i,j\le n. \tag{3.4}$$

We can also write $\mathrm{Toep}_n(\mathbf{x})$ as

$$\mathrm{Toep}_n(\mathbf{x})=\begin{pmatrix} x_1 & x_2 & \cdots & \cdots & \cdots & \cdots & x_n\\ & x_1 & x_2 & \cdots & \cdots & \cdots & x_{n-1}\\ & & x_1 & x_2 & \cdots & \cdots & x_{n-2}\\ & & & \ddots & \ddots & & \vdots\\ & & & & \ddots & \ddots & \vdots\\ & & & & & x_1 & x_2\\ & & & & & & x_1\end{pmatrix}. \tag{3.5}$$

Definition 3.8. *For a non-zero* $\lambda\in\mathbb{C}$ *and* $\mathbf{x}=(x_1,x_2,\dots,x_n)\in\mathbb{C}^n$ *such that* $x_1\neq 0$*, define upper triangular matrix* $\Omega(\lambda,\mathbf{x},n):=[g_{i,j}]_{n\times n}\in \mathrm{GL}(n,\mathbb{C})$ *as follows*

(1) $g_{i,j}=0$ *for all* $1\le i,j\le n$ *such that* $i>j$*,*
(2) $g_{i,n}=x_{n-i+1}$ *for all* $1\le i\le n$*,*
(3) $g_{i,j}=-\lambda^{-1}g_{i+1,j}-\lambda^{-2}g_{i+1,j+1}$ *for all* $1\le i\le j\le n-1$*.*

Note the following example that shows the relationship between $\Omega(\lambda,5)$ and $\Omega(\lambda,\mathbf{x},5)$ when $\lambda=1$.

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Example 3.9. Let $\lambda = 1$ and $\mathbf{x} = (x_1, x_2, \ldots, x_5) \in \mathbb{C}^5$. Note that

$$\begin{pmatrix} x_1 & 3x_1 - x_2 & 3x_1 - 2x_2 + x_3 & x_1 - x_2 + x_3 - x_4 & x_5 \\ & -x_1 & -2x_1 + x_2 & -x_1 + x_2 - x_3 & x_4 \\ & & x_1 & x_1 - x_2 & x_3 \\ & & & -x_1 & x_2 \\ & & & & x_1 \end{pmatrix}$$
$$= \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ & x_1 & x_2 & x_3 & x_4 \\ & & x_1 & x_2 & x_3 \\ & & & x_1 & x_2 \\ & & & & x_1 \end{pmatrix} \begin{pmatrix} 1 & 3 & 3 & 1 & 0 \\ & -1 & -2 & -1 & 0 \\ & & 1 & 1 & 0 \\ & & & -1 & 0 \\ & & & & 1 \end{pmatrix}.$$

Therefore, we have $\Omega(1, \mathbf{x}, 5) = \mathrm{Toep}_5(\mathbf{x})\, \Omega(1, 5)$.

In the following lemma, we generalise Example 3.9 and establish a relationship between $\Omega(\lambda, \mathbf{x}, n)$ and the Jordan block $\mathrm{J}(\lambda, n)$.

Lemma 3.10. *Consider $\lambda \in \mathbb{C}$ and $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{C}^n$ such that $\lambda \neq 0$ and $x_1 \neq 0$. Let $\Omega(\lambda, n), \mathrm{Toep}_n(\mathbf{x})$ and $\Omega(\lambda, \mathbf{x}, n)$ be as defined in Definition 1.3, Definition 3.7 and Definition 3.8, respectively. Then the following statements hold.*

(1) $\Omega(\lambda, \mathbf{x}, n) = \mathrm{Toep}_n(\mathbf{x})\, \Omega(\lambda, n)$.
(2) $\Omega(\lambda, \mathbf{x}, n)\, \mathrm{J}(\lambda^{-1}, n) = \Big(\mathrm{J}(\lambda, n)\Big)^{-1} \Omega(\lambda, \mathbf{x}, n)$.

Proof. Let $\mathrm{Toep}_n(\mathbf{x}) = [x_{i,j}]_{n \times n}$ and $\Omega(\lambda, n) = [y_{i,j}]_{n \times n}$. Then $x_{i,j}$ and $y_{i,j}$ are given by Equations (3.4) and (3.1), respectively, where $1 \leq i, j \leq n$. Let $g = [g_{i,j}]_{n \times n} = \mathrm{Toep}_n(\mathbf{x})\, \Omega(\lambda, n)$. Then g is an upper triangular matrix such that for all $1 \leq i \leq j \leq n$, we have

$$g_{i,j} = \sum_{k=1}^{n} x_{i,k}\, y_{k,j} = \sum_{k=i}^{j} x_{i,k}\, y_{k,j} = \sum_{k=i}^{j} x_{k-i+1} (-1)^{n-k} \binom{n-k-1}{j-k} \lambda^{-2n+k+j}. \tag{3.6}$$

This implies that for all $1 \leq i \leq n$, we have

$$g_{i,i} = x_1 (-1)^{n-i} \lambda^{-2(n-i)} \quad \text{and} \quad g_{i,n} = x_{i,n}\, y_{n,n} = x_{i,n} = x_{n-i+1}. \tag{3.7}$$

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Note that for all $1 \leq i < j \leq n-1$, we have

$$g_{i+1,j} = \sum_{k=i+1}^{j} x_{k-(i+1)+1}(-1)^{n-k} \binom{n-k-1}{j-k} \lambda^{-2n+k+j} \text{ and}$$

$$g_{i+1,j+1} = \Big(\sum_{k=i+1}^{j} x_{k-(i+1)+1}(-1)^{n-k} \binom{n-k-1}{(j+1)-k} \lambda^{-2n+k+j+1} \Big) + x_{j-i+1}(-1)^{n-j-1}\lambda^{-2n+2(j+1)}.$$

This implies that for all $1 \leq i < j \leq n-1$, we have

$$\lambda^{-2} g_{i+1,j+1} + \lambda^{-1} g_{i+1,j} = x_{j-i+1}(-1)^{n-j-1}\, \lambda^{-2n+2j} + \sum_{k=i+1}^{j} x_{k-(i+1)+1}(-1)^{n-k}\, \lambda^{-2n+k+j-1} \left(\binom{n-k-1}{(j-k)+1} + \binom{n-k-1}{j-k} \right).$$

Using Pascal's identity (1), for all $1 \leq i < j \leq n-1$, we have

$$\lambda^{-2} g_{i+1,j+1} + \lambda^{-1} g_{i+1,j} = (-1)^{n-j-1} x_{j-i+1} \lambda^{-2n+2j} + \sum_{k=i+1}^{j} (-1)^{n-k+1} x_{k-(i+1)+1} \lambda^{-2n+k+j-1} \binom{n-k}{j-k+1}.$$

By substituting $r = k-1$, for all $1 \leq i < j \leq n-1$, we have

$$\lambda^{-2} g_{i+1,j+1} + \lambda^{-1} g_{i+1,j} = (-1)^{n-j-1} x_{j-i+1} \lambda^{-2n+2j} + \sum_{r=i}^{j-1} (-1)^{n-r-1} x_{r-i+1}\, \lambda^{-2n+r+j} \binom{n-r-1}{j-r}.$$

This implies

$$\lambda^{-2} g_{i+1,j+1} + \lambda^{-1} g_{i+1,j} = \sum_{r=i}^{j} (-1)^{n-r-1} x_{r-i+1}\, \lambda^{-2n+r+j} \binom{n-r-1}{j-r}, \tag{3.8}$$

where $1 \leq i < j \leq n-1$. Therefore, from Equations (3.6) and (3.8), we have

$$g_{i,j} = -(\lambda^{-2} g_{i+1,j+1} + \lambda^{-1} g_{i+1,j}) \text{ for all } 1 \leq i < j \leq n-1. \tag{3.9}$$

Using Equations (3.7) and (3.9), we have $g = \mathrm{Toep}_n(\mathbf{x})\, \Omega(\lambda, n) = \Omega(\lambda, \mathbf{x}, n)$. This proves the first part of the lemma.

Now, recall the well-known fact that Toeplitz matrix $\mathrm{Toep}_n(\mathbf{x})$ commutes with the Jordan block $\mathrm{J}(\lambda, n)$ and its inverse; see [15, Proposition 3.1.2], [6, Theorem 9.1.1]. Thus,

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using Lemma 3.3, we get

$$\begin{aligned}\Big(\mathrm{Toep}_n(\mathbf{x})\,\Omega(\lambda,n)\Big)\mathrm{J}(\lambda^{-1},n) &= \mathrm{Toep}_n(\mathbf{x})\,\Big(\Omega(\lambda,n)\,\mathrm{J}(\lambda^{-1},n)\Big)\\ &= \mathrm{Toep}_n(\mathbf{x})\,\Big(\Big(\mathrm{J}(\lambda,n)\Big)^{-1}\,\Omega(\lambda,n)\Big) = \Big(\mathrm{J}(\lambda,n)\Big)^{-1}\Big(\mathrm{Toep}_n(\mathbf{x})\,\Omega(\lambda,n)\Big).\end{aligned}$$

Therefore, $\Omega(\lambda,\mathbf{x},n)\,\mathrm{J}(\lambda^{-1},n) = \Big(\mathrm{J}(\lambda,n)\Big)^{-1}\,\Omega(\lambda,\mathbf{x},n)$. This completes the proof. □

The following result gives us the reverser set for certain reversible Jordan forms in $\mathrm{GL}(n,\mathbb{C})$. We refer to Definition 1.3 for the definition of $\Omega(\lambda,n)$.

