COMPOSITIO MATHEMATICA

Positivity, plethysm and hyperbolicity of Siegel varieties in positive characteristic

Thibault Alexandre

Compositio Math. **161** (2025), 257–304.

doi: 10.1112/S0010437X24007607

FOUNDATION COMPOSITIO MATHEMATICA

LONDON MATHEMATICAL SOCIETY
EST. 1865

Compositio Math. 161 (2025) 257–304
doi:10.1112/S0010437X24007607

Positivity, plethysm and hyperbolicity of Siegel varieties in positive characteristic

Thibault Alexandre

Abstract

We study hyperbolicity properties of the moduli space of polarized abelian varieties (also known as the Siegel modular variety) in characteristic p. Our method uses the plethysm operation for Schur functors as a key ingredient and requires a new positivity notion for vector bundles in characteristic p called (φ, D)-ampleness. Generalizing what was known for the Hodge line bundle, we also show that many automorphic vector bundles on the Siegel modular variety are (φ, D)-ample.

1. Introduction

1.1 General picture

Let p be a prime number. This article is concerned with the interplay in characteristic p of the following three topics: positivity of vector bundles in algebraic geometry, the plethysm operation on Schur functors (and symmetric functions) and hyperbolicity properties of the moduli space of polarized abelian varieties (also known as the Siegel modular variety). It is well-known that positive vector bundles can be used to prove hyperbolicity results: a crucial new observation in our work is a link between the plethysm operation and hyperbolicity properties of Siegel varieties.

$$\begin{tikzcd}[column sep=small] \textsc{Positivity of vector bundles} \arrow[dr] & & \textsc{Plethysm operation} \arrow[dl, dashed] \\ & \textsc{Hyperbolicity of Siegel varieties} & \end{tikzcd}$$

1.2 History and motivation

1.2.1 *Hyperbolicity over a number field.* One of the most celebrated results in Diophantine geometry is the following.

Received 13 February 2023, accepted in final form 17 September 2024.
2020 Mathematics Subject Classification 14G35 (primary), 14G17, 14C99 (secondary).
Keywords: positivity of vector bundles; Siegel modular variety; hyperbolicity in characteristic p; plethysm operation; Schur functors; Green–Griffiths–Lang conjecture; representation theory of algebraic groups.

Theorem [Fal83]. *Consider a geometrically integral smooth projective curve C over a number field $K/\mathbb{Q}$. The following three assertions are equivalent.*

(1) *For all finite extension F/K, the set of F-rational points of C is finite.*
(2) *Every holomorphic map $\mathbb{C} \to C_{\mathbb{C}}^{\mathrm{an}}$ is constant.*
(3) *The canonical bundle ω_C is big, equivalently the genus g of C satisfies $g \geq 2$.*

A curve satisfying these assertions is called hyperbolic and generalizing hyperbolicity to higher-dimensional varieties is an open problem. One might be tempted to consider the following three definitions of hyperbolicity which are conjectured to be equivalent [Lan86, Conjecture 5.6/5.8].

Definition. Let X denote a projective variety over a number field $K/\mathbb{Q}$:

(1) X is arithmetically hyperbolic if for all finite extension F/K, the set of F-rational points of X is finite;
(2) X is Brody hyperbolic if every holomorphic map $\mathbb{C} \to X_{\mathbb{C}}^{\mathrm{an}}$ is constant;
(3) X is algebraically hyperbolic[1] if every integral subvariety V of $X_{\mathbb{C}}$ is of general type, i.e. there exists a desingularization $\tilde{V} \to V$ such that $\omega_{\tilde{V}}$ is big.

What can be said about the algebraic hyperbolicity of the moduli space $\mathcal{A}_{g,N} \to \operatorname{Spec} \mathbb{Q}$ of g-dimensional polarized abelian varieties with a full level N-structure? Since this moduli space is not proper, we should replace condition (3) with the condition that all subvarieties are of *log general type*. We recall that a variety V is of log general type if there exists a proper desingularization $V \to \tilde{V}$ and a smooth projective variety W together with an open embedding $\tilde{V} \subset W$ with $D := W \smallsetminus \tilde{V}$ a normal crossing divisor such that $\omega_W(D)$ is big. The moduli space $\mathcal{A}_{g,N}$ is known to be algebraically hyperbolic [Zuo00, Bru18].

1.2.2 *Algebraic hyperbolicity of $\mathcal{A}_{g,N}$ over a field of characteristic p.* Let us replace the base field K by k, an algebraically closed field of characteristic p. Since desingularization techniques do not always exist in characteristic p, we restrict ourselves to *smooth* subvarieties. Assume that p does not divide N and consider a smooth projective toroidal compactification $\mathcal{A}_{g,N}^{\mathrm{tor}}$ of the Siegel modular variety over k and write D for its boundary as a normal crossing divisor. In this context, we say that a smooth subvariety $\iota : V \hookrightarrow \mathcal{A}_{g,N}^{\mathrm{tor}}$ such that $\iota^{-1}D$ remains an effective Cartier divisor is said of log general type with respect to D if the log canonical bundle $\omega_V(\iota^{-1}D)$ is big.

Question (Characteristic p). Is $\mathcal{A}_{g,N}^{\mathrm{tor}}$ algebraically hyperbolic over k? In other words, is every subvariety

$$\iota : V \hookrightarrow \mathcal{A}_{g,N}^{\mathrm{tor}},$$

such that $\iota^{-1}D$ is well defined, of log general type with respect to D?

The answer to this question is, in fact, negative. In [Mor81], Moret-Bailly constructed a non-isotrivial family $A \to \mathbb{P}^1$ of principally polarized supersingular abelian surfaces with a full level N-structure over the projective line over $\mathbb{F}_p$. This family yields a closed immersion $\mathbb{P}^1 \to \mathcal{A}_{2,N}$ which contradicts the hyperbolicity of $\mathcal{A}_{2,N}^{\mathrm{tor}}$. The main objective of this article is to investigate the failure of hyperbolicity of Siegel varieties in positive characteristic.

[1] The terminology *algebraically hyperbolic* is also used by Demailly and many others such as Rousseau and Riedl for another notion of hyperbolicity.

https://doi.org/10.1112/S0010437X24007607

1.3 Our main result

From now on, the letter k will denote an algebraically closed field of characteristic p. To simplify our notation we will denote by Sh the Siegel variety of genus g over k (instead of $\mathcal{A}_{g,N}$) and D_{red} the boundary of a smooth projective toroidal compactification $\mathrm{Sh}^{\mathrm{tor}}$. Motivated by the Green–Griffiths–Lang conjecture (8.8), it is natural to expect that there is some exceptional locus $E \subset \mathrm{Sh}^{\mathrm{tor}}$ such that for any smooth subvariety V not contained in the boundary, V is of log general type if and only if $V \nsubseteq E$. Our main result about the hyperbolicity of the Siegel varieties is the following.

Theorem 1 (Corollary 8.7). *Assume that $p \geq g^2 + 3g + 1$. Any subvariety $\iota : V \hookrightarrow \mathrm{Sh}^{\mathrm{tor}}$ of codimension $\leq g-1$ satisfying the following:*

(1) *V is smooth;*
(2) *$\iota^{-1}D_{\mathrm{red}}$ is a normal crossing divisor;*

is of log general type with respect to D.

This indicates that the hypothetical exceptional locus $E \subset \mathrm{Sh}^{\mathrm{tor}}$ has a codimension strictly larger than $g-1$ and we believe it has exactly codimension g.

Remark.

(1) Theorem 1 is actually a corollary of Theorem 4 which is stated at the end of the introduction.
(2) For simplicity, we have restricted ourselves to smooth subvarieties but Theorem 1 should also hold for non-smooth subvarieties if we use the definition of the logarithmic Kodaira dimension of a variety in positive characteristic which appears in [Abr94, p. 46] and [Luo87, Luo88] and we adapt our arguments using, for example, [Abr94, Lemma 5, p. 46] in the proof of our Lemma 8.3.
(3) When $g = 1$ and $p = 2$ or 3, there exist families of non-isotrivial elliptic curves over the multiplicative group $\mathbb{G}_m$. Specifically, consider the families $y^2 = x^3 + x^2 - t$ in characteristic 3 and $y^2 + xy = x^3 + t$ in characteristic 2 where $t \in \mathbb{G}_m$. In both cases, the j-invariant is $j = 1/t$, so the curves are non-isotrivial. Since $\mathbb{G}_m$ is a smooth curve not of log general type, these examples show that the bound $p \geq g^2 + 3g + 1 = 5$ in Theorem 1 is sharp when $g = 1$. For $g > 1$, it is not known whether counterexamples exist when $p < g^2 + 3g + 1$.
(4) The codimension assumption in Theorem 1 indicates that the Siegel modular variety $\mathrm{Sh}^{\mathrm{tor}}$ exhibits an intermediate form of pseudo-hyperbolicity, as suggested by the intermediate Lang conjectures (see [Lan86]). These conjectures predict that varieties of general type should not contain 'large' subvarieties that are not of general type. Our result aligns with this expectation by showing that any smooth subvariety of codimension $\leq g-1$ intersecting the boundary normally is of log general type.

1.4 A new positivity notion for vector bundles in characteristic *p*

In order to prove Theorem 1, we introduce and study a positivity notion for vector bundles which is weaker than ampleness but stronger that nefness and bigness. Assume that X is a projective scheme over k and D is an effective Cartier divisor on X. Since this positivity notion involves the relative Frobenius map $\varphi : X \to X^{(p)}$, we have decided to call it (φ, D)-ampleness.

Definition. A vector bundle $\mathcal{E}$ over X is said to be (φ, D)-ample if there is an integer $r_0 \geq 1$ such that for all integers $r \geq r_0$, the vector bundle $\mathcal{E}^{(p^r)}(-D) := (\varphi^r)^*(\varphi^r)_* \mathcal{E} \otimes \mathcal{O}_X(-D)$ is ample.

Our main motivation comes from the fact that the Hodge line bundle $\omega := \det \Omega$ is not always ample on a toroidal compactification $\mathrm{Sh}^{\mathrm{tor}}$ but it is nef and big with exceptional locus[2] contained in the boundary D_{red}. In fact, we even know that ω is (φ, D)-ample for some effective Cartier divisor D whose associated reduced divisor is the boundary D_{red}. Compared with nefness and bigness for vector bundles, we show that (φ, D)-ampleness behaves well as it is stable under direct sum, extension,[3] quotient, tensor product, tensor roots, pullback by finite morphism and it satisfies descent along finite surjective morphism. Inspired by a result of Mourougane [Mou97, Théorème 1] over $\mathbb{C}$ about the ampleness of the adjoint bundle $\pi_*(\mathcal{L} \otimes \omega_{Y/X})$ where $\pi : Y \to X$ is a surjective morphism and $\mathcal{L}$ is an ample line bundle on Y, we prove similar results in characteristic p when π is a flag bundle.

More precisely, let G denote a connected split reductive algebraic group over k. Fix a Borel pair (B, T) of G and write ρ for the half-sum of positive roots of G. Let E be a G-torsor over X and $\pi : Y \to X$ the flag bundle that parametrizes B-reduction of E. Recall that we can associate a line bundle $\mathcal{L}_\lambda$ on Y to each character λ of T. We prove the following.

Theorem 2 (Theorems 6.4 and 6.5). *If $\mathcal{L}_{2\lambda+2\rho}$ is ample (respectively, $(\varphi, \pi^{-1}D)$-ample) on Y, then $\pi_* \mathcal{L}_\lambda$ is an ample (respectively, (φ, D)-ample) vector bundle on X.*

Note that since $\omega_{Y/X} = \mathcal{L}_{-2\rho}$, our result can be seen as a characteristic p version of the result of Mourougane.

1.5 Positivity of automorphic vector bundles on the Siegel variety

We explain a direct application of Theorem 2 to automorphic vector bundles defined over the Siegel variety. Recall that the Hodge bundle is a rank g vector bundle over the Siegel variety which is defined as $\Omega = e^*\Omega^1_{A/\mathrm{Sh}}$ where e is the neutral section of the universal abelian scheme $f : A \to \mathrm{Sh}$. Use $\pi : Y \to \mathrm{Sh}$ to denote the flag bundle which parametrizes complete filtration of Ω. Recall that for every character λ of the standard maximal torus of GL_g, we have an associated line bundle $\mathcal{L}_\lambda$ on Y and a costandard automorphic vector bundle $\nabla(\lambda)$ over the Siegel variety which is isomorphic to $\pi_* \mathcal{L}_\lambda$. All these objets can be extended to a toroidal compactification $\mathrm{Sh}^{\mathrm{tor}}$. Following the idea of [BGKS], we know by [Ale24, Theorem 5.11] that certain line bundles $\mathcal{L}_\lambda$ are (φ, D)-ample on Y where D is some fixed[4] effective Cartier divisor whose associated reduced divisor is the boundary D_{red}.

We denote by G[5] the symplectic group Sp_{2g} over k, W the Weyl group of G, $P \subset G$ the parabolic that stabilizes the Hodge filtration on the first de Rham cohomology of $A \to \mathrm{Sh}$, $\Delta \subset \Phi^+ \subset \Phi$ the set of (simple, positive) roots of G, $I \subset \Delta$ the type of P, L the Levi subgroup of P, $\Phi_L^+ \subset \Phi_L$ the set of (positive) roots of L and $\rho_L = 1/2 \sum_{\alpha \in \Phi_L^+} \alpha$. The following result is a direct application of Theorem 2.

Theorem 3 (Theorem 7.20). *Let λ be a dominant character of T. If $\gamma := 2\lambda + 2\rho_L$ is:*

[2]Following [Kee99], the exceptional locus of a nef line bundle $\mathcal{L}$ is the closure, with reduced structure, of the union of all subvarieties V such that $\mathcal{L}_{|V}$ is not big.

[3]Under a regular hypothesis on X.

[4]It depends on the choice of a polarization function on the polyhedral cone decomposition of a toroidal compactification.

[5]It is not the same G as in § 1.4.

(1) *orbitally p-close, i.e.*

$$\max_{\alpha\in\Phi, w\in W, \langle\gamma,\alpha^\vee\rangle\neq 0}\left|\frac{\langle\gamma, w\alpha^\vee\rangle}{\langle\gamma,\alpha^\vee\rangle}\right|\leq p-1;$$

(2) *$\mathcal{Z}_\emptyset$-ample, i.e.*

$$\langle\gamma,\alpha^\vee\rangle>0 \quad \textit{for all } \alpha\in I \textit{ and } \langle\gamma,\alpha^\vee\rangle<0 \quad \textit{for all } \alpha\in\Phi^+\backslash\Phi_L^+;$$

then the automorphic vector bundle $\nabla(\lambda)$ is (φ, D)-ample on $\mathrm{Sh}^{\mathrm{tor}}$.

Remark. Positivity results for automorphic vector bundles were only known for line bundles which corresponds to the case where λ is positive parallel, i.e. $\nabla(\lambda)$ is a positive power of the Hodge line bundle $\omega=\det\Omega=\nabla(-1,\dots,-1)$.

1.6 Schur functor and the plethysm operation

Schur functors are certain endofunctors

$$S:\mathrm{FinVect}_k\to\mathrm{FinVect}_k$$

of the abelian category of finite-dimensional k-vector spaces that generalize the constructions of exterior powers and symmetric powers of a vector space. Schur functors are indexed by integer partition or Young diagrams and they can be defined on the category of finite locally free modules over a scheme. We are interested in these functors because if $\lambda=(k_1\geq\cdots\geq k_g\geq 0)$ is a G-dominant character, we can identify it with a Young diagram where the ith-row has k_i columns and we get an isomorphism

$$S_\lambda\Omega=\nabla(-w_0\lambda),$$

where $w_0\in W$ is the longest element of the Coxeter group W. The strategy to prove Theorem 1 is to show that the bundle $S_\lambda\Omega^1_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}})$ is (φ, D)-ample for specific choices of λ. Since (φ, D)-ampleness is stable by quotients, pullback by finite morphisms and S_λ respects surjections, the (φ, D)-ampleness of $S_\lambda\Omega^1_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}})$ implies that the quotient

$$\iota^*S_\lambda\Omega^1_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}})\twoheadrightarrow S_\lambda\Omega^1_V(\log\iota^{-1}D_{\mathrm{red}})$$

is also $(\varphi,\iota^{-1}D)$-ample for any smooth subvariety $\iota:V\hookrightarrow\mathrm{Sh}^{\mathrm{tor}}$ such that $\iota^{-1}D_{\mathrm{red}}$ is well-defined as a normal crossing divisor. It follows from the general theory of Schur functors that the bundle $S_\lambda\Omega^1_V(\log\iota^{-1}D_{\mathrm{red}})$ is non-zero exactly when the dimension of V is larger than the number of parts (also called the height $\mathrm{ht}(\lambda)$) of λ. In this case, the determinant of $S_\lambda\Omega^1_V(\log\iota^{-1}D_{\mathrm{red}})$ is a tensor power of $\omega_V(\iota^{-1}D_{\mathrm{red}})$ and the (φ, D)-ampleness of $S_\lambda\Omega^1_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}})$ implies that V is of log general type with respect to D_{red}. By the Kodaira–Spencer isomorphism

$$\rho_{\mathrm{KS}}:\mathrm{Sym}^2\,\Omega\xrightarrow{\sim}\Omega^1_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}}),$$

we are reduced to studying the composition of Schur functors $S_\lambda\circ\mathrm{Sym}^2$.

The correct category to study Schur functors in characteristic p such as S_λ is the category of strictly polynomial functors Pol introduced by Friedlander and Suslin in [FS97]. A strictly polynomial functor $T:\mathrm{FinVect}_k\to\mathrm{FinVect}_k$ over a field k is polynomial in the sense that for any finite-dimensional k-vector spaces V, W, the map

$$T_{V,W}:\mathrm{Hom}_k(V,W)\to\mathrm{Hom}_k(T(V),T(W))$$

is a scheme morphism where we have enriched $\mathrm{Hom}_k(V,W)$ and $\mathrm{Hom}_k(T(V),T(W))$ with their natural scheme structure. Equipped with the classical tensor product $\otimes$, the category of strictly polynomial functors is a symmetric monoidal category whose Grothendieck group $K_0(\mathsf{Pol})$ is the ring $\mathcal{R}$ of symmetric functions. A key feature of Pol is that the functor composition $\circ$ defines a second (non-symmetric) monoidal structure on it. Recall that $\mathcal{R}$ possesses a natural basis $\{s_\lambda\}_\lambda$ indexed by the set of integer partition where each s_λ is the class of the Schur functor S_λ. Over $\mathbb{C}$, it is well-known that Pol is semi-simple; in particular, the composition of two Schur functors of partition λ and μ can be split as a direct sum of Schur functors

$$S_\lambda \circ S_\mu = \bigoplus_\eta S_\eta^{\oplus c^\eta_{\lambda,\mu}},$$

where the coefficient $c^\eta_{\lambda,\mu}$ are given by the decomposition of $s_\lambda \circ s_\mu$ in the basis $\{s_\lambda\}_\lambda$ of $\mathcal{R}$. The problem of determining the coefficients $c^\eta_{\lambda,\mu}$ is known as plethysm. Over a field of characteristic p, semi-simplicity of Pol fails but we may ask whether the composition $S_\lambda \circ S_\mu$ admits at least a filtration where the graded pieces are isomorphic to Schur functors S_η. Unfortunately, Boffi [Bof91] and Touzé [Tou13, Corollary 6.10.] have found counter-examples to the existence of such filtrations. For example, the plethysm $\Lambda^2 \circ \Lambda^2$ over $\mathbb{F}_2$ does not admit any such filtration. We avoid these counter-examples with a technical restriction on the prime p.

Proposition (Proposition 3.16). *Let λ and μ be partitions of size $|\lambda|$ and $|\mu|$. If $p \geq 2|\lambda| - 1$, the strict polynomial functor $S_\lambda \circ S_\mu$ admits a finite filtration*

$$0 = T^n \subsetneq T^{n-1} \subsetneq \cdots \subsetneq T^0 = S_\lambda \circ S_\mu$$

by strict polynomial functors of degree $|\lambda||\mu|$ where the graded pieces are Schur functors.

1.7 Plethysm and hyperbolicity

Since (φ, D)-ampleness is stable under extension, we can use Proposition 3.16 to see that $S_\lambda \Omega^1_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}})$ is (φ, D)-ample if the graded pieces $\nabla(\eta)$ that appears in the plethysm $S_\lambda \circ \mathrm{Sym}^2$ are (φ, D)-ample. It is worth pointing out that plethysm computations are really hard and there is no known general combinatorial rule to express the coefficient $c^\eta_{\lambda,\mu}$.[6] Moreover, determining effectively whether an automorphic bundle $\nabla(\eta)$ is (φ, D)-ample with Theorem 3 is also challenging as it involves the orbitally p-closeness condition. It is known since [Wil09, Lemma 7] that the plethysm $\Lambda^k \circ \mathrm{Sym}^2$ belongs to one of the few cases where a general formula is known. With this formula and an upper bound of the orbitally p-closeness condition, we were able to show the following result.

Theorem 4 (Theorem 8.6). *Assume that $p \geq g^2 + 3g + 1$. For all $k \geq g(g-1)/2 + 1$, the bundle $\Lambda^k \mathrm{Sym}^2 \Omega = \Omega^k_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}})$ is (φ, D)-ample.*

In the case $g \in \{2, 3\}$, we also prove that this bound on k is optimal, which is some evidence that the hypothetical exceptional locus $E \subset \mathrm{Sh}^{\mathrm{tor}}$ has codimension g.

1.8 Organization of the paper

In § 2, we recall some general results on algebraic representations of reductive groups in characteristic p. In § 3, we study the plethysm operation for Schur functors in characteristic p. In § 4, we introduce and study the main properties of (φ, D)-ample vector bundles. In particular, we prove many stability properties that are summarized in Table 1. In § 5, we recall the flag

[6] At the beginning, we used a computer to find integer partitions λ such that each automorphic bundle appearing in the plethysm $S_\lambda \circ \mathrm{Sym}^2$ is (φ, D)-ample (see Appendix B). Starting with $g = 2$, we have found that surfaces in the Siegel threefold are of log general type. For each $g \in \{2, 3, 4\}$, the computer was able to find a partition λ of height equal to $\dim \mathrm{Sh} - (g-1)$ such that $S_\lambda \circ \mathrm{Sym}^2 \Omega$ is (φ, D)-ample. For $g = 5$, the plethysm computation was too long to conclude and it became clear that we needed a different method.

https://doi.org/10.1112/S0010437X24007607

bundle construction associated to a general G-torsor. In § 6, we prove that the adjoint bundle of an ample (respectively, (φ, D)-ample) line bundle along a complete flag bundle, is an ample (respectively, (φ, D)-ample) vector bundle. In § 7, we apply our result on the positivity of adjoint bundles to the case of automorphic vector bundles over the Siegel variety. In particular, Figure 1 illustrate our ampleness result in the case $g = 2$. In § 8, we finish the proof of our main theorem about the partial hyperbolicity of the Siegel modular variety in characteristic p.

2. Representations of algebraic groups

Recall that k is an algebraically closed field of characteristic p. In this section, we recall some well-known results about algebraic representations of reductive groups over k that can be found in [Jan03]. Let G be a connected split reductive algebraic group over k. We choose a Borel pair (B, T) of G, i.e. a Borel subgroup $B \subset G$ together with a maximal torus $T \subset G$ defined over k. Denote by $(X^*, \Phi, X_*, \Phi^\vee)$ the root datum of G where X^* is the group of characters of T, X_* is the group of cocharacters of T, Φ is the set of roots of G, $\Phi^\vee$ is the set of coroots of G and

$$\langle \cdot, \cdot \rangle : X^* \times X_* \to \mathbb{Z}$$

is the perfect pairing between the characters and the cocharacters of T. To any root $\alpha \in \Phi$, there is an associated coroot $\alpha^\vee$ such that $\langle \alpha, \alpha^\vee \rangle = 2$. This choice of (B, T) determines a set of positive roots Φ^+ and a set of simple roots $\Delta \subset \Phi^+$. To simplify the statement of Proposition 7.15, we follow a non-standard convention for the positive roots by declaring $\alpha \in \Phi$ to be positive if the root group $U_{-\alpha}$ is contained in B. A character $\lambda \in X^*$ is said to be G-dominant (or simply dominant if there is no ambiguity on the group G) if $\langle \lambda, \alpha^\vee \rangle \geq 0$ for all $\alpha \in \Phi^+$. We denote by ρ the half-sum of the positive roots. We denote by W the Weyl group of G, $l : W \to \mathbb{N}$ its length function and w_0 its longest element. Consider a collection $I \subset \Delta$ of simple roots. We denote by Φ_I (respectively, Φ_I^+) the set of roots (respectively, positive roots) obtained as $\mathbb{Z}$-linear combination of roots in I. We denote by $W_I \subset W$ the subgroup generated by the reflections s_α where $\alpha \in I$ and ${}^I W \subset W$ the set of minimal length representatives of $W_I \backslash W$. We denote by $\varphi : G \to G^{(p)}$ the relative Frobenius morphism of G where $G^{(p)} = G \times_{k,\sigma} k$ is the pullback along the Frobenius map $\sigma : k \to k$ of k. Since any split reductive group is a base change of a split reductive group over $\mathbb{Z}$, the reductive group G is isomorphic to $G^{(p)}$. For any G-module M, we define $M^{(p^r)}$ as the same module M with a G-action twisted by φ^r.

Denote by $\mathrm{Rep}_k(G)$ the category of algebraic representations of G on finite-dimensional k-vector spaces. We use interchangeably the term G-module to denote any representation $V \in \mathrm{Rep}_k(G)$. It is well-known that this category is not semi-simple but we can still define some interesting highest weight representations.

Proposition 2.1 [Jan03, Part II, § 2.4]. *For any dominant T-character λ, there is a unique simple G-module of highest weight λ that we denote by $L(\lambda)$.*

Definition 2.2 [Jan03, Part I, § 5.8]. For any character λ of T, we denote by $\mathcal{L}_\lambda$ the line bundle on the flag variety G/B defined as the B-quotient of the vector bundle $G \times_k \mathbb{A}^1 \to G$ where B acts on $G \times_k \mathbb{A}^1$ by

$$(g, x)b = (gb^{-1}, \lambda(b^{-1})x),$$

and where λ is extended by zero on the unipotent radical of B.

Recall Kempf's vanishing theorem.

Proposition 2.3 [Jan03, Part II, § 4.5]. *Let λ be a dominant character. We have*

$$H^i(G/B, \mathcal{L}_\lambda) = 0$$

for every integer $i > 0$.

Definition 2.4 [Jan03, Part II, § 2]. Let λ be a character of T. The costandard G-module $\nabla(\lambda)$ of highest weight λ is defined as the global section group $H^0(G/B, \mathcal{L}_\lambda)$ where G acts through left translation. The standard G-module $\Delta(\lambda)$ of highest weight λ is defined as $\nabla(-w_0\lambda)^\vee$ where $\vee$ denotes the linear dual in $\mathrm{Rep}_k(G)$.

Proposition 2.5 [Jan03, Part II, § 2.6]. *The G-modules $\nabla(\lambda)$ and $\Delta(\lambda)$ are non-zero exactly when λ is dominant. Moreover, their highest T-weight is λ.*

It follows directly from their definition that $\nabla(\lambda)$ and $\Delta(\lambda)$ have the same weights but they are usually not simple and not isomorphic. We give a condition on the size of the highest weight of a standard/costandard module to be simple.

Proposition 2.6 [Jan03, Part II, § 5.6]. *If λ is a p-small character, i.e.*

$$\forall \alpha \in \Phi^+, \quad \langle \lambda + \rho, \alpha^\vee \rangle \leq p,$$

then we have isomorphisms

$$\nabla(\lambda) = \Delta(\lambda) = L(\lambda).$$

Remark 2.7. If λ is p-small and $\mu \leq \lambda$, then μ is also p-small.

In positive characteristic, there is a very special algebraic representation called the Steinberg representation St_r. The Steinberg representation is a self-dual simple G-module whose highest weight is never p-small.

Definition 2.8 [Jan03, Part II, § 3.18]. Assume $p \neq 2$ or $\rho \in X^*(T)$. For each $r \geq 1$, we define the Steinberg module as

$$\mathrm{St}_r := \nabla((p^r - 1)\rho).$$

Proposition 2.9 [Jan03, Part II, § 3.19]. *We have isomorphisms*

$$\nabla((p^r - 1)\rho) = \Delta((p^r - 1)\rho) = L((p^r - 1)\rho).$$

In particular, St_r is a simple G-module.

We come to the main proposition that justifies our interest in the Steinberg representation.

Proposition 2.10 [Jan03, Part 2, Chapter 3, § 19]. *Let λ be a character and $r \geq 1$ an integer. For all $i \geq 0$, we have an isomorphism of G-modules*

$$H^i(G/B, \mathcal{L}_{(p^r-1)\rho} \otimes \mathcal{L}_{p^r\lambda}) = \mathrm{St}_r \otimes H^i(G/B, \mathcal{L}_\lambda)^{(p^r)}.$$

We recall the notion of ∇-filtration and Δ-filtration.

Definition 2.11. Let V be a G-module. A ∇-filtration is a filtration of V where each graded piece is a costandard module. A Δ-filtration is a filtration of V where each graded piece is a standard module.

Remark 2.12. The category $\mathrm{Rep}_k(G)$ has the structure of a highest weight category.[7] Within this framework, tilting modules are defined as modules that admits both a ∇- and a Δ-filtration.

[7] An introduction to this framework is given in [Ric16].

The following proposition, due to Mathieu, states the existence of a ∇-filtration for the tensor product $\nabla(\lambda) \otimes \nabla(\mu)$ of costandard modules.

Proposition 2.13 [Mat90, Theorem 1]. *Consider two dominant characters λ, μ in $X^*(T)$. Then the tensor product $\nabla(\lambda) \otimes \nabla(\mu)$ admits a ∇-filtration $(V^i)_{i \geq 0}$ with graded pieces*

$$V^i/V^{i+1} \simeq \nabla(\lambda + \mu_i),$$

where $(\mu_i)_i$ is a certain subcollection of weights of $\nabla(\mu)$.

Remark 2.14.

(1) Not all the weights $\mu' \leq \mu$ of $\nabla(\mu)$ such that $\lambda + \mu'$ is dominant will appear in the ∇-filtration.

(2) By duality, we deduce that tensor products of standard modules $\Delta(\lambda) \otimes \Delta(\mu)$ admit a Δ-filtration.

Corollary 2.15. *Consider two G-modules V and W. If V and W admit a ∇-filtration, then $V \otimes W$ admits also a ∇-filtration.*

The following cohomological criterion is very useful to detect when a G-module possesses a ∇-filtration.