Proposition 3.11. *Let $\mu,\lambda \in \mathbb{C}\setminus\{0\}$ such that $\mu \in \{-1,+1\}$ and $\lambda \notin \{-1,+1\}$. Then the following statements hold.*

(1) $\mathcal{R}_{\mathrm{GL}(n,\mathbb{C})}(\mathrm{J}(\mu,n)) = \Big\{\Omega(\mu,\mathbf{x},n) \in \mathrm{GL}(n,\mathbb{C}) \mid \mathbf{x}\in\mathbb{C}^n \text{ with } x_1 \neq 0\Big\}$.

(2) $\mathcal{R}_{\mathrm{GL}(2n,\mathbb{C})}(\mathrm{J}(\lambda,n)\oplus\mathrm{J}(\lambda^{-1},n)) =$

$$\left\{\begin{pmatrix} & \Omega(\lambda,\mathbf{x},n)\\ \Omega(\lambda^{-1},\mathbf{y},n) & \end{pmatrix} \in \mathrm{GL}(2n,\mathbb{C}) \mid \mathbf{x},\mathbf{y}\in\mathbb{C}^n \text{ with } x_1,y_1\neq 0\right\}.$$

Proof. Suppose $A \in \mathrm{GL}(n,\mathbb{C})$ is a reversible element and $h \in \mathcal{R}_{\mathrm{GL}(n,\mathbb{C})}(A)$ is a reverser for A. Recall that the set $\mathcal{R}_{\mathrm{GL}(n,\mathbb{C})}(A)$ of reversers of A is a right coset of the centraliser $\mathcal{Z}_{\mathrm{GL}(n,\mathbb{C})}(A)$ of A. Therefore, if $g \in \mathcal{R}_{\mathrm{GL}(n,\mathbb{C})}(A)$, i.e., $gAg^{-1} = A^{-1}$, then $g = fh$ for some $f \in \mathcal{Z}_{\mathrm{GL}(n,\mathbb{C})}(A))$. In other words,

$$\mathcal{R}_{\mathrm{GL}(n,\mathbb{C})}(A) = \mathcal{Z}_{\mathrm{GL}(n,\mathbb{C})}(A))\,h. \tag{3.10}$$

Now, using the above observation, we prove the result as follows.

Proof of (1). Let $g \in \mathcal{R}_{\mathrm{GL}(n,\mathbb{C})}(\mathrm{J}(\mu,n))$ and $h = \Omega(\mu,n)$. Then Lemma 3.3 implies that $h \in \mathcal{R}_{\mathrm{GL}(n,\mathbb{C})}(\mathrm{J}(\mu,n))$. Using Equation (3.10) and [15, Proposition 3.1.2], there exists a Toeplitz matrix $\mathrm{Toep}_n(\mathbf{x})$ such that

$$g = \mathrm{Toep}_n(\mathbf{x})\,h, \quad \text{where} \;\; \mathbf{x} = (x_1,x_2,\ldots,x_n) \in \mathbb{C}^n \;\; \text{and} \;\; x_1 \neq 0.$$

The first part of the proposition now follows from Lemma 3.10.

Proof of (2). Let $g \in \mathcal{R}_{\mathrm{GL}(2n,\mathbb{C})}(\mathrm{J}(\lambda,n)\oplus\mathrm{J}(\lambda^{-1},n))$ and $h = \begin{pmatrix} & \Omega(\lambda,n)\\ \Omega(\lambda^{-1},n) & \end{pmatrix}$. Using Lemma 3.3 and Lemma 3.4, we have $h \in \mathcal{R}_{\mathrm{GL}(2n,\mathbb{C})}(\mathrm{J}(\lambda,n)\oplus\mathrm{J}(\lambda^{-1},n))$. Since $\lambda \neq \lambda^{-1}$, Equation (3.10) and [6, Theorem 9.1.1, Corollary 9.1.3] imply that there exist

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Toeplitz matrices $\mathrm{Toep}_n(\mathbf{x})$ and $\mathrm{Toep}_n(\mathbf{y})$ in $\mathrm{GL}(n, \mathbb{C})$ such that

$$g = \begin{pmatrix} \mathrm{Toep}_n(\mathbf{x}) & \\ & \mathrm{Toep}_n(\mathbf{y}) \end{pmatrix} \begin{pmatrix} & \Omega(\lambda, n) \\ \Omega(\lambda^{-1}, n) & \end{pmatrix},$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ are in $\mathbb{C}^n$ such that both x_1 and y_1 are non-zero. Now, Lemma 3.10 implies that g has the following form

$$g = \begin{pmatrix} & \Omega(\lambda, \mathbf{x}, n) \\ \Omega(\lambda^{-1}, \mathbf{y}, n) & \end{pmatrix}.$$

This completes the proof. □

For more clarity, using Definition 3.8, we can rewrite Proposition 3.11(1) as in the following result.

Corollary. *Let* $g = [g_{i,j}]_{1 \leq i,j \leq n} \in \mathcal{R}_{\mathrm{GL}(n,\mathbb{C})}(\mathrm{J}(\mu, n))$*, where* $\mu \in \{-1, +1\}$*. The entries of matrix* g *satisfy the following conditions*

(1) $g_{i,j} = 0$ *for all* $1 \leq i, j \leq n$ *such that* $j < i$*,*
(2) $g_{n,n}$ *is a non-zero complex number (since* $\det(g) \neq 0$*),*
(3) $g_{i,n}$ *is an arbitrary complex number for all* $1 \leq i \leq n-1$*,*
(4) *For all* $1 \leq i \leq j \leq n-1$*, we have*

$$g_{i,j} = -\mu^{-1} g_{i+1,j} - \mu^{-2} g_{i+1,j+1}.$$

4. Strong reversibility of certain Jordan forms in $\mathbf{SL}(n, \mathbb{C})$

We will now examine the determinant of involutions in the reverser set obtained in Proposition 3.11.

Lemma 4.1. *Let* $\mu, \lambda \in \mathbb{C} \setminus \{0\}$ *such that* $\mu \in \{-1, +1\}$ *and* $\lambda \notin \{-1, +1\}$*. Then the following statements hold.*

(1) *Let* $g \in \mathcal{R}_{\mathrm{GL}(n,\mathbb{C})}(\mathrm{J}(\mu, n))$ *be an involution. Then*

$$\det(g) = \begin{cases} +1 & \text{if } n = 4k \\ -1 & \text{if } n = 4k+2 \\ \pm 1 & \text{if } n = 4k+1, 4k+3\, (\text{i.e., } n \text{ is odd}), \end{cases}$$

where $k \in \mathbb{N} \cup \{0\}$*.*

(2) *Let* $g \in \mathcal{R}_{\mathrm{GL}(2n,\mathbb{C})}(\mathrm{J}(\lambda, n) \oplus \mathrm{J}(\lambda^{-1}, n))$ *be an involution. Then*

$$\det(g) = (-1)^n = \begin{cases} +1 & \text{if } n \text{ is even} \\ -1 & \text{if } n \text{ is odd.} \end{cases}$$

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Proof. *Proof of (1).* Using Proposition 3.11(1), we can write $g = [g_{i,j}]_{1 \le i,j \le n} = \Omega(\mu, \mathbf{x}, n)$, where $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{C}^n$ such that $x_1 \neq 0$. Since $\mu \in \{-1, +1\}$, using Equation (3.7), we get that g is an upper triangular matrix with diagonal entries $g_{i,i} = x_1(-1)^{(n-i)}$ for all $1 \le i \le n$. This implies

$$\det(g) = (x_1)^n \prod_{i=1}^{n} (-1)^{n-i} = (x_1)^n (-1)^{\sum_{k=1}^{n}(n-i)} = (x_1)^n (-1)^{\frac{n(n-1)}{2}}.$$

Therefore, $\det(g)$ depends only on x_1 and n. Since g is an involution with an upper triangular form, we have $(g_{n,n})^2 = (x_1)^2 = 1$. This implies

$$x_1 \in \{-1, +1\}.$$

The proof now follows from the equation $\det(g) = (x_1)^n \, (-1)^{\frac{n(n-1)}{2}}$.

Proof of (2). Using Proposition 3.11(2), we can write

$$g = \begin{pmatrix} & \Omega(\lambda, \mathbf{x}, n) \\ \Omega(\lambda^{-1}, \mathbf{y}, n) & \end{pmatrix}, \tag{4.1}$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ are in $\mathbb{C}^n$ such that both x_1 and y_1 are non-zero. Let $\Omega(\lambda, \mathbf{x}, n) = [a_{i,j}]_{1 \le i,j \le n}$ and $\Omega(\lambda^{-1}, \mathbf{y}, n) = [b_{i,j}]_{1 \le i,j \le n}$. Note that $\Omega(\lambda, \mathbf{x}, n)$ and $\Omega(\lambda^{-1}, \mathbf{y}, n)$ are both upper triangular matrices with diagonal entries $a_{i,i} = (x_1)(-\lambda^{-2})^{(n-i)}$ and $b_{i,i} = (y_1)(-\lambda^{2})^{(n-i)}$, respectively, where $1 \le i \le n$. This implies

$$\det(\Omega(\lambda, \mathbf{x}, n)) = (x_1)^n \Big[\prod_{i=1}^{n} (-\lambda^{-2})^{(n-i)} \Big] = (x_1)^n \Big[(-\lambda^{-2})^{\sum_{i=1}^{n}(n-i)} \Big].$$

Thus, $\det(\Omega(\lambda, \mathbf{x}, n)) = (x_1)^n \Big[(-\lambda^{-2})^{\frac{n(n-1)}{2}} \Big]$. Similarly, we get

$$\det(\Omega(\lambda^{-1}, \mathbf{y}, n)) = (y_1)^n \Big[(-\lambda^{2})^{\frac{n(n-1)}{2}} \Big].$$

Since $\det(g) = \det \begin{pmatrix} & \Omega(\lambda, \mathbf{x}, n) \\ \Omega(\lambda^{-1}, \mathbf{y}, n) & \end{pmatrix}$, we have

$$\det(g) = \det \begin{pmatrix} \Omega(\lambda, \mathbf{x}, n) & \\ & \Omega(\lambda^{-1}, \mathbf{y}, n) \end{pmatrix} \begin{pmatrix} & \mathrm{I}_n \\ \mathrm{I}_n & \end{pmatrix}.$$

Thus, $\det(g) = \det(\Omega(\lambda, \mathbf{x}, n)) \det(\Omega(\lambda^{-1}, \mathbf{y}, n))(-1)^n$. This implies

$$\det(g) = (-1)^n (-x_1)^n (-y_1)^n = (-1)^n (x_1 y_1)^n.$$

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Since $g \in \mathrm{GL}(2n, \mathbb{C})$ is an involution given in Equation (4.1), where both $\Omega(\lambda, \mathbf{x}, n)$ and $\Omega(\lambda^{-1}, \mathbf{y}, n)$ are upper triangular matrices in $\mathrm{GL}(n, \mathbb{C})$, we have $(g^2)_{2n,2n} = x_1 y_1 = 1$. This implies

$$\det(g) = (-1)^n (x_1 y_1)^n = (-1)^n.$$

This completes the proof. □

Remark 4.2. It is worth mentioning that the converse of Lemma 4.1 also holds, and it follows by constructing a suitable reverser using Table 1 and Proposition 3.11, which is also an involution. This construction is useful in proving the converse of Theorem 1.2.

The next result helps us understand the strong reversibility in $\mathrm{SL}(n, \mathbb{C})$.

Proposition 4.3. *Let $\mu, \lambda \in \mathbb{C} \setminus \{0\}$ such that $\mu \in \{-1, +1\}$ and $\lambda \notin \{-1, +1\}$. Let $A \in \mathrm{SL}(n, \mathbb{C})$ and $B \in \mathrm{SL}(2n, \mathbb{C})$ denote the Jordan forms $\mathrm{J}(\mu, n)$ and $\mathrm{J}(\lambda, n) \oplus \mathrm{J}(\lambda^{-1}, n)$, respectively. Then the following statements hold.*

(1) *A is reversible in $\mathrm{SL}(n, \mathbb{C})$ for all $n \in \mathbb{N}$. Moreover, A is strongly reversible in $\mathrm{SL}(n, \mathbb{C})$ if and only if $n \neq 4k + 2$, where $k \in \mathbb{N} \cup \{0\}$.*
(2) *B is reversible in $\mathrm{SL}(2n, \mathbb{C})$ for all $n \in \mathbb{N}$. Moreover, B is strongly reversible in $\mathrm{SL}(2n, \mathbb{C})$ if and only if n is even.*

Proof. *Proof of (1).* Let $g = \Omega(\lambda, \mathbf{x}, n)$, where $\mathbf{x} = (x_1, 0, \ldots, 0) \in \mathbb{C}^n$ such that

$$x_1 = \begin{cases} -1 & \text{if } n = 4k, 4k+1 \\ +1 & \text{if } n = 4k+3 \\ -\iota & \text{if } n = 4k+2 \end{cases}, \text{where } \iota^2 = -1, k \in \mathbb{N} \cup \{0\}.$$

Then $gAg^{-1} = A^{-1}$ and $g \in \mathrm{SL}(n, \mathbb{C})$. Thus A is reversible in $\mathrm{SL}(n, \mathbb{C})$ for all $n \in \mathbb{N}$. Furthermore, if $n \neq 4k + 2$, then g is also an involution in $\mathrm{SL}(n, \mathbb{C})$, where $k \in \mathbb{N} \cup \{0\}$. This implies that A is strongly reversible if $n \neq 4k + 2$.

Next, consider the case when $n = 4k+2$, where $k \in \mathbb{N} \cup \{0\}$. Suppose that A is strongly reversible. Then there exists $h \in \mathrm{SL}(n, \mathbb{C})$ such that $hAh^{-1} = A^{-1}$ and $h^2 = \mathrm{I}_n$. In view of the Lemma 4.1(1), we have $\det(h) = -1$, which is a contradiction. This proves the first part of the result.

Proof of (2). Let $g = \begin{pmatrix} & \Omega(\lambda, \mathbf{x}, n) \\ \Omega(\lambda^{-1}, \mathbf{y}, n) & \end{pmatrix}$, where $\mathbf{x} = (x_1, 0, \ldots, 0)$ and $\mathbf{y} = (y_1, 0, \ldots, 0)$ are in $\mathbb{C}^n$ such that

$$x_1 = 1 \quad \text{and} \quad y_1 = \begin{cases} 1 & \text{if } n \text{ is even} \\ -1 & \text{if } n \text{ is odd} \end{cases}.$$

Then $gAg^{-1} = A^{-1}$ and $g \in \mathrm{SL}(2n, \mathbb{C})$. Thus A is reversible in $\mathrm{SL}(2n, \mathbb{C})$ for all $n \in \mathbb{N}$. Furthermore, if n is even, then g is also an involution in $\mathrm{SL}(2n, \mathbb{C})$. This implies that A is strongly reversible in $\mathrm{SL}(2n, \mathbb{C})$ if n is even.

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Next, consider the case when n is odd. Suppose that A is strongly reversible. Then there exists $h \in \mathrm{SL}(2n, \mathbb{C})$ such that $hAh^{-1} = A^{-1}$ and $h^2 = \mathrm{I}_{2n}$. In view of Lemma 4.1 (2), we have $\det(h) = -1$, which is a contradiction. This completes the proof. □

4.1. Strongly reversible semisimple elements

The following lemma classifies strongly reversible semisimple elements in $\mathrm{SL}(n, \mathbb{C})$.

Proposition 4.4. *A reversible semisimple element in* $\mathrm{SL}(n, \mathbb{C})$ *is strongly reversible if and only if either* $\{\pm 1\}$ *is an eigenvalue or* $n = 4k$ *for some* $k \in \mathbb{N}$.

Proof. Let $A \in \mathrm{SL}(n, \mathbb{C})$ be a reversible semisimple element. If 1 or -1 is an eigenvalue of A, we can construct an involution in $\mathrm{SL}(n, \mathbb{C})$ that conjugates A to A^{-1}. Therefore, A is strongly reversible in $\mathrm{SL}(n, \mathbb{C})$; see Remark 4.2. Suppose 1 and -1 are not eigenvalues of A. Then Theorem 1.1 implies that $n = 4m$ for some $m \in \mathbb{N}$, and up to conjugacy, we can assume $A = \mathrm{diag}(\lambda_1, \dots, \lambda_{2m}, \lambda_1^{-1}, \dots, \lambda_{2m}^{-1})$. Consider the involution $g = \begin{pmatrix} & \mathrm{I}_{2m} \\ \mathrm{I}_{2m} & \end{pmatrix}$ in $\mathrm{SL}(n, \mathbb{C})$. Then $gAg^{-1} = A^{-1}$. Hence, A is strongly reversible in $\mathrm{SL}(n, \mathbb{C})$.