Proposition 2.16 (Donkyn criterion). *Consider a G-module V. The following assertions are equivalent:*

(1) *V admits a ∇-filtration;*

(2) *for any dominant character λ and $i > 0$, $\mathrm{Ext}^i_G(\Delta(\lambda), V) = 0$;*

(3) *for any dominant character λ, $\mathrm{Ext}^1_G(\Delta(\lambda), V) = 0$.*

Proof. See [Jan03, Part II, § 4.16]. □

Corollary 2.17. *Consider two G-modules V and W. If V admits a ∇-filtration and W is a direct factor of V, then W admits a ∇-filtration.*

3. Plethysm for Schur functors in positive characteristic

3.1 Over the complex numbers

Classically, Schur functors are certain endofunctors

$$S : \mathrm{FinVect}_{\mathbb{C}} \to \mathrm{FinVect}_{\mathbb{C}}$$

of the abelian category of finite-dimensional complex vector spaces. The first example is given by the nth-symmetric power Sym^n which sends a vector space V to the space of $\mathfrak{S}_n$-coinvariants $(V^{\otimes n})_{\mathfrak{S}_n}$ where $\mathfrak{S}_n$ acts on $V^{\otimes n}$ by permuting the factors. A second example is given by the nth-exterior power Λ^n which sends a vector space V to the space of $\mathfrak{S}_n$-coinvariants $(V^{\otimes n})_{\mathfrak{S}_n}$ where an element $\sigma \in \mathfrak{S}_n$ acts on $V^{\otimes n}$ by antisymmetrization

$$\sigma(v_1 \otimes \cdots \otimes v_n) = \varepsilon(\sigma)(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}).^{8}$$

[8] This definition is not correct over a field of characteristic p if $p \geq n$. We should instead consider a quotient by the ideal generated by tensors of the form $x_1 \otimes \cdots \otimes x_n$ where $x_i = x_j$ for some $i \neq j$.

https://doi.org/10.1112/S0010437X24007607

In general, we consider a finite-dimensional representation π of the symmetric group $\mathfrak{S}_n$ for some integer $n \geq 1$ and we define the Schur functor

$$S_\pi : \mathrm{FinVect}_{\mathbb{C}} \to \mathrm{FinVect}_{\mathbb{C}}$$

associated to π as

$$S_\pi(V) = (V^{\otimes n} \otimes \pi)_{\mathfrak{S}_n},$$

where $\mathfrak{S}_n$ acts via permutation on the first factor. It is well-known that irreducible representations π of $\mathfrak{S}_n$ are in bijection with partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 0)$ of n. This bijection is made explicit by sending a partition λ of n to the Specht module Sp_λ of shape λ. We call $S_\lambda = S_{\mathrm{Sp}_\lambda}$, the Schur functor of weight λ. It is well-known that for any two partitions λ and μ, we have a direct sum decomposition

$$S_\lambda \circ S_\mu \simeq \bigoplus_\eta S_\eta^{\oplus c^\eta_{\lambda,\mu}} \tag{1}$$

in the category of endofunctors of $\mathrm{FinVect}_{\mathbb{C}}$. The problem of determining the coefficients $c^\eta_{\lambda,\mu}$ is called plethysm.[9]

Example 3.1. There is no known combinatorial rule for computing the coefficients $c^\eta_{\lambda,\mu}$. To illustrate how difficult the plethysm problem is, we give the following examples:

(1) $S_{(2,1)} \circ S_{(1,1)} = S_{(2,1,1,1,1)} \oplus S_{(2,2,1,1)} \oplus S_{(3,2,1)}$;

(2) the composition $S_{(4,2)} \circ S_{(3,1)}$ involves $1,238$ different partitions η with a maximum multiplicity $c^\eta_{\lambda,\mu}$ of 8408; counted with multiplicity, there are 958,705 endofunctors in the direct sum;

(3) the composition $S_{(3,2,1)} \circ S_{(4,2)}$ involves 11,938 different partitions η with a maximum multiplicity $c^\eta_{\lambda,\mu}$ of 9,496,674; counted with multiplicity, there are 4,966,079,903 endofunctors in the direct sum.

3.2 Strict polynomial functors

In their founder article [FS97], Friedlander and Suslin introduced the category of strict polynomial functors Pol over k. This functor category is well-behaved compared with the category of endofunctors of $\mathrm{FinVect}_k$. In particular, when $n \geq d$, they prove an equivalence of categories

$$\mathsf{Pol}_d \simeq S(n,d) - \mathrm{Mod}$$

between the category Pol_d of strict polynomial functors homogeneous of degree d and the category of modules over the Schur algebra $S(n,d)$.[10] If V, W are finite-dimensional vector spaces over k, we denote by $\mathrm{Hom}_{\mathrm{pol}}(V,W)$ the abelian group of scheme morphisms over k between $\underline{V}$ and $\underline{W}$. To clarify, we have $\mathrm{Hom}_{\mathrm{pol}}(V,W) = \mathrm{Sym}^*(V^\vee) \otimes_k W$ and elements of $\mathrm{Hom}_{\mathrm{pol}}(V,W)$ are called polynomial maps between V and W.

[9] The term 'plethysm' was suggested to Littlewood by M. L. Clark after the Greek word plethysmos, or $\pi\lambda\eta\theta\upsilon\sigma\mu o\varsigma$, which means 'multiplication' in modern Greek (though apparently the meaning goes back to ancient Greek). The related term plethys in Greek means 'a big number' or 'a throng', and this, in turn, comes from the Greek verb plethein, which means 'to be full', 'to increase', 'to fill', etc.

[10] Let $A_n = k[\mathrm{Mat}_n]$ denote the Hopf algebra freely generated by the k-vector space Mat_n of $n \times n$ matrices where the non-commutative comultiplication is induced by the matrix multiplication on Mat_n. Let $A(n,d) \subset A_n$ denote the subalgebra of homogeneous polynomials of degree d. The Schur algebra $S(n,d)$ is then defined as the linear dual of $A(n,d)$.

Definition 3.2 [FS97, Definition 2.1]. A strict polynomial functor

$$T : \mathrm{FinVect}_k \to \mathrm{FinVect}_k$$

is a pair of functions, the first of which assigns to each $V \in \mathrm{FinVect}_k$ a vector space $T(V) \in \mathrm{FinVect}_k$ and the second assigns a polynomial map

$$T_{V,W} \in \mathrm{Hom}_{\mathrm{pol}}(\mathrm{Hom}_k(V,W), \mathrm{Hom}_k(T(V), T(W)))$$

to each V, W. These two functions should satisfy the usual conditions of the definition of a functor:

(1) for any vector space $V \in \mathrm{FinVect}_k$, we have $T_{V,V}(\mathrm{id}_V) = \mathrm{id}_{T(V)}$;
(2) for any U, V, W, the following diagram of polynomial maps commutes.

$$\begin{tikzcd}
\mathrm{Hom}_k(V,W) \times \mathrm{Hom}_k(U,V) \arrow[r] \arrow[d, "T_{V,W} \times T_{U,V}"] & \mathrm{Hom}_k(U,W) \arrow[d, "T_{U,W}"] \\
\mathrm{Hom}_k(T(V),T(W)) \times \mathrm{Hom}_k(T(U),T(V)) \arrow[r] & \mathrm{Hom}_k(T(U),T(W))
\end{tikzcd}$$

Let $T : \mathrm{FinVect}_k \to \mathrm{FinVect}_k$ be a strict polynomial functor. We say that T is homogeneous of degree d if for all vector spaces V, W, the polynomial map

$$T_{V,W} \in \mathrm{Hom}_{\mathrm{pol}}(\mathrm{Hom}_k(V,W), \mathrm{Hom}_k(T(V), T(W)))$$

has degree d. We denote by Pol the category of strict polynomial functors of finite degree where the morphism are morphism between the underlying functors.

Proposition 3.3 [FS97, Proposition 2.6]. *The category of strict polynomial functors (of finite degree) decomposes*

$$\mathsf{Pol} = \bigoplus_{d \geq 0} \mathsf{Pol}_d,$$

where Pol_d is the full subcategory of Pol consisting of strict polynomial functors homogeneous of degree d. In particular, there are no extension between two strict polynomial functors homogeneous of different degrees.

Example 3.4. We give some simple example of strict polynomial functors.

(1) The nth-tensor power $(\cdot)^{\otimes n}$ which sends a k-vector space V to $V^{\otimes n}$ is homogeneous of degree n.
(2) The nth-symmetric power Sym^n which sends a k-vector space V to the space of $\mathfrak{S}_n$-coinvariants $(V^{\otimes n})_{\mathfrak{S}_n}$ where $\mathfrak{S}_n$ acts on $V^{\otimes n}$ by permuting the factors is homogeneous of degree n.
(3) The nth-divided power Γ^n which sends a k-vector space V to the space of $\mathfrak{S}_n$-invariants $(V^{\otimes n})^{\mathfrak{S}_n}$ where $\mathfrak{S}_n$ acts on $V^{\otimes n}$ by permuting the factors is homogeneous of degree n.
(4) The nth-exterior power Λ^n which sends a vector space V to the quotient space $V^{\otimes n}/I$ where I is the ideal generated by elements $x_1 \otimes \cdots \otimes x_n$ such that $x_i = x_j$ for some $i \neq j$ is homogeneous of degree n.
(5) If $\mathrm{char}(k) = p$, the Frobenius twist $(\cdot)^{(p)}$ which sends a k-vector space V to its pullback $V \otimes_{k,\sigma} k$ by the Frobenius map $\sigma : k \to k$ is homogeneous of degree p.

Remark 3.5. The functors Sym^n and Γ^n are isomorphic over a field of characteristic 0 but not over a field of characteristic $p > 0$ when $n \geq p$.

Following [ABW82], we now define Schur functors and Weyl functors that are indexed by partitions λ as strict polynomial functors.

Definition 3.6. Given a partition $\lambda = (k_1 \geq k_2 \geq \cdots \geq k_r > 0)$, we write $|\lambda| = \sum_{i=1}^{r} k_r$ for its size of λ and $\mathrm{ht}(\lambda) = r$ for its height.

Definition 3.7. We represent a partition $\lambda = (k_1 \geq k_2 \geq \cdots \geq k_r > 0)$ with a diagram containing r rows and such that for each i, the ith row contains k_i columns. Such a representation is called a Young diagram.

Example 3.8. The Young diagram of the partition $\lambda = (4, 2, 1)$ is as follows.

$$\begin{array}{|c|c|c|c|}
\hline
 & & & \\
\hline
 & \\
\cline{1-2}
 \\
\cline{1-1}
\end{array}$$

Its size is 7 and its height is 3.

Definition 3.9. Given a partition $\lambda = (k_1 \geq k_2 \geq \cdots \geq k_r > 0)$, we define its conjugate partition $\lambda' = (k'_1 \geq k'_2 \geq \cdots \geq k'_s > 0)$ as the partition where k'_i is the number of terms of k_j that are greater or equal to i. Note that λ and λ' have the same size. Any integer l between 1 and $|\lambda|$ determines a unique position (i, j) in the Young diagram of λ such that $l = k_1 + \cdots k_{i-1} + j$. Then, we define a permutation on $|\lambda|$-letters $\sigma_\lambda \in \mathfrak{S}_{|\lambda|}$ by setting

$$\sigma_\lambda(l) = k'_1 + \cdots k'_{i-1} + j.$$

Note that we have $\sigma_{\lambda'} = \sigma_\lambda^{-1}$.

Example 3.10. The conjugate partition of $\lambda = (8, 4, 2)$

$$\lambda = \begin{array}{|c|c|c|c|c|c|c|c|}
\hline
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\hline
9 & 10 & 11 & 12 \\
\cline{1-4}
13 & 14 \\
\cline{1-2}
\end{array}$$

is $\lambda' = (3, 3, 2, 2, 1, 1, 1, 1)$,

$$\lambda' = \begin{array}{|c|c|c|}
\hline
1 & 2 & 3 \\
\hline
4 & 5 & 6 \\
\hline
7 & 8 \\
\cline{1-2}
9 & 10 \\
\cline{1-2}
11 \\
\cline{1-1}
12 \\
\cline{1-1}
13 \\
\cline{1-1}
14 \\
\cline{1-1}
\end{array}$$

and we have

$$\sigma_\lambda = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ 1 & 4 & 7 & 9 & 11 & 12 & 13 & 14 & 2 & 5 & 8 & 10 & 3 & 6 \end{pmatrix}.$$

Definition 3.11. Let $\lambda = (k_1 \geq k_2 \geq \cdots \geq k_r > 0)$ denote a partition, $\lambda' = (k'_1 \geq k'_2 \geq \cdots \geq k'_s > 0)$ its conjugate partition and V a finite-dimensional vector space over k. We define $S_\lambda V$ as the image of the map

$$\bigotimes_{1\leq j\leq s} \Lambda^{k'_j} V \xrightarrow{\Delta^{\otimes s}} V^{\otimes|\lambda|} \xrightarrow{\sigma_\lambda} V^{\otimes|\lambda|} \xrightarrow{\nabla^{\otimes r}} \bigotimes_{1\leq i\leq r} \mathrm{Sym}^{k_i} V$$

where $\Delta : \Lambda^l V \to V^{\otimes l}$ is the comultiplication given by

$$\Delta(v_1 \wedge \cdots \wedge v_l) = \sum_{\sigma\in\mathfrak{S}_l} \epsilon(\sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(l)},$$

$\nabla : V^{\otimes l} \to \mathrm{Sym}^l V$ is the multiplication given by

$$\nabla(v_1 \otimes \cdots \otimes v_l) = v_1 \cdots v_l,$$

and $\sigma_\lambda : V^{\otimes|\lambda|} \to V^{\otimes|\lambda|}$ is given by

$$\sigma_\lambda(v_1 \otimes \cdots v_{|\lambda|}) = v_{\sigma_\lambda(1)} \otimes \cdots \otimes v_{\sigma_\lambda(|\lambda|)}.$$

We define $W_\lambda V$ as the image of the map

$$\bigotimes_{1\leq i\leq r} \Gamma^{k_i} V \xrightarrow{\Delta^{\otimes r}} V^{\otimes|\lambda|} \xrightarrow{\sigma_{\lambda'}} V^{\otimes|\lambda|} \xrightarrow{\nabla^{\otimes s}} \bigotimes_{1\leq j\leq s} \Lambda^{k'_j} V$$

where $\nabla : V^{\otimes l} \to \Lambda^l V$ is the canonical quotient map and $\Delta : \Gamma^l V \to V^{\otimes l}$ is the canonical inclusion. Note that we consider both the exterior and the symmetric algebras as Hopf algebras. These construction are functorial in V and define strict polynomial functors S_λ and W_λ that are homogeneous of degree $|\lambda|$.

Example 3.12. We give the following examples.

(1) If $\lambda = (n)$, then $S_\lambda = \mathrm{Sym}^n$ and $W_\lambda = \Gamma^n$.
(2) If $\lambda = (1, \ldots, 1)$ is a partition of n, then $S_\lambda = W_\lambda = \Lambda^n$.

Proposition 3.13. *Let λ be a partition of d and $V \in \mathrm{FinVect}_k$. We have an isomorphism*

$$S_\lambda(V)^\vee = W_\lambda(V^\vee),$$

which is functorial in V.

Proof. Follows from the fact that $\mathrm{Sym}^n(V)^\vee = \Gamma^n(V^\vee)$ and $\Lambda^n(V)^\vee = \Lambda^n(V^\vee)$. □

We state the main result of this section.

Proposition 3.14 [FS97, Lemma 3.4]. *Let $d \geq 1$ be an integer and V a vector space of dimension n. If $n \geq d$, the evaluation functor at V*

$$\mathrm{ev}_V : \mathsf{Pol} \longrightarrow \mathrm{Rep}_k(GL(V))$$

$$T \longmapsto T(V)$$

restricts to an equivalence of category between Pol_d and the category $\mathrm{Rep}_k(GL(V))_d^{\mathsf{Pol}}$ of polynomial[11] *representations of $\mathrm{GL}(V)$ where $\mathbb{G}_m$ acts by $z \mapsto z^d$. Moreover, through this equivalence, the Schur functor S_λ maps to the costandard module $\nabla(\lambda)$ and the Weyl functor W_λ maps to the standard module $\Delta(\lambda)$ where we see $\lambda = (k_1, \ldots, k_r)$ as a character of the standard maximal*

[11] Recall that a rational representation M of $GL(V)$ is polynomial if its action of the algebraic group $GL(V)$ extends to an action of the algebraic monoid $\mathrm{End}(V)$.

torus of $\mathrm{GL}(V)$

$$\lambda : \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \mapsto t_1^{k_1} \cdots t_r^{k_r}.$$

Remark 3.15. All the results of this section are valid if we replace $\mathrm{FinVect}_k$ with the category $\mathrm{Loc}(X)$ of locally free sheaves of finite rank over a k-scheme X.

3.3 A plethysm in positive characteristic under additional hypothesis

Take two Schur functors S_λ and S_μ as strict polynomial functors over k and consider the composition $S_\lambda \circ S_\mu$. It is a strict polynomial functor homogeneous of degree $|\lambda|\,|\mu|$. Since the category of algebraic representation of GL_n is not semi-simple, we have no reason to hope for a decomposition of $S_\lambda \circ S_\mu$ as a direct sum of Schur functors. One might hope, that there exists at least a filtration of $S_\lambda \circ S_\mu$ where the graded pieces are Schur functors. Unfortunately, Boffi [Bof91] and Touzé [Tou13, Corollary 6.10.] have found counter-examples to the existence of such filtrations for plethysm of the form $\mathrm{Sym}^k \circ \mathrm{Sym}^d$, $\Lambda^k \circ \mathrm{Sym}^d$ and $\mathrm{Sym}^k \circ \Lambda^d$ with $d \geq 3$ and $p|k$. More precisely, Touzé has found an obstruction to the existence of such filtration that lives in the p-torsion of the homology of the Eilenberg–Mac Lane space $K(\mathbb{Z}, d)$. In this section, we prove the following existence result.