Conversely, suppose that A is strongly reversible $\mathrm{SL}(n, \mathbb{C})$ such that 1 and -1 are not eigenvalues of A. In view of Theorem 1.1, up to conjugacy, we can assume that A has the form

$$A = \left(\lambda_1 \mathrm{I}_{m_1} \oplus {\lambda_1}^{-1} \mathrm{I}_{m_1}\right) \oplus \left(\lambda_2 \mathrm{I}_{m_2} \oplus {\lambda_2}^{-1} \mathrm{I}_{m_2}\right) \oplus \cdots \oplus \left(\lambda_k \mathrm{I}_{m_k} \oplus {\lambda_k}^{-1} \mathrm{I}_{m_k}\right),$$

where $\sum_{i=1}^{k} 2(m_i) = n$, $\lambda_s \neq \lambda_t$ or λ_t^{-1} for $s \neq t$, and $\lambda_i \notin \{-1, +1\}$ for all $1 \leq i, s, t \leq k$. Let $g \in \mathrm{SL}(n, \mathbb{C})$ be an involution such that $gAg^{-1} = A^{-1}$. By comparing each entry of the matrix equation $gA = A^{-1}g$ and using the conditions satisfied by each λ_i, we obtain that g has the following block diagonal form

$$g = \begin{pmatrix} & g_1 \\ \widetilde{g}_1 & \end{pmatrix} \oplus \begin{pmatrix} & g_2 \\ \widetilde{g}_2 & \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} & g_k \\ \widetilde{g}_k & \end{pmatrix},$$

where $g_i, \widetilde{g}_i \in \mathrm{GL}(m_i, \mathbb{C})$ for all $1 \leq i \leq k$. Since g is an involution, we have $\widetilde{g}_i = g_i^{-1}$ for all $1 \leq i \leq k$. This implies

$$\det(g) = \prod_{i=1}^{k} \det \begin{pmatrix} & g_i \\ g_i^{-1} & \end{pmatrix} = \prod_{i=1}^{k} \det \begin{pmatrix} g_i & \\ & g_i^{-1} \end{pmatrix} \det \begin{pmatrix} & \mathrm{I}_{m_i} \\ \mathrm{I}_{m_i} & \end{pmatrix}.$$

Therefore, $\det(g) = \prod_{i=1}^{k} (-1)^{m_i} = (-1)^{\sum_{i=1}^{k} m_i} = (-1)^{\frac{n}{2}}$. Since $\det g = 1$, n is even. This completes the proof. □

5. Strong reversibility of unipotent elements in $\mathrm{SL}(n, \mathbb{C})$

In this section, we will investigate the strong reversibility of elements in $\mathrm{SL}(n, \mathbb{C})$ with eigenvalues in $\{-1, +1\}$. First, we will focus on unipotent elements in $\mathrm{SL}(n, \mathbb{C})$. Before

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proceeding, we will consider an example demonstrating the complexities of studying the strong reversibility of unipotent elements in $\mathrm{SL}(n,\mathbb{C})$.

Example 5.1. Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \oplus \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \oplus \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ be a unipotent element in $\mathrm{SL}(6,\mathbb{C})$. Then A is not strongly reversible in $\mathrm{SL}(6,\mathbb{C})$. To see this, suppose that A is strongly reversible. Then there exists an involution $g \in \mathrm{SL}(6,\mathbb{C})$ such that $gAg^{-1} = A^{-1}$. Using the equation $gA = A^{-1}g$, we get

$$g = \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} \\ 0 & -x_{11} & 0 & -x_{13} & 0 & -x_{15} \\ x_{31} & x_{32} & x_{33} & x_{34} & x_{35} & x_{36} \\ 0 & -x_{31} & 0 & -x_{33} & 0 & -x_{35} \\ x_{51} & x_{52} & x_{53} & x_{54} & x_{55} & x_{56} \\ 0 & -x_{51} & 0 & -x_{53} & 0 & -x_{55} \end{pmatrix}. \tag{5.1}$$

Note that from Equation (5.1), it is not straightforward to show that if g is an involution, then $g \notin \mathrm{SL}(6,\mathbb{C})$. However, after suitably permuting rows and columns of matrix g, we get

$$\widetilde{g} = SgS^{-1} = \begin{pmatrix} x_{11} & x_{13} & x_{15} & x_{12} & x_{14} & x_{16} \\ x_{31} & x_{33} & x_{35} & x_{32} & x_{34} & x_{36} \\ x_{51} & x_{53} & x_{55} & x_{52} & x_{54} & x_{56} \\ 0 & 0 & 0 & -x_{11} & -x_{13} & -x_{15} \\ 0 & 0 & 0 & -x_{31} & -x_{33} & -x_{35} \\ 0 & 0 & 0 & -x_{51} & -x_{53} & -x_{55} \end{pmatrix}, \quad \text{where } \ S \in \mathrm{SL}(6,\mathbb{C}).$$

Thus, we can write $\widetilde{g} = \left(\begin{array}{c|c} P & Q \\ \hline & -P \end{array}\right)$, where $P \in \mathrm{GL}(3,\mathbb{C})$ and $Q \in \mathrm{M}(3,\mathbb{C})$. Since g is an involution in $\mathrm{SL}(6,\mathbb{C})$ such that $\widetilde{g} = SgS^{-1}$, we have $\widetilde{g} \in \mathrm{SL}(6,\mathbb{C})$ and $(\widetilde{g})^2 = \mathrm{I}_6$. Using $(\widetilde{g})^2 = \mathrm{I}_6$, i.e., $P^2 = \mathrm{I}_3$, we have $\det(\widetilde{g}) = \det(P)\det(-P) = \det(P^2)(-1)^3 = (-1)^3$. This is a contradiction. Hence, A is not strongly reversible in $\mathrm{SL}(6,\mathbb{C})$.

Remark 5.2. Recall that every unipotent matrix in $\mathrm{SL}(n,\mathbb{C})$ is reversible, and we know the structure of the set of corresponding reversers; see Theorem 1.1 and § 3. We are interested in finding the necessary conditions for strong reversibility of a unipotent matrix. In Example 5.1, we transform the reverser g into a block upper triangular form by appropriately permuting its rows and columns. This step is crucial in proving that the unipotent matrix considered in Example 5.1 is not strongly reversible. For a unipotent matrix A in Jordan form with Jordan blocks of unequal sizes or a large number of diagonal Jordan blocks, checking the existence of an involution in $\mathcal{R}_{\mathrm{GL}(n,\mathbb{C})}(A) \cap \mathrm{SL}(n,\mathbb{C})$ becomes challenging. Therefore, using the Jordan canonical form of a matrix to study strongly reversible elements in $\mathrm{SL}(n,\mathbb{C})$ is not an efficient approach. Instead, we can use the Weyr canonical form, which has a more suitable centraliser (and reverser) than the Jordan

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canonical form; see Proposition 2.10. In particular, if a reversible element of $\mathrm{SL}(n,\mathbb{C})$ is in the Weyr canonical form, every reverser has a block upper triangular form.

Now, using the notion of the Weyr form, we investigate the strong reversibility of the unipotent Jordan form, where all the Jordan blocks are of the same size. The following result generalises Example 5.1.

Lemma 5.3. *Let $A = \bigoplus_{i=1}^{k} \mathrm{J}(1,2m)$ be a unipotent Jordan form in $\mathrm{SL}(2mk,\mathbb{C})$. Then the following statements hold.*

(i) *If $gAg^{-1} = A^{-1}$ and $g^2 = \mathrm{I}_{2mk}$, then $\det(g) = (-1)^{mk}$.*
(ii) *If there are an odd number of Jordan blocks of size $2 \pmod 4$ (i.e., m and k both are odd), then A can not be strongly reversible in $\mathrm{SL}(2mk,\mathbb{C})$.*

Proof. Let $A_W \in \mathrm{SL}(2mk,\mathbb{C})$ be the Weyr form corresponding to the Jordan form A; see Definition 2.7. Then using Theorem 2.11, we have $A_W = \tau A \tau^{-1}$for some $\tau \in \mathrm{SL}(2mk,\mathbb{C})$. Moreover, A_W has the Weyr structure $(\underbrace{k,k,\dots,k}_{2m\text{-times}})$ and can be written as follows

$$A_W = \begin{pmatrix} \mathrm{I}_k & \mathrm{I}_k & & & \\ & \mathrm{I}_k & \mathrm{I}_k & & \\ & & \ddots & \ddots & \\ & & & \mathrm{I}_k & \mathrm{I}_k \\ & & & & \mathrm{I}_k \end{pmatrix}. \tag{5.2}$$

Now, define $\Omega(\mathrm{I}_k, 2m) := [X_{i,j}]_{1\le i,j\le 2m} \in \mathrm{GL}(2mk,\mathbb{C})$ such that

(1) $X_{i,j} = \mathrm{O}_k$ for all $1 \le i,j \le m$ such that $j < i$, where O_k denotes the $k \times k$ zero matrix.
(2) $X_{2m,2m} = \mathrm{I}_k$ and $X_{i,2m} = \mathrm{O}_k$ for all $1 \le i \le 2m-1$,
(3) $X_{i,j} = -X_{i+1,j} - X_{i+1,j+1}$ for all $1 \le i \le j \le 2m-1$.

By using a similar argument as in Lemma 3.3, we have

$$\Omega(\mathrm{I}_k, 2m)\, A_W = A_W^{-1}\, \Omega(\mathrm{I}_k, 2m). \tag{5.3}$$

Furthermore, if $B \in \mathrm{GL}(2mk,\mathbb{C})$ commutes with A_W, then Proposition 2.10 implies that

$$B = \mathrm{Toep}_{2m}(\mathbf{K}) := \begin{pmatrix} K_{1,1} & K_{1,2} & \cdots & \cdots & \cdots & K_{1,2m} \\ & K_{1,1} & K_{1,2} & \cdots & \cdots & K_{1,2m-1} \\ & & \ddots & \ddots & & \vdots \\ & & & \ddots & \ddots & \vdots \\ & & & & K_{1,1} & K_{1,2} \\ & & & & & K_{1,1} \end{pmatrix}, \tag{5.4}$$

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where $\mathbf{K} = (K_{1,1}, K_{1,2}, \dots, K_{1,2m})$ is a $2m$-tuple of matrices such that $K_{1,1} \in \mathrm{GL}(k, \mathbb{C})$ and $K_{1,j} \in \mathrm{M}(k, \mathbb{C})$ for all $2 \leq j \leq 2m$.

Let $h = [Y_{i,j}] = \tau g \tau^{-1} \in \mathrm{GL}(2mk, \mathbb{C})$, where $Y_{i,j} \in \mathrm{M}(k, \mathbb{C})$ for all $1 \leq i, j \leq 2m$. Since $g \in \mathrm{GL}(2mk, \mathbb{C})$ such that $gAg^{-1} = A^{-1}$ and $g^2 = \mathrm{I}_{2mk}$, we have

$$hA_W h^{-1} = A_W^{-1} \text{ and } h^2 = \mathrm{I}_{2mk}.$$

Note that the set of reversers of A_W is a right coset of the centraliser of A_W. Therefore, using Equations (5.3) and (5.4), we have

$$h = \mathrm{Toep}_{2m}(\mathbf{K})\, \Omega(\mathrm{I}_k, 2m) \text{ and } h^2 = \mathrm{I}_{2mk}. \tag{5}$$

where $\det(K_{1,1}) \neq 0$. It follows that h is a block upper-triangular matrix with diagonal blocks $Y_{i,i} = (-1)^{(2m-i)} K_{1,1}$ for all $1 \leq i \leq 2m$ such that $(K_{1,1})^2 = \mathrm{I}_k$. This implies

$$\det(h) = \prod_{i=1}^{2m} \det\Big((-1)^{(2m-i)} K_{1,1}\Big) = ((-1)^k)^m \left(\det\Big((K_{1,1})^2\Big)\right)^m = (-1)^{km}.$$

Since $h = \tau g \tau^{-1}$, we have $\det(g) = (-1)^{mk}$. Therefore, if m and k are odd, then $\det(g) = -1$. Hence, A is not strongly reversible in $\mathrm{SL}(2mk, \mathbb{C})$ if m and k are odd. This completes the proof. $\square$

Next, we consider an arbitrary unipotent Jordan form in $\mathrm{SL}(n, \mathbb{C})$. Before that, note the following example, which gives an idea of the proof of the general result (cf. Proposition 5.6).

Example 5.4. Let A be a unipotent element in $\mathrm{SL}(10, \mathbb{C})$ with Jordan structure $(4, 4, 2)$ as given in Example 2.12. Then A is not strongly reversible in $\mathrm{SL}(10, \mathbb{C})$. To see this, recall that A has the Weyr structure $(3, 3, 2, 2)$. Therefore, the Weyr form A_W of A and its inverse A_W^{-1} can be given as

$$A_W = \left(\begin{array}{c|ccc} \mathrm{I}_3 & \mathrm{I}_3 & & \\ \hline & \mathrm{I}_3 & \mathrm{I}_{3,2} & \\ \hline & & \mathrm{I}_2 & \mathrm{I}_2 \\ \hline & & & \mathrm{I}_2 \end{array}\right) \text{ and } A_W^{-1} = \left(\begin{array}{c|ccc} \mathrm{I}_3 & -\mathrm{I}_3 & \mathrm{I}_{3,2} & -\mathrm{I}_{3,2} \\ \hline & \mathrm{I}_3 & -\mathrm{I}_{3,2} & \mathrm{I}_{3,2} \\ \hline & & \mathrm{I}_2 & -\mathrm{I}_2 \\ \hline & & & \mathrm{I}_2 \end{array}\right).$$

In view of Equation (2.2), any element $f \in \mathrm{GL}(10, \mathbb{C})$ that satisfies $fA_W = A_W f$ has the following block upper triangular form

$$f = \left(\begin{array}{c|ccc} \begin{pmatrix} P & * \\ & Q \end{pmatrix} & * & * & * \\ \hline & \begin{pmatrix} P & * \\ & Q \end{pmatrix} & * & * \\ \hline & & P & * \\ \hline & & & P \end{array}\right), \text{ where } P = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, \mathbb{C}),\ Q = \big(g\big) \in \mathrm{GL}(1, \mathbb{C}).$$

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Let $h = \left(\begin{array}{c|c|c|c} -\mathrm{I}_3 & -2\mathrm{I}_3 & -\mathrm{I}_{3,2} & \\ \hline & \mathrm{I}_3 & \mathrm{I}_{3,2} & \\ \hline & & -\mathrm{I}_2 & \\ \hline & & & \mathrm{I}_2 \end{array}\right)$. Then we have $hA_W = A_W^{-1}h = \left(\begin{array}{c|ccc} -\mathrm{I}_3 & -3\mathrm{I}_3 & -3\mathrm{I}_{3,2} & -\mathrm{I}_{3,2} \\ \hline & \mathrm{I}_3 & 2\mathrm{I}_{3,2} & \mathrm{I}_{3,2} \\ \hline & & -\mathrm{I}_2 & -\mathrm{I}_2 \\ \hline & & & \mathrm{I}_2 \end{array}\right)$. Therefore, $hA_Wh^{-1} = A_W^{-1}$. Since the set of reversers of A_W is a right coset of the centraliser of A_W, every reverser τ of A_W has the following form

$$\tau = fh = \left(\begin{array}{c|c|c|c} \begin{pmatrix} -P & * \\ & -Q \end{pmatrix} & * & * & * \\ \hline & \begin{pmatrix} P & * \\ & Q \end{pmatrix} & * & * \\ \hline & & -P & * \\ \hline & & & P \end{array}\right).$$

This implies that $\det(\tau) = ((-1)^2)^2\det(P^4)(-1)^1\det(Q^2) = (-1)\det(P^4)\det(Q^2)$. Moreover, if τ is an involution, then both P and Q are also involutions, and thus $\det(P^4) = \det(Q^2) = 1$. Therefore, if $\tau \in \mathrm{GL}(10, \mathbb{C})$ such that $\tau A_W \tau^{-1} = A_W^{-1}$ and $\tau^2 = \mathrm{I}_{10}$, then $\det(\tau) = -1$. Hence, A is not strongly reversible in $\mathrm{SL}(10, \mathbb{C})$.

Now, we will generalise Example 5.4. The following result follows from the proof of [8, Theorem 4.6]. However, we will provide an alternative proof using the notion of the Weyr form. We refer to § 2.1 for notation of the partition used in the following result.

Proposition 5.5. *Let $A \in \mathrm{SL}(n, \mathbb{C})$ be a unipotent element such that the Jordan decomposition of A is represented by the partition $\mathbf{d}(n) = [d_1^{t_{d_1}}, \ldots, d_s^{t_{d_s}}]$, where d_k is even for all $1 \le k \le s$. If g is an involution in $\mathrm{GL}(n, \mathbb{C})$ such that $gAg^{-1} = A^{-1}$, then $\det(g) = (-1)^{|\mathbb{E}^2_{\mathbf{d}(n)}|}$.*

Proof. Let A_W denote the Weyr form of A. Using Lemma 2.3, the partition $\overline{\mathbf{d}}(n)$ representing the Weyr form A_W is given by

$$\overline{\mathbf{d}}(n) = \left[(t_{d_1} + t_{d_2} + \cdots + t_{d_s})^{d_s}, (t_{d_1} + t_{d_2} + \cdots + t_{d_{s-1}})^{d_{s-1} - d_s}, \ldots, (t_{d_1} + t_{d_2})^{d_2 - d_3}, (t_{d_1})^{d_1 - d_2}\right].$$

Therefore, A_W is a block matrix with $(d_1)^2$ many blocks. Moreover, for all $1 \le i \le d_1$, the size n_i of the i-th diagonal block of A_W is given by

$$n_i = \begin{cases} t_{d_1} + t_{d_2} + \cdots + t_{d_s} & \text{if } 1 \le i \le d_s \\ t_{d_1} + t_{d_2} + \cdots + t_{d_{s-r-1}} & \text{if } d_{s-r} + 1 \le i \le d_{s-r-1}, \text{where } 0 \le r \le s-2 \end{cases}.$$