Proposition 3.16. *Let λ and μ be partitions. If $p \geq 2|\lambda| - 1$, the strict polynomial functor $S_\lambda \circ S_\mu$ admits a finite filtration*

$$0 = T^n \subsetneq T^{n-1} \subsetneq \cdots \subsetneq T^0 = S_\lambda \circ S_\mu$$

by strict polynomial functors of degree $|\lambda||\mu|$ where the graded pieces are Schur functors.

We start with the following lemma.

Lemma 3.17. *If $p \geq 2|\lambda| - 1$, then S_λ is a direct summand of $(\cdot)^{\otimes|\lambda|}$ in $\mathsf{Pol}_{|\lambda|}$.*

Remark 3.18. *If $S_\lambda = \Lambda^n$, then it is enough to ask that $p > n$.*

Proof. Write $\lambda = (k_1 \geq \cdots \geq k_r > 0)$. By Proposition 3.14, it is enough to prove that $S_\lambda V$ is a direct summand of $V^{\otimes|\lambda|}$ in the category of $\mathrm{GL}(V)$-modules for one k-vector space of dimension greater than $|\lambda|$. Consider a vector space V of dimension n. Note that the surjection

$$\nabla^{\otimes r} : V^{\otimes|\lambda|} \to \mathrm{Sym}^\lambda V := \bigotimes_{1 \leq i \leq r} \mathrm{Sym}^{k_i} V$$

admits a section when $p > \max_i k_i = k_1$. Indeed, we define it as $s = s_1 \otimes \cdots \otimes s_r$ where

$$s_i(v_1 v_2 \cdots v_r) = \frac{1}{k_i!} \sum_{\sigma \in \mathfrak{S}_{k_i}} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k_i)}.$$

By definition, $S_\lambda V$ is a sub-$\mathrm{GL}(V)$-module of $\mathrm{Sym}^\lambda V$ and we would like to find a condition on p that guarantees it is also a direct summand. The following exact sequence of $\mathrm{GL}(V)$-modules

$$0 \longrightarrow S_\lambda V \longrightarrow \mathrm{Sym}^\lambda V \longrightarrow \mathrm{Sym}^\lambda V / S_\lambda V \longrightarrow 0$$

is split if we can show that $\mathrm{Ext}^1_{\mathrm{GL}(V)}(\mathrm{Sym}^\lambda V/S_\lambda V, S_\lambda V)$ vanishes. Let $\tilde{\lambda}$ denote the character $(|\lambda|, 0, \ldots, 0)$ of the standard maximal torus of $\mathrm{GL}(V)$. Since $\mathrm{Sym}^\lambda V$ is of highest weight $\tilde{\lambda}$, the $\mathrm{GL}(V)$-modules $S_\lambda V$ and $\mathrm{Sym}^\lambda V/S_\lambda V$ are filtered by simple modules $L(\nu)$ with $\nu \leq \tilde{\lambda}$. Under the assumption that $\tilde{\lambda}$ is p-small, Proposition 2.6 implies that we have isomorphisms

$$L(\nu) = \nabla(\nu) = \Delta(\nu)$$

for all characters ν satisfying $\nu \leq \tilde{\lambda}$. Since $\mathrm{Ext}^1_{\mathrm{GL}(V)}(\mathrm{Sym}^\lambda V/S_\lambda V, S_\lambda V)$ is the limit of a spectral sequence involving Ext groups

$$\mathrm{Ext}^1_{\mathrm{GL}(V)}(L(\nu), L(\nu')) = \mathrm{Ext}^1_{\mathrm{GL}(V)}(\Delta(\nu), \nabla(\nu')),$$

that vanishes by Proposition 2.16, $\mathrm{Ext}^1_{\mathrm{GL}(V)}(\mathrm{Sym}^\lambda V/S_\lambda V, S_\lambda V)$ must vanish. In conclusion, we have the desired splitting, provided that $\tilde{\lambda}$ is p-small, i.e.

$$\begin{aligned} p \geq \max_{\alpha \in \Phi^+} \langle \tilde{\lambda} + \rho, \alpha^\vee \rangle &= \max_{1 \leq i < j \leq n} \langle \tilde{\lambda} + \rho, \varepsilon_i - \varepsilon_j \rangle \\ &= \max_{1 \leq i < j \leq n} \left\langle \left(|\lambda| + \frac{n-1}{2}, \frac{n-3}{2}, \ldots, -\frac{n-1}{2} \right), \varepsilon_i - \varepsilon_j \right\rangle \\ &= |\lambda| + n - 1. \end{aligned}$$

Since our argument is valid only when $n \geq |\lambda|$, we get the bound $p \geq 2|\lambda| - 1$. □

Proof of Proposition 3.16. By Proposition 3.14, it is enough to show that $S_\lambda \circ S_\mu(V)$ admits a ∇-filtration as a $\mathrm{GL}(V)$-module where V is one vector space of dimension greater that $|\lambda|\,|\mu|$. Consider a vector space V of dimension $n \geq |\lambda|\,|\mu|$. By Lemma 3.17, $S_\lambda(S_\mu V)$ is a direct summand of $(S_\mu V)^{\otimes|\lambda|}$ as $\mathrm{GL}(S_\mu V)$-modules. After restriction to the category of $\mathrm{GL}(V)$-modules through the map $\mathrm{GL}(V) \to \mathrm{GL}(S_\mu V)$ induced by S_μ, $S_\lambda(S_\mu V)$ is again a direct summand of $(S_\mu V)^{\otimes|\lambda|}$. By Corollary 2.15, the $\mathrm{GL}(V)$-module $(S_\mu V)^{\otimes|\lambda|}$ admits a ∇-filtration. By Corollary 2.17, the $\mathrm{GL}(V)$-module $S_\lambda \circ S_\mu(V)$ admits a ∇-filtration. □

Remark 3.19. Under the assumption of Proposition 3.16, the partitions (counted with multiplicity) of the Schur functors appearing in the graded pieces of the filtration of $S_\lambda \circ S_\mu$ are the same as that appearing in the decomposition (1) over the complex numbers. This is just a consequence of the $\mathbb{Z}$-linearity of the $\nabla(\lambda)$ in the space $X^*(T)^W$ of W-invariants characters, but we reprove it directly. First note that the weights of the $\mathrm{GL}(V)$-module $S_\lambda \circ S_\mu(V)$ where V is a vector space of dimension $\geq |\lambda|\,|\mu|$ do not depend on the characteristic of the base field of V. Then, we are left to check that a direct sum $M = \bigoplus_\lambda \nabla(\lambda)^{\oplus c_\lambda}$ of costandard modules is uniquely determined by its characters $\mathrm{ch}(M) = \sum_\eta d_\eta \eta$. We prove it with a descending induction on the number of distinct factors of M. Consider the highest weight η_0 appearing in the sum $\mathrm{ch}(M)$. Clearly, $\nabla(\eta_0)$ is a direct factor of the module M because η_0 cannot appear in the weights of a costandard module $\nabla(\lambda)$ with $\lambda < \eta_0$. Moreover, the multiplicity of $\nabla(\eta_0)$ in M is exactly d_{η_0} and we can pursue the induction with $M' = \bigoplus_{\lambda < \eta_0} \nabla(\lambda)^{\oplus c_\lambda}$.

4. Positive vector bundles

In the positive characteristic, Hartshorne has defined in [Har66] two non-equivalent notions of ampleness for vector bundles. The first notion is simply called ampleness, the second, strictly stronger, is called p-ampleness. Furthermore, Kleiman has defined in [Kle69] a third notion, again strictly stronger, called cohomological p-ampleness. For the convenience of the reader, we recall some well-known results about globally generated sheaves and ampleness notions in positive characteristic.

In § 4.3, we consider an effective Cartier divisor D and we define a positivity notion for vector bundles called (φ, D)-ampleness. In the case of line bundles, this notion is equivalent to being nef and big with D as exceptional divisor. Let X be a projective scheme over k. We write

$\varphi: X \to X^{(p)}$ for the relative geometric Frobenius of X. If $\mathcal{F}$ is a sheaf on X and $r \geq 1$ is an integer, we write $\mathcal{F}^{(p^r)} := (\varphi^r)^*(\varphi^r)_* \mathcal{F}$. We endow the finite-dimensional $\mathbb{R}$-vector space $A_1(X)$ of 1-cycles on X modulo linear equivalence with a norm $\|\cdot\|$. If C is a projective curve and $\mathcal{E}$ is a vector bundle on C, we denote by $\delta(\mathcal{E})$ the minimum of the degrees of quotient line bundles of $\mathcal{E}$.

4.1 Globally generated sheaves

Definition 4.1. We say that a coherent sheaf $\mathcal{F}$ is globally generated at $x \in X$ if the canonical map

$$H^0(X, \mathcal{F}) \otimes_k \mathcal{O}_X \to \mathcal{F}$$

is surjective at $x \in X$. We say $\mathcal{F}$ is globally generated over $U \subset X$ if it is globally generated at x for all $x \in U$.

The following lemma is well-known.

Lemma 4.2. *Let x be a point of X. We have the following assertions.*

(1) *The direct sum of two globally generated sheaves at x is globally generated at x.*
(2) *Let $\mathcal{F} \to \mathcal{F}'$ be a morphism of coherent sheaves which is surjective at x. If $\mathcal{F}$ is globally generated at x, then so is $\mathcal{F}'$.*
(3) *The tensor product of two globally generated sheaves at x is globally generated at x.*
(4) *The pullback of a globally generated sheaf at x is globally generated at x.*

Proof. Left to the reader. □

4.2 Ample bundles

Definition 4.3. We say that a line bundle $\mathcal{L}$ over X is ample if the following equivalent propositions are satisfied.

(1) For all coherent sheaf $\mathcal{F}$ on X, there is an integer n_0 such that $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is globally generated for all $n \geq n_0$.
(2) For all coherent sheaf $\mathcal{F}$ on X, there is an integer n_0 such that the cohomology groups $H^i(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n})$ vanishes for all $i > 0$, $n \geq n_0$.
(3) For any subvariety $V \subset X$, we have
$$c_1(\mathcal{L})^{\dim V} \cdot [V] > 0$$
in the Chow ring of X.

Proof. For the equivalence of the definitions, see [Har66, Prop 1.1/1.2/1.4]. □

From now on, we fix an ample line bundle $\mathcal{O}_X(1)$ on X and we write $\mathcal{F}(m)$ instead of $\mathcal{F} \otimes \mathcal{O}_X(1)^{\otimes m}$ for any coherent sheaf $\mathcal{F}$ on X and integer m. We recall the definition of relative ample line bundles.

Definition 4.4. *Let Y be a projective scheme over a base scheme S. Write $f: Y \to S$ for the structure morphism. We say that a line bundle $\mathcal{L}$ on Y is f-ample if the following equivalent propositions are satisfied.*

(1) *For all coherent sheaf $\mathcal{F}$ on Y, there is an integer n_0 such that the adjunction morphism $f^* f_*(\mathcal{F} \otimes \mathcal{L}^{\otimes n}) \to \mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is surjective for all $n \geq n_0$.*

https://doi.org/10.1112/S0010437X24007607

(2) *For all coherent sheaf $\mathcal{F}$ on Y, there is an integer n_0 such that the higher direct image sheaves $R^i f_*(\mathcal{F} \otimes \mathcal{L}^{\otimes n})$ vanishes for all $i > 0$, $n \geq n_0$.*

Proof. For the equivalence of the definitions, see [Laz04b, Theorem 1.7.6] or [Sta21, Lemma 02O1]. □

DEFINITION 4.5 [Har66]. We say that a vector bundle $\mathcal{E}$ over X is ample if the universal line bundle $\mathcal{O}(1)$ is ample on the projective bundle $\mathbb{P}(\mathcal{E})$. Note that the universal line bundle $\mathcal{O}(1)$ is equal to the associated line bundle $\mathcal{L}_\lambda$ with $\lambda = (1, 0, \ldots, 0)$ for the canonical isomorphism $X^*(T) \simeq \mathbb{Z}^n$ where T is the standard maximal torus of GL_n.

PROPOSITION 4.6. *Let $\mathcal{E}$ be a vector bundle on X. The following assertions are equivalent.*

(1) *The vector bundle $\mathcal{E}$ is ample on X.*
(2) *For all coherent sheaf $\mathcal{F}$ on X, there is an integer n_0 such that $\mathcal{F} \otimes \operatorname{Sym}^n \mathcal{E}$ is globally generated for all $n \geq n_0$.*
(3) *For all coherent sheaf $\mathcal{F}$ on X, there is an integer n_0 such that the cohomology groups $H^i(X, \mathcal{F} \otimes \operatorname{Sym}^n \mathcal{E})$ vanishes for all $i > 0$, $n \geq n_0$.*
(4) *There exists a real number $\varepsilon > 0$ such that for all finite morphism $g : C \to X$ where C is a curve, we have*

$$\delta(g^* \mathcal{E}) \geq \varepsilon \| g_* C \|.$$

Recall that $\delta(g^* \mathcal{E})$ is the minimum of the degrees of quotient line bundles of $g^* \mathcal{E}$ and $\|\cdot\|$ denotes a norm on $A_1(X)$, the k-vector space of 1-cycles modulo linear equivalence.

Proof. See [Har66, Proposition 3.2/3.3] for a complete proof of

$$(1) \Leftrightarrow (2) \Leftrightarrow (3).$$

For $(1) \Leftrightarrow (4)$, this numerical criterion is due to Barton [Bar71]. □

PROPOSITION 4.7. *We have the following assertions*

(1) *Let $\mathcal{E}$ and $\mathcal{E}'$ be two ample vector bundles on X. Then $\mathcal{E} \oplus \mathcal{E}'$ is ample.*
(2) *Consider an extension of vector bundles on X*

$$0 \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}_2 \longrightarrow 0$$

where $\mathcal{E}_1$ and $\mathcal{E}_2$ are ample. Then $\mathcal{E}$ is ample.
(3) *Let $\mathcal{E}$ and $\mathcal{E}'$ be two vector bundles on X such that $\mathcal{E}$ is ample and $\mathcal{E}'$ is globally generated over X. Then the tensor product $\mathcal{E} \otimes \mathcal{E}'$ is an ample vector bundle.*
(4) *Let $\mathcal{E} \to \mathcal{E}'$ be a surjective morphism of $\mathcal{O}_X$-modules between two vector bundles. If $\mathcal{E}$ is ample, then so is $\mathcal{E}'$.*
(5) *The tensor product of ample vector bundles over X is ample.*

Proof. See [Har66, Proposition 2.2/Corollary 2.5] for assertions (1), (3) and (4), [Har66, Corollary 3.4] for assertion (4) and [Bar71, Theorem 3.3] for assertion (5). □

PROPOSITION 4.8. *If $\mathcal{E}$ is a vector bundle such that $\mathcal{E}^{\otimes n}$ is ample for some $n \geq 1$, then $\mathcal{E}$ is also ample.*

Proof. Assume that $\mathcal{E}^{\otimes n}$ is ample. As a quotient of $\mathcal{E}^{\otimes n}$, $\operatorname{Sym}^n \mathcal{E}$ is ample and we conclude with [Har66, Proposition 2.4]. □

Proposition 4.9. *Let $f : Y \to X$ be a finite morphism of projective schemes and $\mathcal{E}$ be an ample vector bundle on X. If $\mathcal{E}$ is ample on X, then $f^* \mathcal{E}$ is ample on Y. If, furthermore, f is assumed surjective, then the converse holds.*

Proof. See [Laz04a, Proposition 1.2.9] and [Laz04a, Corollary 1.2.24]. □

Corollary 4.10. *Let $\mathcal{E}$ be a vector bundle and $r \geq 1$ an integer. Then $\mathcal{E}$ is ample if and only if $\mathcal{E}^{(p^r)}$ is ample.*

Proof. Since the Frobenius map is finite surjective, it follows from the previous proposition. □

Definition 4.11 [Har66]. We say that a vector bundle $\mathcal{E}$ on X is p-ample if for all coherent sheaf $\mathcal{F}$ on X, there is an integer r_0 such that $\mathcal{F} \otimes \mathcal{E}^{(p^r)}$ is globally generated for all $r \geq r_0$.