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This implies that the (i,j)-th block of A_W has size $n_i \times n_j$, where $1 \leq i,j \leq d_1$; see § 2.2. Furthermore, the (i,j)-th blocks of the Weyr form A_W and its inverse A_W^{-1} are given by

$$(A_W)_{i,j} = \begin{cases} \mathrm{I}_{n_i} & \text{if } i = j \\ \mathrm{I}_{n_i,n_{i+1}} & \text{if } i+1 = j \\ \mathrm{O}_{n_i \times n_j} & \text{otherwise} \end{cases} \text{ and } (A_W^{-1})_{i,j} = \begin{cases} \mathrm{I}_{n_i} & \text{if } i = j \\ (-1)^{(j-i)}\, \mathrm{I}_{n_i,n_j} & \text{if } i < j \\ \mathrm{O}_{n_i \times n_j} & \text{otherwise} \end{cases},$$

where $1 \leq i,j \leq d_1$. Consider $\Omega_W := [X_{i,j}]_{1 \leq i,j \leq d_1} \in \mathrm{GL}(n,\mathbb{C})$ such that

$$X_{i,j} = \begin{cases} \mathrm{O}_{n_i \times n_j} & \text{if } j < i \\ \mathrm{O}_{n_i \times n_j} & \text{if } j = d_1, i \neq d_1 \\ (-1)^{d_1 - i}\, \mathrm{I}_{n_i} & \text{if } j = i \\ (-1)^{d_1 - i} \binom{d_1 - i - 1}{j - i} \mathrm{I}_{n_i,n_j} & \text{if } i < j, j \neq d_1 \end{cases}, \tag{5.5}$$

where $\binom{d_1 - i - 1}{j - i}$ denotes the binomial coefficients. Then by using a similar argument as in Lemma 3.3, we have

$$\Omega_W A_W \Omega_W^{-1} = A_W^{-1}.$$

Let $f = [P_{i,j}]_{1 \leq i,j \leq d_1} \in \mathrm{GL}(n,\mathbb{C})$ be an $n \times n$ matrix commuting with Weyr form A_W such both f and A has the same block structure given by the partition $\overline{\mathbf{d}}(n)$. Then using Proposition 2.10, we can conclude that f is an upper triangular block matrix, and the i-th diagonal block $P_{i,i}$ of f has the following form

(1) if $1 \leq i \leq d_s$, then $P_{i,i} = \begin{pmatrix} P_1 & * & * & * & \cdots & * \\ & P_2 & * & \cdots & \cdots & * \\ & & \ddots & \ddots & & \vdots \\ & & & \ddots & \ddots & \vdots \\ & & & & P_{s-1} & * \\ & & & & & P_s \end{pmatrix}$,

(2) if $d_{s-r} + 1 \leq i \leq d_{s-r-1}$, then

$$P_{i,i} = \begin{pmatrix} P_1 & * & * & * & \cdots & * \\ & P_2 & * & \cdots & \cdots & * \\ & & \ddots & \ddots & & \vdots \\ & & & \ddots & \ddots & \vdots \\ & & & & P_{s-r-2} & * \\ & & & & & P_{s-r-1} \end{pmatrix},$$

(3) if $d_2 + 1 \leq i \leq d_1$, then $P_{i,i} = P_1$,

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where $0 \leq r \leq s-3$ and $P_k \in \mathrm{GL}(t_{d_k}, \mathbb{C})$ for all $1 \leq k \leq s$. It is worth noting that each matrix P_k appears d_k times in the diagonal blocks of matrix f, where $1 \leq k \leq s$.

Since A_W is the Weyr form of A, there exists a $\tau \in \mathrm{GL}(n, \mathbb{C})$ such that $A_W = \tau A \tau^{-1}$. Consider $h = \tau g \tau^{-1} = [Y_{i,j}]_{1 \leq i,j \leq d_1} \in \mathrm{GL}(n, \mathbb{C})$. Since g is an involution in $\mathrm{GL}(n, \mathbb{C})$ such that $gAg^{-1} = A^{-1}$, it follows that h is an involution in $\mathrm{GL}(n, \mathbb{C})$ such that

$$hA_W h^{-1} = A_W^{-1} \text{ and } \det(h) = \det(g). \tag{5.6}$$

Note that the set of reversers of A_W is a right coset of the centraliser of A_W. Therefore,

$$h = f\,\Omega_W.$$

This implies h is an upper triangular block matrix such that diagonal blocks of h are given by

$$Y_{i,i} = (-1)^{d_1 - i} P_{i,i} \text{ where } 1 \leq i \leq d_1.$$

Therefore, $\det(h) = \prod_{i=1}^{d_1} \det((-1)^{d_1-i} P_{i,i})$. Since d_k is even for all $1 \leq k \leq s$, we have

$$\det(h) = ((-1)^{t_{d_1}})^{\frac{d_1}{2}} \det((P_1)^2)^{\frac{d_1}{2}} ((-1)^{t_{d_2}})^{\frac{d_2}{2}} \det((P_2)^2)^{\frac{d_2}{2}} \cdots ((-1)^{t_{d_s}})^{\frac{d_s}{2}} \det((P_s)^2)^{\frac{d_s}{2}}.$$

Since h is an involution, P_k is an involution for all $1 \leq k \leq s$. This implies

$$\det(h) = ((-1)^{t_{d_1}})^{\frac{d_1}{2}} ((-1)^{t_{d_2}})^{\frac{d_2}{2}} \cdots ((-1)^{t_{d_s}})^{\frac{d_s}{2}}.$$

Observe that if $d_k = 0 \pmod 4$ for some $1 \leq k \leq s$, then $((-1)^{t_{d_k}})^{\frac{d_k}{2}} = 1$. Thus, we have

$$\det(h) = \prod_{d_k \equiv 2 \pmod 4} ((-1)^{t_{d_k}})^{\frac{d_k}{2}} = \prod_{d_k \equiv 2 \pmod 4} (-1)^{t_{d_k}} = (-1)^{\sum_{d_k \in \mathbb{E}^2_{\mathbf{d}(n)}} t_{d_k}} = (-1)^{|\mathbb{E}^2_{\mathbf{d}(n)}|},$$

where $\mathbb{E}^2_{\mathbf{d}(n)} = \{d_k \mid d_k \equiv 2 \pmod 4\}$. The proof now follows from Equation (5.6). □

Finally, we classify the strongly reversible unipotent elements in $\mathrm{SL}(n, \mathbb{C})$. This result is also proved in [8] using an infinitesimal version of the notion of the classical reversibility or reality, known as *adjoint reality*.

Proposition 5.6. Gongopadhyay and Maity [8, Theorem 4.6] *Let $A \in \mathrm{SL}(n, \mathbb{C})$ be a unipotent element. Then A is strongly reversible if and only if at least one of the following conditions holds.*

(1) There is a Jordan block $\mathrm{J}(1, 2r+1)$ of odd size in the Jordan decomposition of A.
(2) The total number of Jordan blocks of the form $\mathrm{J}(1, 4k+2)$ in the Jordan decomposition of A is even.

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Proof. Up to conjugacy, we can assume that A is in Jordan form. Let $\mathbf{d}(n) = [d_1^{t_{d_1}}, \dots, d_s^{t_{d_s}}]$ be the partition representing the Jordan form of A. Note that if A has a Jordan block of odd size, then using Remark 4.2 and Proposition 4.3, we can construct a suitable involution g in $\mathrm{SL}(n,\mathbb{C})$ such that $gAg^{-1} = A^{-1}$. Let A be a strongly reversible element in $\mathrm{SL}(n,\mathbb{C})$. Then there exists an involution $g \in \mathrm{SL}(n,\mathbb{C})$ such that $gAg^{-1} = A^{-1}$. Suppose that there does not exist any Jordan block of odd size in the Jordan decomposition of A, otherwise we are done. Then d_k is even for all $1 \le k \le s$. Using Proposition 5.5, we have

$$\det(g) = (-1)^{|\mathbb{E}^2_{\mathbf{d}(n)}|} = 1.$$

This implies that $|\mathbb{E}^2_{\mathbf{d}(n)}|$ is even, where $\mathbb{E}^2_{\mathbf{d}(n)} = \{d_k \mid d_k \equiv 2 \pmod 4\}$ is as defined in Definition 2.2. Thus, the forward direction of the result is proven.

Conversely, let at least one of the conditions (1) or (2) of Proposition 5.6 holds. Then using Lemma 2.5, Remark 4.2 and Proposition 4.3, we can construct a suitable involution g in $\mathrm{SL}(n,\mathbb{C})$ such that $gAg^{-1} = A^{-1}$. Therefore, A is strongly reversible in $\mathrm{SL}(n,\mathbb{C})$. This completes the proof. □

The study of strong reversibility of an element in $\mathrm{SL}(n,\mathbb{C})$ with -1 as its only eigenvalue is analogous to the unipotent case. Similar to Proposition 5.5, we have the following result, which will be used in proving Theorem 1.2.

Proposition 5.7. *Let $A \in \mathrm{SL}(n,\mathbb{C})$ be an element with -1 as its only eigenvalue such that the Jordan decomposition of A is represented by the partition $\mathbf{d}(n) = [d_1^{t_{d_1}}, \dots, d_s^{t_{d_s}}]$, where d_k is even for all $1 \le k \le s$. If g is an involution in $\mathrm{GL}(n,\mathbb{C})$ such that $gAg^{-1} = A^{-1}$, then $\det(g) = (-1)^{|\mathbb{E}^2_{\mathbf{d}(n)}|}$.*

Proof. The proof follows using a similar line of arguments as in Proposition 5.5. □

The next result classify strongly reversible elements in $\mathrm{SL}(n,\mathbb{C})$ with -1 as their only eigenvalue.

Proposition 5.8. *Let A be an element of $\mathrm{SL}(n,\mathbb{C})$ with -1 as its only eigenvalue. Then A is strongly reversible if and only if at least one of the following conditions holds.*

(1) *There is a Jordan block $\mathrm{J}(-1, 2r+1)$ of odd size in the Jordan decomposition of A.*
(2) *The total number of Jordan blocks of the form $\mathrm{J}(-1, 4k+2)$ in the Jordan decomposition of A is even.*

Proof. The proof follows using a similar line of arguments as in Proposition 5.6. □

6. Strong reversibility in $\mathrm{SL}(n,\mathbb{C})$

In this section, we will investigate the strong reversibility of reversible elements of $\mathrm{SL}(n,\mathbb{C})$ having eigenvalues λ and λ^{-1}, where $\lambda \neq \pm 1$ and prove our main result Theorem 1.2.

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Proposition 6.1. *Let $\lambda \neq \pm 1$ and $A = A_1 \oplus A_2$ be an element of $\mathrm{SL}(2n, \mathbb{C})$ such that $A_1 \in \mathrm{GL}(n, \mathbb{C})$ and $A_2 \in \mathrm{GL}(n, \mathbb{C})$ have eigenvalues λ and λ^{-1}, respectively. If $g \in \mathrm{GL}(n, \mathbb{C})$ be an involution such that $gAg^{-1} = A^{-1}$, then $\det(g) = (-1)^n$.*

Proof. Up to conjugacy, we can assume that A_1 and A_2 are in Jordan form. Since $A = A_1 \oplus A_2$ is reversible, Theorem 1.1 implies that A_1 and A_2 have the same Jordan structure; see Definition 2.6. Let A_W denote the Weyr form of A such that $A = (A_1)_W \oplus (A_2)_W$. Then $A_W = \tau A \tau^{-1}$ for some $\tau \in \mathrm{SL}(2n, \mathbb{C})$. Moreover, the Weyr forms $(A_1)_W$ and $(A_2)_W$ have the same Weyr structure, say $(n_1, n_2, \ldots, n_r)$; see Definition 2.7. Then the (i, j)-th blocks of the Weyr forms $(A_1)_W$ and $(A_2)_W$ are given by

$$((A_1)_W)_{i,j} = \begin{cases} \lambda \mathrm{I}_{n_i} & \text{if } j = i \\ \mathrm{I}_{n_i, n_{i+1}} & \text{if } j = i+1 \\ \mathrm{O}_{n_i \times n_j} & \text{otherwise} \end{cases} \quad \text{and} \quad ((A_2)_W)_{i,j} = \begin{cases} \lambda^{-1} \mathrm{I}_{n_i} & \text{if } j = i \\ \mathrm{I}_{n_i, n_{i+1}} & \text{if } j = i+1 \\ \mathrm{O}_{n_i \times n_j} & \text{otherwise} \end{cases},$$

where $1 \leq i, j \leq r$. Moreover, the (i, j)-th block of the upper triangular block matrix $(A_1)_W^{-1}$ can be written as follows

$$((A_1)_W)_{i,j}^{-1} = \begin{cases} \lambda^{-1} \mathrm{I}_{n_i} & \text{if } j = i \\ (-1)^k \lambda^{-(k+1)} \mathrm{I}_{n_i, n_{i+k}} & \text{if } j = i+k, 1 \leq k \leq r-i \\ \mathrm{O}_{n_i \times n_j} & \text{otherwise} \end{cases}, \text{where } 1 \leq i, j \leq r.$$

Similarly, we can write

$$((A_2)_W)_{i,j}^{-1} = \begin{cases} \lambda \mathrm{I}_{n_i} & j = i \\ (-1)^k \lambda^{k+1} \mathrm{I}_{n_i, n_{i+k}} & j = i+k, 1 \leq k \leq r-i \\ \mathrm{O}_{n_i \times n_j} & \text{otherwise} \end{cases}, \text{where } 1 \leq i, j \leq r.$$

Consider $\Omega_W = \begin{pmatrix} & \Omega_1 \\ \Omega_2 & \end{pmatrix}$ in $\mathrm{GL}(2n, \mathbb{C})$ such that $\Omega_1 = [X_{i,j}]_{1 \leq i,j \leq r} \in \mathrm{GL}(n, \mathbb{C})$ and $\Omega_2 = [Y_{i,j}]_{1 \leq i,j \leq r} \in \mathrm{GL}(n, \mathbb{C})$ are defined as follows

$$X_{i,j} = \begin{cases} \mathrm{O}_{n_i \times n_j} & \text{if } j < i \\ \mathrm{O}_{n_i \times n_j} & \text{if } j = r, i \neq r \\ (-1)^{r-i} \, \lambda^{-2(r-i)} \, \mathrm{I}_{n_i} & \text{if } j = i \\ (-1)^{r-i} \binom{r-i-1}{j-i} \lambda^{-2r+i+j} \, \mathrm{I}_{n_i, n_j} & \text{if } i < j, j \neq r \end{cases}, \text{and}$$

$$Y_{i,j} = \begin{cases} \mathrm{O}_{n_i \times n_j} & \text{if } j < i \\ \mathrm{O}_{n_i \times n_j} & \text{if } j = r, i \neq r \\ (-1)^{r-i} \, \lambda^{2(r-i)} \, \mathrm{I}_{n_i} & \text{if } j = i \\ (-1)^{r-i} \binom{r-i-1}{j-i} \lambda^{2r-i-j} \, \mathrm{I}_{n_i, n_j} & \text{if } i < j, j \neq r \end{cases},$$

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where $\binom{r-i-1}{j-i}$ denotes the binomial coefficients. Then by using a similar argument as in Lemma 3.3, we have

$$\Omega_W A_W \Omega_W^{-1} = A_W^{-1}.$$

Let $f \in \mathrm{GL}(2n,\mathbb{C})$ satisfies $fA = Af$. Since $A = A_1 \oplus A_2$ such that A_1 and A_2 have no common eigenvalues, [15, Proposition 3.1.1] implies that $f = \begin{pmatrix} B_1 & \\ & B_2 \end{pmatrix}$, where $B_1, B_2 \in \mathrm{GL}(n,\mathbb{C})$ such that $B_1 A_1 = A_1 B_1$ and $B_2 A_2 = A_2 B_2$. Moreover, since A_1 and A_2 are basic Weyr matrices with the same Weyr structure, Proposition 2.10 implies that B_1 and B_2 are block upper triangular matrices with the same block structure $(n_1, n_2, \ldots, n_r)$.

Consider $h = \tau g \tau^{-1}$ in $\mathrm{GL}(2n,\mathbb{C})$, where $\tau \in \mathrm{SL}(2n,\mathbb{C})$ such that $A_W = \tau A \tau^{-1}$. Since g is an involution in $\mathrm{GL}(2n,\mathbb{C})$ such that $gAg^{-1} = A^{-1}$, it follows that h is an involution in $\mathrm{GL}(2n,\mathbb{C})$ such that

$$hA_W h^{-1} = A_W^{-1} \quad \text{and} \quad \det(h) = \det(g). \tag{6.1}$$

Since the set of reversers of A_W is a right coset of the centraliser of A_W, we have $h = f\,\Omega_W$. Therefore, $h = \begin{pmatrix} B_1 & \\ & B_2 \end{pmatrix} \begin{pmatrix} & \Omega_1 \\ \Omega_2 & \end{pmatrix} = \begin{pmatrix} & B_1\Omega_1 \\ B_2\Omega_2 & \end{pmatrix}$. This implies that

$$\det(h) = \det \begin{pmatrix} B_1\Omega_1 & \\ & B_2\Omega_2 \end{pmatrix} \det \begin{pmatrix} & \mathrm{I}_n \\ \mathrm{I}_n & \end{pmatrix} = (-1)^n \det \begin{pmatrix} B_1\Omega_1 & \\ & B_2\Omega_2 \end{pmatrix}.$$

Since h is an involution, we get $\det \begin{pmatrix} B_1\Omega_1 & \\ & B_2\Omega_2 \end{pmatrix} = \det(B_1\Omega_1)\det(B_2\Omega_2) = 1$. Therefore, $\det(h) = (-1)^n$. The proof now follows from Equation (6.1). □

The following result generalises Proposition 4.3(2).

Proposition 6.2. *Let $\lambda \neq \pm 1$ and $A = A_1 \oplus A_2$ be a reversible element of* $\mathrm{SL}(2n,\mathbb{C})$ *such that all eigenvalues of $A_1 \in \mathrm{GL}(n,\mathbb{C})$ and $A_2 \in \mathrm{GL}(n,\mathbb{C})$ are λ and λ^{-1}, respectively. Then A is strongly reversible if and only if n is even.*

Proof. If A is strongly reversible in $\mathrm{SL}(2n,\mathbb{C})$, then there exists an involution g in $\mathrm{SL}(n,\mathbb{C})$ such that $gAg^{-1} = A^{-1}$. Using Proposition 6.1, we have $\det g = (-1)^n = 1$. This implies that n is even.