Lemma 4.12. *For any coherent sheaf $\mathcal{F}$ and $m \geq 0$ large enough, we can write $\mathcal{F}$ as a quotient of $\mathcal{O}_X(-m)^{\oplus s}$ for a suitable $s \geq 1$.*

Proof. Choose $m \geq 0$ large enough such that $\mathcal{F}(m)$ is globally generated over X. We get a surjective morphism

$$\mathcal{O}_X^{\oplus s} \to \mathcal{F}(m)$$

for some $s \geq 1$ and then we tensor by $\mathcal{O}_X(-m)$. □

Proposition 4.13. *In the Definition 4.11, we can restrict ourselves to coherent sheaves of the form $\mathcal{F} = \mathcal{O}_X(-m)$ for all $m \geq 0$ large enough.*

Proof. We use Lemma 4.12 to write $\mathcal{F}$ as a quotient of $\mathcal{O}_X(-m)^{\oplus s}$ for a suitable $s \geq 1$. Take n large enough such that $\mathcal{O}_X(-m) \otimes \mathcal{E}^{(p^r)}$ is globally generated. Since the quotient of a globally generated sheaf is globally generated, we get that $\mathcal{F} \otimes \mathcal{E}^{(p^r)}$ is globally generated. □

Proposition 4.14. *If $\mathcal{E}$ is p-ample on X, then $\mathcal{E}$ is ample.*

Proof. Choose n large enough such that $\mathcal{E}^{(p^n)}(-1)$ is globally generated. We deduce that $\mathcal{E}^{(p^n)}$ is quotient of $\mathcal{O}_X(1)^{\oplus s}$ for a suitable $s \geq 1$. By assertion (3) of Proposition 4.7, $\mathcal{E}^{(p^n)}$ is ample and by Corollary 4.10, $\mathcal{E}$ is ample. □

Remark 4.15. The converse to the previous proposition is false in general (see [Gie71] for a counter-example). However, in the special case where $\mathcal{E}$ is a line bundle or X is curve, it holds by [Har66, Proposition 7.3].

Proposition 4.16. *We have the following assertions.*

(1) *Let $\mathcal{E}$ and $\mathcal{E}'$ be two p-ample vector bundles on X. Then $\mathcal{E} \oplus \mathcal{E}'$ is p-ample.*
(2) *Let $\mathcal{E}$ and $\mathcal{E}'$ be two vector bundles on X such that $\mathcal{E}$ is p-ample and $\mathcal{E}'$ is globally generated over X. Then, the tensor product $\mathcal{E} \otimes \mathcal{E}'$ is a p-ample vector bundle.*
(3) *Let $\mathcal{E} \to \mathcal{E}'$ be a surjective morphism of $\mathcal{O}_X$-modules between two vector bundles. If $\mathcal{E}$ is p-ample, then $\mathcal{E}'$ is also p-ample.*
(4) *The tensor product of p-ample vector bundles over X is p-ample.*

Proof. See [Har66, Proposition 6.4/Corollary 6.7] for assertions (1), (2) and (4). Hartshorne does not state assertion (3), so we give a proof. Let $\mathcal{F}$ be a coherent sheaf and $r_0 \geq 1$ be an integer such that $\mathcal{F} \otimes \mathcal{E}^{(p^r)}$ is globally generated for all $r \geq r_0$. For all $r \geq r_0$, the surjective morphism $\mathcal{E} \to \mathcal{E}'$ induces a surjective morphism of $\mathcal{O}_X$-modules

$$\mathcal{F} \otimes \mathcal{E}^{(p^r)} \to \mathcal{F} \otimes (\mathcal{E}')^{(p^r)}$$

and from assertion (2) of Lemma 4.2, the module $\mathcal{F}\otimes(\mathcal{E}')^{(p^r)}$ is globally generated over X. □

There is no known cohomological criterion for p-ampleness. However, Kleiman has defined in [Kle69] the strictly[12] stronger notion of cohomological p-ampleness.

DEFINITION 4.17. We say that a vector bundle $\mathcal{E}$ on X is cohomologically p-ample if for all coherent sheaves $\mathcal{F}$ on X, there is an integer r_0 such that the cohomology groups $H^i(X,\mathcal{F}\otimes\mathcal{E}^{(p^r)})$ vanishes for all $i>0$, $r\geq r_0$.

PROPOSITION 4.18. *If $\mathcal{E}$ is cohomologically p-ample on X, then $\mathcal{E}$ is p-ample.*

Proof. See [Kle69, Proposition 9]. □

To the best of the authors' knowledge, the following statements do not appear in the literature so we state them and provide a proof.

LEMMA 4.19. *A direct sum of cohomologically p-ample vector bundle is cohomologically p-ample.*

Proof. The proof follows directly from the isomorphism

$$H^i(X,\mathcal{F}\otimes(\mathcal{E}\oplus\mathcal{E}')^{(p^r)}) = H^i(X,\mathcal{F}\otimes\mathcal{E}^{(p^r)})\oplus H^i(X,\mathcal{F}\otimes\mathcal{E}'^{(p^r)}).$$ □

LEMMA 4.20. *Let $f:Y\to X$ be a finite morphism of projective schemes and $\mathcal{E}$ be a cohomologically p-ample vector bundle on X. Then $f^*\mathcal{E}$ is cohomologically p-ample on Y.*

Proof. Let $\mathcal{F}$ be a coherent sheaf on Y. Since f is finite, the Leray spectral sequence degenerates at page 2 and we have isomorphisms

$$H^i(X, f_*(\mathcal{F}\otimes f^*\mathcal{E}^{(p^r)})) = H^i(Y,\mathcal{F}\otimes f^*\mathcal{E}^{(p^r)})$$

for all $i\geq 0$ and $r\geq 0$. Since f is finite, the pushforward $f_*\mathcal{F}$ is a coherent $\mathcal{O}_X$-module and the projection formula implies that

$$f_*(\mathcal{F}\otimes f^*\mathcal{E}^{(p^r)}) = f_*\mathcal{F}\otimes\mathcal{E}^{(p^r)}.$$

Since $\mathcal{E}$ is cohomologically p-ample on X, there is an integer $r_0\geq 1$ such that

$$H^i(X, f_*\mathcal{F}\otimes\mathcal{E}^{(p^r)}) = 0 = H^i(Y,\mathcal{F}\otimes(f^*\mathcal{E})^{(p^r)})$$

for all $i>0$ and $r\geq r_0$. In particular, $f^*\mathcal{E}$ is cohomologically p-ample on Y. □

4.3 The (φ, D)-ample bundles

If D is a Cartier divisor, we write $\mathcal{O}_X(D)$ for the associated line bundle. If $\mathcal{F}$ is a coherent sheaf on X, then we simply write $\mathcal{F}(D)$ instead of $\mathcal{F}\otimes\mathcal{O}_X(D)$. We consider an *effective* Cartier divisor D on X and we define the notion of (φ, D)-ampleness for vector bundles over X.

DEFINITION 4.21. Let $\mathcal{E}$ be a vector bundle over X. We say that $\mathcal{E}$ is (φ, D)-ample if there is an integer $r_0\geq 1$ such that for all integer $r\geq r_0$, the vector bundle $\mathcal{E}^{(p^r)}(-D)$ is ample.

In the case of line bundles, (φ, D)-ampleness has the following characterization.

PROPOSITION 4.22. *Let $\mathcal{L}$ be a line bundle over X. Then $\mathcal{L}$ is (φ, D)-ample if and only if $\mathcal{L}$ is nef and there is an integer $n_0\geq 1$ such that $\mathcal{L}^{\otimes n_0}(-D)$ is ample.*

Proof. Note that $\mathcal{L}^{(p^r)} = \mathcal{L}^{\otimes p^r}$ for all $r\geq 0$. Assume that $r_0\geq 1$ is an integer such that $\mathcal{L}^{\otimes p^r}(-D)$ is ample for all $r\geq r_0$. If $\mathcal{L}$ was not nef, we could find a subcurve $C\subset X$ such that the intersection

[12] See again [Gie71] for an example of p-ample vector bundle that is not cohomologically p-ample.

https://doi.org/10.1112/S0010437X24007607

product

$$c_1(\mathcal{L}) \cdot [C]$$

is negative. It would imply that the intersection product

$$c_1(\mathcal{L}^{\otimes p^r}(-D)) \cdot [C] = p^r \underbrace{(c_1(\mathcal{L}) \cdot [C])}_{<0} - D \cdot [C]$$

is negative for some $r \geq r_0$ large enough, which contradicts the ampleness of $\mathcal{L}^{\otimes p^r}(-D)$. Inversely, we assume that $\mathcal{L}$ is nef and there exists an integer $n_0 \geq 1$ such that $\mathcal{L}^{\otimes n_0}(-D)$ is ample. Let r be an integer such that

$$r \geq \log_p n_0$$

and consider

$$\mathcal{L}^{(p^r)}(-D) = \mathcal{L}^{\otimes p^r}(-D) = \mathcal{L}^{\otimes n_0}(-D) \otimes \mathcal{L}^{\otimes p^r - n_0},$$

which is ample as the tensor product of an ample line bundle with a nef line bundle. □

Remark 4.23. In the case of line bundles we will drop the φ from the notation and simply say that the line bundle is D-ample.

Proposition 4.24. *Let $\mathcal{L}$ be a line bundle over X. The following propositions are equivalent:*

(1) *$\mathcal{L}$ is nef and big;*
(2) *there exists an effective Cartier divisor H on X, such that $\mathcal{L}$ is H-ample.*

Proof. Assume that there exists an effective Cartier divisor H on X such that $\mathcal{L}$ is H-ample. We have seen in Proposition 4.22 that $\mathcal{L}$ is nef and there is an integer $n_0 \geq 1$ such that $\mathcal{L}^{\otimes n_0}(-H)$ is ample. Moreover, since we can write $\mathcal{L}^{\otimes n_0}$ as a tensor product

$$\mathcal{L}^{\otimes n_0} = \mathcal{L}^{\otimes n_0}(-H) \otimes \mathcal{O}_X(H)$$

of an ample line bundle with an effective line bundle, $\mathcal{L}$ is big. We are left to show the implication $(1) \Rightarrow (2)$. Since $\mathcal{L}$ is big, there exists an integer $n_0 \geq 1$ and an ample line bundle $\mathcal{A}$ such that $\mathcal{L}^{\otimes n_0} \otimes \mathcal{A}^{-1} = \mathcal{O}_X(H)$ with H an effective divisor. In particular, the line bundle $\mathcal{L}^{\otimes n_0}(-H)$ is ample. We conclude with Proposition 4.22. □

We prove some stability properties of (φ, D)-ample vector bundles. We first prove the following easy lemma.

Lemma 4.25. *Let C be a projective curve and $\mathcal{E}$ be a vector bundle on C. Recall that $\delta(\mathcal{E})$ denotes the minimum of degrees of quotient line bundles of $\mathcal{E}$. Then, we have the following.*

(1) *If $\mathcal{L}$ is a line bundle on C, then $\delta(\mathcal{E} \otimes \mathcal{L}) = \delta(\mathcal{E}) + \deg \mathcal{L}$.*
(2) *If $f: C' \to C$ is a finite morphism of degree d with C' a projective curve, then $d\delta(\mathcal{E}) \geq \delta(f^*\mathcal{E})$.*

Proof of Lemma 4.25. For condition (1), take a line bundle $\mathcal{E} \twoheadrightarrow \mathcal{L}'$ such that $\delta(\mathcal{E}) = \deg \mathcal{L}'$. If we tensor it by $\mathcal{L}$, we get $\delta(\mathcal{E} \otimes \mathcal{L}) \leq \deg \mathcal{L}' + \deg \mathcal{L} = \delta(\mathcal{E}) + \deg \mathcal{L}$. The same argument applied to $\mathcal{E} \otimes \mathcal{L}^{-1}$ shows the reverse inequality. For condition (2), take a line bundle $\mathcal{E} \twoheadrightarrow \mathcal{L}'$ such that $\delta(\mathcal{E}) = \deg \mathcal{L}'$. The pullback f^* induces a quotient map $f^*\mathcal{E} \twoheadrightarrow f^*\mathcal{L}' = \mathcal{L}'^{\otimes d}$ which shows that $\delta(f^*\mathcal{E}) \leq d \deg \mathcal{L}' = d\delta(\mathcal{E})$. □

Proposition 4.26. *Let $\mathcal{E}$ be a vector bundle on X and $n \geq 1$ an integer. The following assertions are equivalent:*

https://doi.org/10.1112/S0010437X24007607

(1) $\mathcal{E}$ *is* (φ, D)*-ample;*

(2) $\mathcal{E}$ *is* (φ, nD)*-ample.*

Proof. Assume that $\mathcal{E}$ is (φ, D)-ample and consider $r_0 \geq 1$ such that

$$\mathcal{E}^{(p^r)}(-D)$$

is ample for all $r \geq r_0$. By Barton's numerical criterion of ampleness recalled in assertion (4) of Proposition 4.6, for all $r \geq r_0$, we have a real number $\varepsilon_r > 0$ such that for all finite morphism $g : C \to X$ where C is a smooth projective curve over k, we have

$$\delta(g^* \mathcal{E}^{(p^r)}(-D)) \geq \varepsilon_r \|g_* C\|,$$

which is equivalent to

$$\delta(g^* \mathcal{E}^{(p^r)}) - D \cdot C \geq \varepsilon_r \|g_* C\|,$$

where $D \cdot C$ is the degree of the line bundle $g^* \mathcal{O}_X(D) = \mathcal{O}_X(D)_{|C}$ (it is also equal to the intersection number of D with C). If $D \cdot C \leq 0$, then

$$\begin{aligned}\delta(g^* \mathcal{E}^{(p^r)}(-nD)) &= \delta(g^* \mathcal{E}^{(p^r)}) - D \cdot C - (n-1) D \cdot C \\ &\geq \delta(g^* \mathcal{E}^{(p^r)}) - D \cdot C \\ &\geq \varepsilon_r \|g_* C\|,\end{aligned}$$

for all $r \geq r_0$. If $D \cdot C > 0$, we take $r_1 \geq r_0$ such that $r_1 \geq r_0 + \log_p(n)$ and let $r \geq r_1$ be an integer. Since $\mathcal{E}^{(p^r)}(-D)$ is ample, the bundle

$$(\mathcal{E}^{(p^{r_0})}(-D))^{(p^{r-r_0})} = \mathcal{E}^{(p^r)}(-p^{r-r_0} D)$$

is ample and we have

$$\delta(g^* \mathcal{E}^{(p^r)}(-p^{r-r_0} D)) \geq \varepsilon'_r \|g_* C\|$$

for some real number $\varepsilon'_r > 0$. Thus,

$$\begin{aligned}\delta(g^* \mathcal{E}^{(p^r)}(-nD)) &= \delta(g^* \mathcal{E}^{(p^r)}(-p^{r-r_0} D)) + (p^{r-r_0} - n) D \cdot C \\ &\geq \delta(g^* \mathcal{E}^{(p^r)}(-p^{r-r_0} D)) \\ &\geq \varepsilon'_r \|g_* C\|.\end{aligned}$$