Conversely, let n be even. Since A is reversible, using Theorem 1.1, we can partition the blocks in the Jordan form of A into pairs $\{\mathrm{J}(\lambda, r), \mathrm{J}(\lambda^{-1}, r)\}$. Since n is even, there are an even number of pairs $\{\mathrm{J}(\lambda, r), \mathrm{J}(\lambda^{-1}, r)\}$ with r is odd. Therefore, using Remark 4.2 and Proposition 4.3, we can construct a suitable involution g in $\mathrm{SL}(n,\mathbb{C})$ such that $gAg^{-1} = A^{-1}$. Therefore, A is strongly reversible in $\mathrm{SL}(n,\mathbb{C})$. This proves the result. □

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6.1. Proof of Theorem 1.2

Up to conjugacy, we can assume that A is in Jordan form. Since $A \in \mathrm{SL}(n, \mathbb{C})$ is reversible, Theorem 1.1 implies that the Jordan blocks in the Jordan form A can be partitioned into singletons $\{\mathrm{J}(\mu, k)\}$ or pairs $\{\mathrm{J}(\lambda, m), \mathrm{J}(\lambda^{-1}, m)\}$, where $\mu \in \{-1, +1\}$, $\lambda \notin \{-1, +1\}$. Let $\lambda_1, \lambda_2, \ldots, \lambda_\ell$ and their inverses $\lambda_1^{-1}, \lambda_2^{-1}, \ldots, \lambda_\ell^{-1}$ be distinct eigenvalues of A such that $\lambda_i \notin \{-1, +1\}$ and $\lambda_j \neq \lambda_{j'}$ or $\lambda_{j'}^{-1}$ for all $j \neq j'$, where $1 \leq i, j, j' \leq \ell$. Furthermore, suppose that both λ_i and λ_i^{-1} have multiplicity m_i for all $1 \leq i \leq \ell$. Therefore, up to conjugacy, we can assume that

$$A = P \oplus Q \oplus \begin{pmatrix} R_1 & \\ & R_1' \end{pmatrix} \oplus \begin{pmatrix} R_2 & \\ & R_2' \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} R_\ell & \\ & R_\ell' \end{pmatrix}, \tag{6.2}$$

such that $P \in \mathrm{GL}(p, \mathbb{C})$, $Q \in \mathrm{GL}(q, \mathbb{C})$, $R_i \in \mathrm{GL}(m_i, \mathbb{C})$ and $R_i' \in \mathrm{GL}(m_i, \mathbb{C})$ are Jordan matrices corresponding to eigenvalues $+1$, -1, λ_i and λ_i^{-1}, respectively, where $1 \leq i \leq \ell$. Then the Weyr form A_W corresponding to the Jordan form A can be given by

$$A_W = P_W \oplus Q_W \oplus \begin{pmatrix} (R_1)_W & \\ & (R_1')_W \end{pmatrix} \oplus \begin{pmatrix} (R_2)_W & \\ & (R_2')_W \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} (R_\ell)_W & \\ & (R_\ell')_W \end{pmatrix}, \tag{6.3}$$

such that $P_W, Q_W, (R_i)_W$ and $(R_i')_W$ are basic Weyr matrices corresponding to Jordan matrices P, Q, R_i and R_i', respectively, where $1 \leq i \leq \ell$. Since diagonal block matrices in Weyr form A_W do not have common eigenvalues, [15, Proposition 3.1.1] implies that any $f \in \mathrm{GL}(n, \mathbb{C})$ commuting with A_W has the following form

$$f = B \oplus C \oplus \begin{pmatrix} D_1 & \\ & D_1' \end{pmatrix} \oplus \begin{pmatrix} D_2 & \\ & D_2' \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} D_\ell & \\ & D_\ell' \end{pmatrix}, \tag{6.4}$$

where $B \in \mathrm{GL}(p, \mathbb{C}), C \in \mathrm{GL}(q, \mathbb{C}), D_i \in \mathrm{GL}(m_i, \mathbb{C})$, $D_i' \in \mathrm{M}(m_i, \mathbb{C})$ such that $BP_W = P_W B$, $CQ_W = Q_W C$, $D_i(R_i)_W = (R_i)_W D_i$, $D_i'(R_i')_W = (R_i')_W D_i'$, respectively, where $1 \leq i \leq \ell$.

Suppose that A is strongly reversible in $\mathrm{SL}(n, \mathbb{C})$. Since A_W is the Weyr form of A, there exists $\tau \in \mathrm{SL}(n, \mathbb{C})$ such that $A_W = \tau A \tau^{-1}$. Therefore, A_W is strongly reversible in $\mathrm{SL}(n, \mathbb{C})$. This implies that there exists an involution $g \in \mathrm{SL}(n, \mathbb{C})$ such that $g A_W g^{-1} = A_W^{-1}$. Note that using a similar argument as in Proposition 5.6, Proposition 5.8 and Proposition 6.2, we can find a suitable reverser of A_W given in Equation (6.3). Since the set of reversers is a right coset of the centraliser, it follows that g has the following form

$$g = \alpha \oplus \beta \oplus g_1 \oplus \cdots \oplus g_\ell, \tag{6.5}$$

where $\alpha \in \mathrm{GL}(p, \mathbb{C}), \beta \in \mathrm{GL}(q, \mathbb{C})$, $g_i \in \mathrm{GL}(2m_i, \mathbb{C})$ such that $\alpha P_W \alpha^{-1} = P_W^{-1}$, $\beta Q_W \beta^{-1} = Q_W^{-1}$, $g_i \begin{pmatrix} (R_i)_W & \\ & (R_i')_W \end{pmatrix} g_i^{-1} = \begin{pmatrix} (R_i)_W & \\ & (R_i')_W \end{pmatrix}^{-1}$, respectively, where $1 \leq i \leq t$.

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Let s denote the number of Jordan blocks of the form $\{\mathrm{J}(\mu,4k+2)\}$, $\mu\in\{-1,+1\}$, and t denote the number of pairs of the form $\{\mathrm{J}(\lambda,2m+1),\mathrm{J}(\lambda^{-1},2m+1)\}$, $\lambda\notin\{-1,+1\}$, in the Jordan form A. Note that if condition (1) of Theorem 1.2 holds or $s=t=0$, then using Remark 4.2 and Proposition 4.3, we can construct an involution $h\in\mathrm{SL}(n,\mathbb{C})$ such that $hAh^{-1}=A^{-1}$.

Now, assume that A does not satisfy condition (1) of Theorem 1.2 (i.e., both $\mathbb{O}_{\mathbf{d}(p)}$ and $\mathbb{O}_{\mathbf{d}(q)}$ are empty) and $s+t\neq 0$, otherwise we are done. Using Proposition 5.5, Proposition 5.7, Proposition 6.1 and Equation (6.5), we have

$$\det(g)=(-1)^{|\mathbb{E}^2_{\mathbf{d}(p)}|}(-1)^{|\mathbb{E}^2_{\mathbf{d}(q)}|}\prod_{i=1}^{\ell}(-1)^{m_i}=(-1)^{|\mathbb{E}^2_{\mathbf{d}(p)}|+|\mathbb{E}^2_{\mathbf{d}(q)}|+\sum_{i=1}^{\ell}m_i},$$

where $p+q+2\sum_{i=1}^{\ell}m_i=n$. Since $g\in\mathrm{SL}(n,\mathbb{C})$, $\det(g)=(-1)^{|\mathbb{E}^2_{\mathbf{d}(p)}|+|\mathbb{E}^2_{\mathbf{d}(q)}|+\sum_{i=1}^{\ell}m_i}=1$. This implies that

$$\left(|\mathbb{E}^2_{\mathbf{d}(p)}|+|\mathbb{E}^2_{\mathbf{d}(q)}|+\sum_{i=1}^{\ell}m_i\right)\text{ is even.}$$

Since $s=|\mathbb{E}^2_{\mathbf{d}(p)}|+|\mathbb{E}^2_{\mathbf{d}(q)}|$, we get

$$\left(s+\sum_{i=1}^{\ell}m_i\right)\text{ is even.}$$

This implies that $s+t$ is even, and thus A satisfies condition (2) of Theorem 1.2. Therefore, the forward direction of the theorem is proven.

Conversely, up to conjugacy, we can assume that A is in Jordan form as given by Equation (6.2). If A satisfies at least one of the conditions (1) and (2) of Theorem 1.2, then using Remark 4.2 and Proposition 4.3, we can construct a suitable involution g in $\mathrm{SL}(n,\mathbb{C})$ such that $gAg^{-1}=A^{-1}$. Therefore, A is strongly reversible in $\mathrm{SL}(n,\mathbb{C})$. This completes the proof.

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