In conclusion, we have

$$\delta(g^* \mathcal{E}^{(p^r)}(-nD)) \geq \min(\varepsilon_r, \varepsilon'_r) \|g_* C\|$$

for all $r \geq r_1$ and all $g : C \to X$, which means that $\mathcal{E}$ is (φ, nD)-ample. Inversely, consider an integer $r_0 \geq 1$ such that for all $r \geq r_0$, we have a real number $\varepsilon_r > 0$ such that for all finite morphisms $g : C \to X$ where C is a smooth projective curve over k, we have

$$\delta(g^* \mathcal{E}^{(p^r)}) - nD \cdot C \geq \varepsilon_r \|g_* C\|.$$

If $D \cdot C \geq 0$, we have

$$\begin{aligned}\delta(g^* \mathcal{E}^{(p^r)}(-D)) &= \delta(g^* \mathcal{E}^{(p^r)}(-nD)) + (n-1) D \cdot C \\ &\geq \delta(g^* \mathcal{E}^{(p^r)}(-nD)) \\ &\geq \varepsilon_r \|g_* C\|\end{aligned}$$

for all $r \geq r_0$. Consider an integer $r_1 \geq r_0 + \log_p n$. If $D \cdot C < 0$, we have

$$p^{r-r_0} \delta(g^* \mathcal{E}^{(p^{r_0})}(-D)) \geq \delta(g^* \mathcal{E}^{(p^r)}(-p^{r-r_0} D))$$

https://doi.org/10.1112/S0010437X24007607

$$\begin{aligned}&\geq \delta(g^*\,\mathcal{E}^{(p^r)}(-nD)) + (n - p^{r-r_0})D\cdot C\\ &\geq \delta(g^*\,\mathcal{E}^{(p^r)}(-nD))\\ &\geq \varepsilon_r \|g_*C\|\end{aligned}$$

for all $r \geq r_1$. In conclusion, we have

$$\delta(g^*\,\mathcal{E}^{(p^r)}(-D)) \geq \frac{\varepsilon_r}{p^{r-r_0}}\|g_*C\|$$

for all $r \geq r_1$ and all $g : C \to X$, which means that $\mathcal{E}$ is (φ, D)-ample. □

Proposition 4.27. *Let $\mathcal{E}$ and $\mathcal{E}'$ be two (φ, D)-ample vector bundle on X. Then, $\mathcal{E} \oplus \mathcal{E}'$ is (φ, D)-ample.*

Proof. Let $r_0 \geq 1$ be an integer such that for all $r \geq r_0$, the bundles $\mathcal{E}^{(p^r)}(-D)$ and $(\mathcal{E}')^{(p^r)}(-D)$ are ample. For all $r \geq r_0$, we have

$$(\mathcal{E} \oplus \mathcal{E}')^{(p^r)}(-D) = \mathcal{E}^{(p^r)}(-D) \oplus (\mathcal{E}')^{(p^r)}(-D),$$

which is ample by assertion (1) of Proposition 4.7. □

Proposition 4.28. *Consider an extension of vector bundles on X*

$$0 \longrightarrow {}_1 \longrightarrow \quad \longrightarrow {}_2 \longrightarrow 0$$

where $\mathcal{E}_1$ and $\mathcal{E}_2$ are (φ, D)-ample and assume that X is regular over k. Then $\mathcal{E}$ is (φ, D)-ample.

Proof. On a regular scheme, the Frobenius morphism is flat by [Kun69] or [Sta21, Lemma 0EC0]. As a consequence, we have an integer $r_0 \geq 1$ and an exact sequence

$$0 \longrightarrow (\mathcal{E}_1)^{(p^r)}(-D) \longrightarrow (\mathcal{E})^{(p^r)}(-D) \longrightarrow (\mathcal{E}_2)^{(p^r)}(-D) \longrightarrow 0$$

of vector bundles on X where $(\mathcal{E}_1)^{(p^r)}(-D)$ and $(\mathcal{E}_2)^{(p^r)}(-D)$ are ample for all $r \geq r_0$. We conclude with assertion (2) of Proposition 4.7. □

Proposition 4.29. *Let $\mathcal{E} \to \mathcal{E}'$ be a surjective morphism of $\mathcal{O}_X$-modules between two vector bundles. If $\mathcal{E}$ is (φ, D)-ample, then $\mathcal{E}'$ is also (φ, D)-ample.*

Proof. Let $r_0 \geq 1$ be an integer such that for all $r \geq r_0$, the bundle $\mathcal{E}^{(p^r)}(-D)$ is ample. For all $r \geq r_0$, the surjective morphism $\mathcal{E} \to \mathcal{E}'$ induces a surjection

$$\mathcal{E}^{(p^r)}(-D) \to (\mathcal{E}')^{(p^r)}(-D)$$

and we conclude with assertion (3) of Proposition 4.7. □

Proposition 4.30. *The tensor product of (φ, D)-ample vector bundles is (φ, D)-ample.*

Proof. Let $r_0 \geq 1$ be an integer such that for all $r \geq r_0$, the bundles $\mathcal{E}^{(p^r)}(-D)$ and $(\mathcal{E}')^{(p^r)}(-D)$ are ample. For all $r \geq r_0$, we have

$$(\mathcal{E} \otimes \mathcal{E}')^{(p^r)}(-2D) = \mathcal{E}^{(p^r)}(-D) \otimes (\mathcal{E}')^{(p^r)}(-D)$$

which is ample by assertion (5) of Proposition 4.7. It shows that $\mathcal{E} \otimes \mathcal{E}'$ is $(\varphi, 2D)$-ample and we conclude with Proposition 4.26. □

Proposition 4.31. *If $\mathcal{E}$ is a vector bundle such that $\mathcal{E}^{\otimes n}$ is (φ, D)-ample for some $n \geq 1$, then $\mathcal{E}$ is also (φ, D)-ample.*

https://doi.org/10.1112/S0010437X24007607

Table 1. Main properties of the different positivity notions, from the strongest to the weakest.

	Cohomologically p-ample	p-ample	Ample	(φ, D)-ample
Stability of direct sum	4.19	4.16	4.7	4.27
Stability of extension	?	?	4.7	4.28 (X regular)
Stability of quotient	?	4.16	4.7	4.29
Stability of tensor product	?	4.16	4.7	4.30
Stability of tensor roots	?	?	4.8	4.31
Stability of pullback by finite morphism	4.20	?	4.9	4.32 ($f^{-1}D$ defined)
Descent along finite surjective morphism	?	?	4.9	4.32

Proof. Assume that $\mathcal{E}^{\otimes n}$ is (φ, D)-ample. By Proposition 4.26, $\mathcal{E}^{\otimes n}$ is also (φ, nD)-ample. Let $r_0 \geq 1$ be an integer such that for all $r \geq r_0$, the bundle $(\mathcal{E}^{\otimes n})^{(p^r)}(-nD)$ is ample. For all $r \geq r_0$, the bundle

$$(\mathcal{E}^{(p^r)}(-D))^{\otimes n} = (\mathcal{E}^{\otimes n})^{(p^r)}(-nD)$$

is ample. Thus, the bundle $\mathcal{E}^{(p^r)}(-D)$ is ample for all $r \geq r_0$ by Proposition 4.8. $\square$

Proposition 4.32. *Let $f : Y \to X$ be a finite morphism of projective schemes such that the pullback $f^{-1}D$ is defined as an effective Cartier divisor of Y*[13] *and $\mathcal{E}$ be a (φ, D)-ample bundle on X. Then $f^*\mathcal{E}$ is $(\varphi, f^{-1}D)$-ample on Y. If, furthermore, f is assumed surjective, then the converse holds.*

Proof. Let $r_0 \geq 1$ be an integer such that for all $r \geq r_0$, the bundle $\mathcal{E}^{(p^r)}(-D)$ is ample. For all $r \geq r_0$, the bundle

$$f^*(\mathcal{E}^{(p^r)}(-D)) = (f^*\mathcal{E})^{(p^r)}(-f^{-1}D)$$

is ample by Proposition 4.9. If f is assumed surjective, the converse holds by Proposition 4.9 again. $\square$

Table 1 summarizes the different stability properties of ampleness, p-ampleness, cohomological p-ampleness and (φ, D)-ampleness.

We explain the relationship between (φ, D)-ampleness and other positivity notions.

Proposition 4.33. *Let $\mathcal{E}$ be a vector bundle on X. Then,*

$$\begin{array}{ccccc} & & \mathcal{E} \ \textit{is } L\textit{-big} & & \\ & & \Uparrow & & \\ \mathcal{E} \ \textit{is ample} & \Longrightarrow & \mathcal{E} \ \textit{is } (\varphi, D)\textit{-ample} & \Longrightarrow & \mathcal{E} \ \textit{is nef} \\ & & \Downarrow & & \\ & & \mathcal{E}^{(p^r)} \ \textit{is } V\textit{-big for some } r \geq 1 & & \end{array}$$

Proof. The first implication follows directly from [Bar71, Proposition 3.1]. Now, assume that $\mathcal{E}$ is (φ, D)-ample and consider the universal line bundle $\mathcal{O}(1)$ on the projective bundle $\mathbb{P}(\mathcal{E})$. We have a surjective map $\pi^*\mathcal{E} \to \mathcal{O}(1)$ and since (φ, D)-ampleness is stable under quotient by Proposition 4.29, $\mathcal{O}(1)$ is, in particular, nef and big by Proposition 4.24. It shows that $\mathcal{E}$ is nef and L-big. Take $r \geq 1$ such that $\mathcal{E}^{(p^r)}(-D)$ is ample. We deduce that there is an integer $n \geq 1$

[13] By [Sta21, Lemma 02OO], it is the case when $f(x) \notin D$ for any weakly associated point x of X or when f is flat.

https://doi.org/10.1112/S0010437X24007607

such that

$$\mathrm{Sym}^n(\mathcal{E}^{(p^r)}(-D))\otimes\mathcal{O}_X(-1)=\mathrm{Sym}^n(\mathcal{E}^{(p^r)})(-nD)\otimes\mathcal{O}_X(-1)$$

is globally generated. Since $\mathrm{Sym}^n(\mathcal{E}^{(p^r)})\otimes\mathcal{O}_X(-1)$ can be expressed as a tensor product of a globally generated vector bundle with $\mathcal{O}_X(nD)$, it is globally generated on the complementary open subset of the support of D. It implies that the augmented base locus of $\mathcal{E}^{(p^r)}$ is not equal to X, i.e. that $\mathcal{E}^{(p^r)}$ is V-big. □

5. Flag bundle associated to a G-torsor

5.1 Higher direct image

In this subsection only, $\pi:Y\to X$ is a general scheme morphism. We recall some generalities about cohomology and higher direct image.

Proposition 5.1. *For any $\mathcal{O}_Y$-module $\mathcal{F}$, there is a spectral sequence starting at page 2:*

$$E_2^{i,j}=H^i(X,R^i\pi_*(\mathcal{F}))\Rightarrow H^{i+j}(Y,\mathcal{F}).$$

Proof. See [Sta21, Lemma 01F2]. □

We recall the projection formula.

Proposition 5.2. *Let $\mathcal{F}$ be a $\mathcal{O}_Y$-module, $\mathcal{E}$ a locally free $\mathcal{O}_X$-module of finite rank and $i\geq 0$ an integer. The natural map*

$$R^i\pi_*\,\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{E}\to R^i\pi_*(\mathcal{F}\otimes_{\mathcal{O}_Y}\pi^*\,\mathcal{E})$$

is an isomorphism.

Proof. See [Sta21, Lemma 01E8]. □

We recall the following lemma that appears also in [Ale24].

Lemma 5.3. *Consider two Artin stacks $\mathcal{X}$ and $\mathcal{Y}$ over k and a proper representable morphism $\pi:\mathcal{Y}\to\mathcal{X}$. Consider a coherent sheaf $\mathcal{F}$ over $\mathcal{Y}$ which is flat over $\mathcal{X}$ and such that for any geometric point $x:\mathrm{Spec}\,K\to\mathcal{X}$ fitting in a cartesian diagram*

$$\begin{tikzcd} \mathcal{Y}_x:=\mathcal{Y}\times_{\mathcal{X},x}\mathrm{Spec}\,K \arrow[r, "i"] \arrow[d, "\pi_x"] & \mathcal{Y} \arrow[d, "\pi"] \\ \mathrm{Spec}\,K \arrow[r, "x"] & \mathcal{X} \end{tikzcd}$$

the complex $R(\pi_x)_*\mathcal{F}_{|\mathcal{Y}_x}$ is concentrated in degree 0. Then, the complex $R\pi_*\mathcal{F}$ is also concentrated in degree 0.

Proof. See [Ale24, Lemma 3.19]. □

5.2 G-torsors

In this subsection, k can be an algebraically closed field of any characteristic, G is a connected split reductive group over k, $P\subset G$ is a parabolic subgroup and X is a k-scheme. If Y is a scheme, we denote by $\mathrm{Mod}(\mathcal{O}_Y)$ the abelian category of $\mathcal{O}_Y$-module on Y and $\mathrm{Loc}(\mathcal{O}_Y)\subset\mathrm{Mod}(\mathcal{O}_Y)$ the fully faithful additive subcategory of locally free $\mathcal{O}_Y$-module of finite rank.

https://doi.org/10.1112/S0010437X24007607

Definition 5.4. Let E be a G-torsor over X. We define the flag bundle of type P of E to be the scheme $\mathcal{F}_P(E)$ over X that represents the functor whose S-points are P-reduction of $E\times_X S$ over S.

Definition 5.5. Let V be an algebraic representation of G and E a G-torsor over X. We define the contracted product of E and V over G to be the representable quotient X-scheme

$$V\times^G E := \underline{V}\times_k E/G,$$

where $\underline{V}$ is the k-vector space scheme associated to V and G acts on functorial points by $g(v,e)=(gv,ge)$. Note that the structure of k-vector space on V endows $V\times^G E$ with a structure of vector bundle of rank $\dim_k V$ over X.

Definition 5.6. Let E be a G-torsor over X. We define a functor

$$\mathcal{W}:\mathrm{Rep}(G)\to \mathrm{Loc}(\mathcal{O}_X)$$

through the formula

$$\mathcal{W}(V)=V\times^G E,$$

where V is an algebraic representation of G.

Definition 5.7. Let E be a G-torsor over X and $\pi:\mathcal{F}_P(E)\to X$ the flag bundle of type P of E. We define a functor

$$\mathcal{L}:\mathrm{Rep}(P)\to \mathrm{Loc}(\mathcal{O}_{\mathcal{F}_P(E)})$$

through the formula

$$\mathcal{L}(V)=V\times^P H,$$

where V is an algebraic representation of P and H is the universal P-torsor on $\mathcal{F}_P(E)$.

Remark 5.8. If $\lambda\in X^*(P)$ is a character of P, we simply write $\mathcal{L}_\lambda$ for the associated line bundle on $\mathcal{F}_P(E)$. We also write $\mathcal{W}_\lambda$ for the vector bundle associated to the G-representation $H^0(G/P,\mathcal{L}_\lambda)=\nabla(\lambda)$. We simply denote by St_r the image of the Steinberg module by $\mathcal{W}$.

Proposition 5.9. *The functor $\mathcal{W}$ and $\mathcal{L}$ are monoidal and exact.*

Proof. This is a general result on associated sheaves [Jan03, Part 1, Chapter 5]. □

Proposition 5.10. *Let E be a G-torsor over X. Then, the following diagram commutes where $\mathrm{Ind}_P^G:\mathrm{Rep}_k(P)\to\mathrm{Rep}_k(G)$ and $\mathrm{Res}_P^G:\mathrm{Rep}_k(G)\to\mathrm{Rep}_k(P)$ are the induction and restriction functors.*

$$\begin{tikzcd}
\mathrm{Rep}(P) \arrow[r, "\mathcal{L}"] \arrow[d, "\mathrm{Ind}_P^G"] & \mathrm{Loc}(\mathcal{O}_{\mathcal{F}_P(E)}) \arrow[d, "\pi_*"] \\
\mathrm{Rep}(G) \arrow[r, "\mathcal{W}"] \arrow[d, "\mathrm{Res}_P^G"] & \mathrm{Loc}(\mathcal{O}_X) \arrow[d, "\pi^*"] \\
\mathrm{Rep}(P) \arrow[r, "\mathcal{L}"] & \mathrm{Loc}(\mathcal{O}_{\mathcal{F}_P(E)})
\end{tikzcd}$$

Moreover, if λ is a dominant character of P, then $R\pi_*\mathcal{L}_\lambda$ is isomorphic to $\mathcal{W}_\lambda$ concentrated in degree 0.

Proof. The commutativity of the lower square follows directly from the definitions. We focus on the commutativity of the upper square. Consider a representation V of P. We have a cartesian

https://doi.org/10.1112/S0010437X24007607

diagram

$$\begin{tikzcd} \mathcal{F}_P(E) \arrow[r, "\zeta_P"] \arrow[d, "\pi"] & \lfloor P\backslash * \rfloor \arrow[d, "\tilde{\pi}"] \\ X \arrow[r, "\zeta"] & \lfloor G\backslash * \rfloor \end{tikzcd}$$

where the map ζ is induced by E, the map ζ_P is induced by the universal P-reduction of E on $\mathcal{F}_P(E)$ and the vertical arrow $\tilde{\pi}$ between the classifying stacks is induced by the inclusion $P \subset G$. Denote by $\tilde{\mathcal{L}}(V)$ the vector bundle on the classifying stack of P associated to V and $\tilde{\mathcal{W}}(V)$ the vector bundle on the classifying stack of G associated to the G-module $\mathrm{Ind}_P^G(V)$. It follows directly from the definitions that

$$\begin{cases} \tilde{\pi}_* \tilde{\mathcal{L}}(V) = \tilde{\mathcal{W}}(V), \\ \zeta_P^* \tilde{\mathcal{L}}(V) = \mathcal{L}(V), \\ \zeta^* \tilde{\mathcal{W}}(V) = \mathcal{W}(V), \end{cases}$$

as sheaves on the stack $\lfloor G\backslash * \rfloor$. Since ζ is a flat morphism of algebraic stacks, the base change theorem in the derived category of quasi-coherent sheaves over X tells us that the map

$$\zeta^* \circ R\tilde{\pi}_* \tilde{\mathcal{L}}(V) \xrightarrow{\sim} R\pi_* \circ \zeta_P^* \tilde{\mathcal{L}}(V) \tag{2}$$

is an isomorphism. Taking global sections in (2) yields the following isomorphism over X.

$$\mathcal{W}(V) = \zeta^* \tilde{\pi}_* \tilde{\mathcal{L}}(V) \xrightarrow{\simeq} \pi_* \mathcal{L}(V)$$

Now assume that λ is P-dominant. By Kempf's vanishing theorem from Proposition 2.3 combined with Lemma 5.3, we deduce

$$\begin{cases} R\pi_* \mathcal{L}_\lambda = \pi_* \mathcal{L}_\lambda, \\ R\tilde{\pi}_* \mathcal{L}_\lambda = \tilde{\pi}_* \mathcal{L}_\lambda, \end{cases}$$

and we get an isomorphism

$$R\pi_* \mathcal{L}_\lambda \simeq \mathcal{W}_\lambda[0]. \qquad \square$$

Proposition 5.11. *Let λ be a character. For all $r \geq 1$, we have isomorphisms*

$$\pi_*(\mathcal{L}_{p^r(\lambda+\rho)-\rho}) = \mathrm{St}_r \otimes \mathcal{W}_\lambda^{(p^r)}.$$

Proof. This is a direct consequence of Propositions 2.10 and 5.10. $\square$

Proposition 5.12. *Let P_I be a standard parabolic subgroup of G of type I. Let E be a G-torsor over X and $\pi : \mathcal{F}_{P_I}(E) \to X$ the flag bundle of type P_I of X. We have an isomorphism*

$$\Omega^{top}_{\mathcal{F}_{P_I}(E)/X} \simeq \mathcal{L}_{-2\rho_I},$$

where top denotes the relative dimension of π and $\rho_I = \frac{1}{2}\sum_{\alpha \in \Phi_I^+} \alpha$.

Proof. From the cartesian diagram

$$\begin{tikzcd} \mathcal{F}_{P_I}(E) \arrow[r, "\tilde{\zeta}"] \arrow[d, "\pi"] & \lfloor P_I\backslash * \rfloor \arrow[d, "\tilde{\pi}"] \\ X \arrow[r, "\zeta"] & \lfloor G\backslash * \rfloor \end{tikzcd}$$

we deduce an isomorphism $\tilde{\zeta}^*\Omega^1_{\tilde{\pi}} = \Omega^1_{\pi}$. We know that

$$\Omega^1_{\tilde{\pi}} \simeq \mathcal{L}(\mathrm{Lie}(G)/\mathrm{Lie}(P_I)^\vee),$$

hence

$$\Omega^{top}_{\tilde{\pi}} \simeq \mathcal{L}(\Lambda^{top}\,\mathrm{Lie}(G)/\mathrm{Lie}(P_I)^\vee).$$

The weights of the T-action on $\mathrm{Lie}(G)/\mathrm{Lie}(P_I)$ are the roots $-\Phi_I^+$, so $\Lambda^{top}\,\mathrm{Lie}(G)/\mathrm{Lie}(P_I)$ is a one-dimensional module of weight $-2\rho_I$ and by taking the linear dual, we get an isomorphism $\Omega^{top}_{\pi} = \tilde{\zeta}^*\Omega^{top}_{\tilde{\pi}} = \mathcal{L}_{-2\rho_I}$. □

6. Pushforward of positive line bundles

Recall that k is an algebraically closed field of characteristic p. Let X be a projective scheme over k and D an effective Cartier divisor on X. Let E be a G-torsor and write $\pi : Y \to X$ for the flag bundle of type B of E as defined in Definition 5.4. We also write D for the Cartier divisor $\pi^{-1}(D)$ on Y. Recall that we have fixed an ample line bundle $\mathcal{O}_X(1)$ on X and that we write $\mathcal{F}(m)$ instead of $\mathcal{F} \otimes \mathcal{O}_X(1)^{\otimes m}$ for any coherent sheaf $\mathcal{F}$ on X and integer m. We start this section with some preliminary results.

Lemma 6.1. *Consider a finite surjective morphism $g : G' \to G$ of algebraic groups with central kernel. Then there exists a projective scheme X' and a finite surjective morphism $f : X' \to X$ such that the pullback of the G-torsor f^*E reduces to a G'-torsor on X'.*

Proof. Let us denote by $BG = [\mathrm{Spec}\,k/G]$ and $BG' = [\mathrm{Spec}\,k/G']$ the classifying stacks of G and G'. The G-torsor E on X corresponds to a map $b_E : X \to BG$ and $g : G' \to G$ induces a map $b_g : BG' \to BG$ on the classifying stacks. Let K denote the kernel of g. We consider a cartesian product

$$\begin{tikzcd} X' \arrow[dr, dashed, "h"] \arrow[ddr, dashed, bend right, "\alpha\circ h"'] & & \\ & \mathcal{X} \arrow[r, "\beta"] \arrow[d, "\alpha"] & BG' \arrow[d, "b_g"] \\ & X \arrow[r, "b_E"] & BG \end{tikzcd}$$

in the category of Artin stacks over k and the objective is now to prove that there exists a scheme X' over k and a morphism $h : X' \to \mathcal{X}$ such that $\alpha \circ h : X' \to X$ is finite surjective. The first step is to show that α is quasi-finite, proper[14] and surjective. By base change along b_E, it is enough to show it for b_g. Since K is central, we have a cartesian product

$$\begin{tikzcd} BK \arrow[r] \arrow[d] & BG' \arrow[d, "b_g"] \\ \mathrm{Spec}\,k \arrow[r, "b_G"] & BG \end{tikzcd}$$

where b_G is the classifying map of the trivial G-torsor on $\mathrm{Spec}\,k$. We claim the map $BK \to \mathrm{Spec}\,k$ is proper, quasi-finite and surjective. The only non-trivial part is to show that $BK \to \mathrm{Spec}\,k$ is

[14] See [Ols16, § 10.1] for a reference on properness for non-representable morphisms of stacks.

https://doi.org/10.1112/S0010437X24007607

separated, i.e. that its diagonal is proper. We have a cartesian product

$$\begin{array}{ccc} K & \longrightarrow & \operatorname{Spec} k \\ \downarrow & & \downarrow \\ BK & \longrightarrow & BK \times_k BK \end{array}$$

and since K is finite, the diagonal $BK \to BK \times_k BK$ is also finite by faithfully flat descent. By faithfully flat descent along b_G, it implies that the map b_g is quasi-finite, proper and surjective. The second step is to find a finite surjective morphism $h : X' \to \mathcal{X}$ approximating the Artin stack $\mathcal{X}$. By [Ryd15, Theorem B], we have to check that the diagonal of $\alpha : \mathcal{X} \to X$ is quasi-finite and separated. By base change and faithfully flat descent, it follows from the fact that the diagonal of $BK \to \operatorname{Spec} k$ is finite. Combining the two steps, the composition $\alpha \circ h$ is finite surjective. □

We give a sufficient cohomological condition for a vector bundle to be ample.

Proposition 6.2. *Let $\mathcal{E}$ be a vector bundle over X. Let $\lambda \in X^*$ be a character. If for all coherent sheafs $\mathcal{F}$, there is an integer r_0 such that*

$$H^i(X, \mathcal{F} \otimes \mathrm{St}_r \otimes \mathcal{W}_\lambda^{(p^r)}) = 0$$

for all $i > 0$ and $r \geq r_0$, then $\mathcal{W}_\lambda$ is ample.

Proof. We consider a coherent sheaf $\mathcal{F} = \mathcal{O}_X(-m)$ with $m \geq 0$ and we write

$$\mathcal{G}_r = \mathrm{St}_r \otimes \mathcal{W}_\lambda^{(p^r)} \otimes \mathcal{O}_X(-m).$$

Let $x \in X$ be a closed point. From our hypothesis, there is an integer r_0 such that

$$H^1(X, \mathcal{G}_{r_0} \otimes \mathcal{I}_x) = 0,$$

where $\mathcal{I}_x$ is the ideal sheaf defining the closed point x. From the long exact sequence of cohomology associated to the exact sequence

$$0 \longrightarrow \mathcal{G}_{r_0} \otimes \mathcal{I}_x \longrightarrow \mathcal{G}_{r_0} \longrightarrow \mathcal{G}_{r_0} \otimes k(x) \longrightarrow 0$$

we deduce that the map

$$H^0(X, \mathcal{G}_{r_0}) \to H^0(X, \mathcal{G}_{r_0} \otimes k(x))$$

is surjective. In other words, $\mathcal{G}_{r_0}$ is globally generated at x. It implies there exists an open U containing x such that $\mathcal{G}_{r_0}$ is globally generated over U. Since St_{r_0} is self dual, there is a canonical surjective map

$$\mathrm{St}_{r_0}^{\otimes 2} \to \mathcal{O}_X\,.$$

Since the tensor product of globally generated sheaves over U is again globally generated over U, we deduce that

$$\mathcal{G}_{r_0}^{\otimes 2} = \mathrm{St}_{r_0}^{\otimes 2} \otimes (\mathcal{W}_\lambda^{\otimes 2})^{(p^{r_0})} \otimes \mathcal{O}_X(-2m)$$

is a globally generated over U. Since the quotient of a globally generated sheaf over U is globally generated over U, we know that

$$(\mathcal{W}_\lambda^{\otimes 2})^{(p^{r_0})}(-2m)$$

is globally generated sheaf over U. Now, let $r \geq r_0$ be an integer. From the equality

$$(\mathcal{W}_\lambda^{\otimes 2})^{(p^r)}(-2p^{r-r_0}m) = ((\mathcal{W}_\lambda^{\otimes 2})^{(p^{r_0})}(-2m))^{(p^{r-r_0})},$$

we deduce that $(\mathcal{W}_\lambda^{\otimes 2})^{(p^r)}(-2p^{r-r_0}m)$ is globally generated over U. Now take r_1 large enough to have $\mathcal{O}_X((2p^{r_1-r_0}-1)m)$ globally generated. We deduce that

$$(\mathcal{W}_\lambda^{\otimes 2})^{(p^r)}(-2p^{r-r_0}m)\otimes\mathcal{O}_X((2p^{r-r_0}-1)m)=(\mathcal{W}_\lambda^{\otimes 2})^{(p^r)}(-m)$$

is globally generated over U for all $r\geq r_1$. Since X is quasi-compact, we can find an integer $r_2\geq r_1$ such that $(\mathcal{W}_\lambda^{\otimes 2})^{(p^r)}(-m)$ is globally generated over X for all $r\geq r_2$. We use Proposition 4.13 to deduce that $\mathcal{W}_\lambda^{\otimes 2}$ is p-ample. By Proposition 4.14 and 4.8, it implies that $\mathcal{W}_\lambda$ is ample. $\square$

We give a sufficient cohomological condition for a vector bundle to be (φ, D)-ample.

Proposition 6.3. *Let $\mathcal{E}$ be a vector bundle over X. Let $\lambda\in X^*(T)$ be a character. If there exists an effective Cartier divisor D and an integer $r_0\geq 1$ such that for all $r\geq r_0$ and all coherent sheaf $\mathcal{F}$ over X, there is an integer $r_1\geq 1$ such that for all $r'\geq r_1$ and $i>0$, we have*

$$H^i(X,\mathcal{F}\otimes\mathrm{St}_{r+r'}\otimes\mathcal{W}_\lambda^{(p^{r+r'})}(-p^{r'}D))=0,$$

then $\mathcal{W}_\lambda$ is (φ, D)-ample.

Proof. Using Propositions 4.13 and 4.14, it is sufficient to see that there exists integers $n\geq 1$ and $r_0\geq 1$ such that for all $r\geq r_0$, the bundle $\mathcal{W}_\lambda^{(p^r)}(-nD)$ is p-ample. In other words, it is sufficient to see that there exists integers $n\geq 1$ and $r_0\geq 1$ such that for all $r\geq r_0$ and all $m\geq 1$, there exists $r_1\geq 1$, such that for all $r'\geq r_1$, the bundle

$$\mathcal{W}_\lambda^{(p^{r+r'})}(-p^{r'}nD)(-m)$$

is globally generated over X. By hypothesis, we have a Cartier divisor D and $r_0\geq 1$ an integer. Consider two integers $r\geq r_0$, $m\geq 0$ and write

$$\mathcal{G}_{r'}=\mathrm{St}_{r+r'}\otimes\mathcal{W}_\lambda^{(p^{r+r'})}(-p^{r'}D)\otimes\mathcal{O}_X(-m).$$

Let $x\in X$ be a closed point. By hypothesis, we have an integer $r_1\geq 1$ such that

$$H^1(X,\mathcal{G}_{r_1}\otimes\mathcal{I}_x)=0,$$

where $\mathcal{I}_x$ is the ideal sheaf defining the closed point x. From the long exact sequence of cohomology associated to the exact sequence

$$0\longrightarrow\mathcal{G}_{r_1}\otimes\mathcal{I}_x\longrightarrow\mathcal{G}_{r_1}\longrightarrow\mathcal{G}_{r_1}\otimes k(x)\longrightarrow 0$$

we deduce that the map

$$H^0(X,\mathcal{G}_{r_1})\to H^0(X,\mathcal{G}_{r_1}\otimes k(x))$$

is surjective. In other words, $\mathcal{G}_{r_1}$ is globally generated at x. It implies there exists an open U containing x such that $\mathcal{G}_{r_1}$ is globally generated over U. Since the Steinberg module is self-dual, there is a canonical surjective map

$$\mathrm{St}_{r+r_1}^{\otimes 2}\to\mathcal{O}_X\,.$$

Since the tensor product of globally generated sheaves over U is again globally generated over U, we deduce that

$$\mathcal{G}_{r_1}^{\otimes 2}=\mathrm{St}_{r+r_1}^{\otimes 2}\otimes(\mathcal{W}_\lambda^{\otimes 2})^{(p^{r+r_1})}(-2p^{r_1}D)\otimes\mathcal{O}_X(-2m)$$

is globally generated over U. Since the quotient of a globally generated sheaf over U is globally generated over U, we know that

$$(\mathcal{W}_\lambda^{\otimes 2})^{(p^{r+r_1})}(-2p^{r_1}D)\otimes\mathcal{O}_X(-2m)$$

https://doi.org/10.1112/S0010437X24007607

is a globally generated sheaf over U. From the equality

$$((\mathcal{W}_\lambda^{\otimes 2})^{(p^{r+r_1})}(-2p^{r_1}D)(-2m))^{(p^{r'-r_1})} = (\mathcal{W}_\lambda^{\otimes 2})^{(p^{r+r'})}(-2p^{r'}D)(-2p^{r'-r_1}m),$$

we deduce that $(\mathcal{W}_\lambda^{\otimes 2})^{(p^{r+r'})}(-2p^{r'}D)(-2p^{r'-r_1}m)$ is globally generated over U. Now take $r_2 \geq r_1$ large enough to have $\mathcal{O}_X((2p^{r'-r_1}-1)m)$ globally generated for all $r' \geq r_2$. We deduce that

$$\begin{aligned}(\mathcal{W}_\lambda^{\otimes 2})^{(p^{r+r'})}&(-2p^{r'}D)(-2p^{r'-r_1}m) \otimes \mathcal{O}_X((2p^{r'-r_1}-1)m)\\ &= (\mathcal{W}_\lambda^{\otimes 2})^{(p^{r+r'})}(-2p^{r'}D)(-m)\end{aligned}$$

is globally generated over U for all $r' \geq r_2$. Since X is quasi-compact, we can find an integer $r_3 \geq r_2$ such that $(\mathcal{W}_\lambda^{\otimes 2})^{(p^{r+r'})}(-2p^{r'}D)(-m)$ is globally generated over X for all $r' \geq r_3$. In conclusion, we have proven that $\mathcal{W}_\lambda^{\otimes 2}$ is $(\varphi, 2D)$-ample, which is equivalent to (φ, D)-ample by Proposition 4.26. Then, we use Proposition 4.31 to deduce that $\mathcal{W}_\lambda$ is (φ, D)-ample. □

Theorem 6.4. *Let $\lambda \in X^*(T)$ be character. If $\mathcal{L}_{2\lambda+2\rho}$ is ample on Y, then $\mathcal{W}_\lambda$ is ample on X.*

Proof. Consider a semi-simple cover of G (the existence is proved in [Jan03]), i.e. a finite surjective morphism $h: G' \to G$ of reductive groups with central kernel such that $G' = G_{\mathrm{sc}} \times T_1$ is a product of a semi-simple simply connected group G_{sc} with a torus T_1. Since ampleness can be tested after a pullback by a finite surjective morphism by Proposition 4.9, we can use Lemma 6.1 to assume that ρ is a genuine character. Assume that $\mathcal{L}_{\lambda+\rho}$ is ample and consider a coherent sheaf $\mathcal{F}$ on X. We have a Leray spectral sequence starting at the second page

$$E_2^{i,j} = H^i(X, \mathcal{F} \otimes R^j\pi_*(\mathcal{L}_{\lambda+\rho}^{\otimes p^r} \otimes \mathcal{L}_{-\rho})) \Rightarrow H^{i+j}(Y, \pi^*\mathcal{F} \otimes \mathcal{L}_{\lambda+\rho}^{\otimes p^r} \otimes \mathcal{L}_{-\rho}).$$

Since $\mathcal{L}_{\lambda+\rho}$ is ample on Y, it is also π-ample and we have

$$R^j\pi_*(\mathcal{L}_{\lambda+\rho}^{\otimes p^r} \otimes \mathcal{L}_{-\rho}) = 0$$

for all $j > 0$ and r large enough. We deduce that the spectral sequence degenerates at page 2 and we get isomorphisms

$$H^i(X, \mathcal{F} \otimes \pi_*(\mathcal{L}_{\lambda+\rho}^{\otimes p^r} \otimes \mathcal{L}_{-\rho})) = H^i(Y, \pi^*\mathcal{F} \otimes \mathcal{L}_{\lambda+\rho}^{\otimes p^r} \otimes \mathcal{L}_{-\rho})$$

for all $i \geq 0$ and r large enough. Moreover, since $\mathcal{L}_{\lambda+\rho}$ is ample, the right-hand side vanishes for $i > 0$ and r large enough. From Proposition 2.10, we know that

$$\pi_*(\mathcal{L}_{p^r(\lambda+\rho)-\rho}) = \mathrm{St}_r \otimes \mathcal{W}_\lambda^{(p^r)}$$

and from Proposition 6.2, we deduce that $\mathcal{W}_\lambda$ is ample. □

Theorem 6.5. *Let $\lambda \in X^*(T)$ be character. If $\mathcal{L}_{2\lambda+2\rho}$ is D-ample over Y, then $\mathcal{W}_\lambda$ is (φ, D)-ample on X.*

Proof. Since (φ, D)-ampleness can be tested after a pullback by a finite surjective morphism by Proposition 4.32, we use the same trick as in Theorem 6.4 to assume that ρ is a genuine character. Consider $r_0 \geq 1$ large enough such that $\mathcal{L}_{\lambda+\rho}^{\otimes p^r}(-D)$ is ample for all $r \geq r_0$. Let $\mathcal{F}$ be a coherent sheaf on X and $r \geq r_0$ integer. For all integers $r' \geq 1$, we have a Leray spectral sequence starting at the second page

$$\begin{aligned}E_2^{i,j} &= H^i(X, \mathcal{F} \otimes R^j\pi_*(\mathcal{L}_{\lambda+\rho}^{\otimes p^{r+r'}}(-p^{r'}D) \otimes \mathcal{L}_{-\rho}))\\ &\Rightarrow H^{i+j}(Y, \pi^*\mathcal{F} \otimes \mathcal{L}_{\lambda+\rho}^{\otimes p^{r+r'}}(-p^{r'}D) \otimes \mathcal{L}_{-\rho}).\end{aligned}$$

Since $\mathcal{L}_{\lambda+\rho}^{\otimes p^r}(-D)$ is π-ample, there is a $r_1 \geq 1$ large enough such that

$$R^j\pi_*(\mathcal{L}_{\lambda+\rho}^{\otimes p^{r+r'}}(-p^{r'}D)\otimes\mathcal{L}_{-\rho}) = R^j\pi_*((\mathcal{L}_{\lambda+\rho}^{\otimes p^r}(-D))^{\otimes p^{r'}}\otimes\mathcal{L}_{-\rho}) = 0$$

for all $j > 0$ and $r' \geq r_1$. We deduce that the spectral sequence degenerates at page 2 and we get isomorphisms

$$H^i(X, \mathcal{F}\otimes\pi_*(\mathcal{L}_{\lambda+\rho}^{\otimes p^{r+r'}}(-p^{r'}D)\otimes\mathcal{L}_{-\rho})) = H^i(Y, \pi^*\mathcal{F}\otimes\mathcal{L}_{\lambda+\rho}^{\otimes p^{r+r'}}(-p^{r'}D)\otimes\mathcal{L}_{-\rho})$$

for all $i \geq 0$ and $r' \geq r_1$. Since $\mathcal{L}_{\lambda+\rho}^{\otimes p^r}(-D)$ is ample on Y, there exists $r_2 \geq r_1$ such that we have

$$H^i(Y, \pi^*\mathcal{F}\otimes(\mathcal{L}_{\lambda+\rho}^{\otimes p^r}(-D))^{\otimes p^{r'}}\otimes\mathcal{L}_{-\rho}) = 0$$

for all $i > 0$ and $r' \geq r_2$. From Proposition 2.10, we know that

$$\pi_*(\mathcal{L}_{p^{r+r'}(\lambda+\rho)-\rho}(-p^{r'}D)) = \mathrm{St}_{r+r'}\otimes\mathcal{W}_\lambda^{(p^{r+r'})}(-p^{r'}D),$$

which implies that

$$H^i(X, \mathcal{F}\otimes\mathrm{St}_{r+r'}\otimes\mathcal{W}_\lambda^{(p^{r+r'})}(-p^{r'}D)) = 0$$

for all $r' \geq r_2$. We deduce with the technical Proposition 6.3 that $\mathcal{W}_\lambda$ is (φ, D)-ample. $\square$

7. Positivity of automorphic vector bundles on the Siegel variety

In this section, we prove that certain automorphic bundles on the Siegel modular variety are (φ, D)-ample for some effective Cartier divisor D.

7.1 Recollection on Siegel modular varieties

We start by recalling some well-known results from [FC90] on Siegel modular varieties and their toroidal compactifications. We denote by Sch_R the category of schemes over a ring R.

Definition 7.1. Let V be the $\mathbb{Z}$-module $\mathbb{Z}^{2g}$ endowed with the standard non-degenerate symplectic pairing

$$\psi : V\times V \longrightarrow \mathbb{Z}$$
$$(x,y) \longmapsto {}^t x J y$$

where

$$J = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}.$$

We denote by Sp_{2g} the algebraic group over $\mathbb{Z}$ of $2g\times 2g$ matrices M that preserve the symplectic pairing ψ, i.e. such that

$${}^t MJM = J.$$

Definition 7.2 [FC90]. Let N be a positive integer such that $p \nmid N$. Recall that k is an algebraically closed field of characteristic p. Consider the fibered category in groupoids $\mathcal{A}_{g,N}$ on Sch_k whose S-points are groupoids with the following.

– Objects: (A, λ, ψ_N) where $A \to S$ is abelian scheme over S of relative dimension g, $\lambda : A \to A^\vee$ is a principal polarization and

$$\psi_N : A[N] \xrightarrow{\sim} \underline{(\mathbb{Z}/N\mathbb{Z})}_S^2$$

https://doi.org/10.1112/S0010437X24007607

is a basis over S of the N-torsion of A.

– Morphisms: A morphism $(A,\lambda,\psi_N)\to(A',\lambda',\psi'_N)$ is a scheme morphism $\alpha: A\to A'$ over S such that the diagram

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\lambda"] & A' \arrow[d, "\lambda'"] \\ A^\vee & A'^\vee \arrow[l, "\alpha^\vee"] \end{tikzcd}$$

is commutative and the pullback of ψ_N by α is ψ'_N.

Proposition 7.3 [FC90]. *For any integer $N\geq 3$ such that $p\nmid N$, the fibered category in groupoids $\mathcal{A}_{g,N}$ is representable by a smooth integral quasi-projective scheme over k.*

Notation 7.4. We denote by G the base change of the algebraic group Sp_{2g} over k. We fix a genus $g\geq 1$ and a level $N\geq 3$ such that $p\nmid N$. We denote simply by Sh the Siegel modular variety $\mathcal{A}_{g,N}$. Let μ be the following minuscule cocharacter of G.

$$\mu:\mathbb{G}_m \longrightarrow G$$

$$z \longmapsto \begin{pmatrix} zI_g & 0 \\ 0 & z^{-1}I_g \end{pmatrix}.$$

We denote by $P^+ := P_\mu$ and $P := P_{-\mu}$ the associated opposite parabolic subgroups with common Levi subgroup $L=\mathrm{GL}_g$ over k. We denote by $B\subset P$ the Borel of upper triangular matrices in $G=\mathrm{Sp}_{2g}$ over k. We denote by Φ_L (respectively, Φ_L^+) the corresponding roots of L (respectively, positive roots of L).

Definition 7.5 [FC90]. As a fine moduli space, the Siegel variety Sh is endowed with a universal principally polarized abelian scheme of relative dimension g

$$\begin{tikzcd} A \arrow[r, "f"] & \mathrm{Sh} \arrow[l, bend left, "e"] \end{tikzcd}$$

where $e:\mathrm{Sh}\to A$ is the neutral section. Recall the following associated objects on Sh.

(1) We denote by $\mathcal{H}^1_{\mathrm{dR}} := R^1f_*(\Omega^\bullet_{A/\,\mathrm{Sh}})$ the de Rham cohomology vector bundle of rank $2g$ over Sh.

(2) We denote by $\Omega = e^*\Omega^1_{A/\,\mathrm{Sh}}$ the Hodge vector bundle of rank g over Sh.

Note that the Weil paring and the principal polarization on the universal abelian scheme $f: A\to\mathrm{Sh}$ induce a symplectic pairing of the same type as ψ on $\mathcal{H}^1_{\mathrm{dR}}$. In other words, the de Rham cohomology is equivalent to the data of a G-torsor on Sh.

Proposition 7.6 [DI87]. *The Hodge–de Rham spectral sequence*

$$E_1^{i,j} = R^jf_*(\Omega^i_{A/\,\mathrm{Sh}}) \Rightarrow R^{i+j}f_*(\Omega^\bullet_{A/\,\mathrm{Sh}})$$

degenerates at page 1 which proves the existence of the Hodge–de Rham filtration

$$0 \longrightarrow \Omega \longrightarrow \mathcal{H}^1_{\mathrm{dR}} \longrightarrow R^1f_*\mathcal{O}_A \longrightarrow 0.$$

Moreover, the Hodge bundle Ω is totally isotropic for the symplectic pairing on $\mathcal{H}^1_{\mathrm{dR}}$ which implies that the Hodge–de Rham filtration is equivalent to the data of a P-reduction of the G-torsor $\mathcal{H}^1_{\mathrm{dR}}$ on the Siegel variety.

In the next definition, we recall the main properties of toroidal compactifications of Siegel varieties.

Definition 7.7 [FC90, Chapter 4], [Lan12, Th. 2.15]. Let C denote the cone of all positive semi-definite symmetric bilinear forms on $X^* \otimes_{\mathbb{Z}} \mathbb{R}$ with radicals defined over $\mathbb{Q}$. Following the definitions [FC90, Chapter 4, Definition 2.2/2.3], we consider a smooth $GL(X^*)$-admissible decomposition $\Sigma = \{\sigma_\alpha\}_\alpha$ in polyhedral cones of C. Following Definition [FC90, Chapter 4, Definition 2.4], we assume furthermore that Σ admits a $GL(X^*(T))$-equivariant polarization function. The existence of a polyhedral cone decomposition Σ satisfying these assumptions is ensured by [AMRT10] and [KKMS73]. Denote by $\mathrm{Sh}^{\mathrm{tor}}$ the toroidal compactification of the Siegel variety associated to Σ. It follows from the assumptions on Σ that $\mathrm{Sh}^{\mathrm{tor}}$ is a smooth projective scheme over k satisfying the following assertions.

(1) The boundary $D_{\mathrm{red}} = \mathrm{Sh}^{\mathrm{tor}} - \mathrm{Sh}$ with its reduced structure is an effective Cartier divisor with normal crossings.

(2) The universal abelian scheme $f : A \to \mathrm{Sh}$ extends to a semi-abelian scheme $f^{\mathrm{tor}} : A^{\mathrm{tor}} \to \mathrm{Sh}^{\mathrm{tor}}$.

(3) The sheaf $\Omega^{\mathrm{tor}} := e^*\Omega^1_{A^{\mathrm{tor}}/\mathrm{Sh}^{\mathrm{tor},\Sigma}}$ is a vector bundle of rank g that extends the Hodge bundle Ω to $\mathrm{Sh}^{\mathrm{tor},\Sigma}$.

(4) By [FC90, Chapter 4] or [Lan12, Th. 2.15, (2)] there exists a log-smooth projective compactification $\bar{f}^{\mathrm{tor}} : \bar{A}^{\mathrm{tor}} \to \mathrm{Sh}^{\mathrm{tor}}$ of the semi-abelian scheme $f^{\mathrm{tor}} : A^{\mathrm{tor}} \to \mathrm{Sh}^{\mathrm{tor}}$ and we again denote by D_{red} the divisor with normal crossings $\bar{A}^{\mathrm{tor}} - A$.

(5) By [FC90, Chapter 4] or [Lan12, Th. 2.15, (3)], the log-de Rham cohomology

$$\mathcal{H}^1_{\log-\mathrm{dR}} := R^1(\bar{f}^{\mathrm{tor}})_*\bar{\Omega}^\bullet_{\bar{A}^{\mathrm{tor}}/\mathrm{Sh}^{\mathrm{tor}}},$$

where $\bar{\Omega}^\bullet_{\bar{A}^{\mathrm{tor}}/\mathrm{Sh}^{\mathrm{tor}}}$ is the complex of log-differentials

$$\begin{aligned}\bar{\Omega}^i_{\bar{A}^{\mathrm{tor}}/\mathrm{Sh}^{\mathrm{tor}}} &= \Lambda^i\bar{\Omega}^1_{\bar{A}^{\mathrm{tor}}/\mathrm{Sh}^{\mathrm{tor}}}\\ &= \Lambda^i\Omega^1_{\bar{A}^{\mathrm{tor}}}(\log D_{\mathrm{red}})/(\bar{f}^{\mathrm{tor}})^*\Omega^1_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}}),\end{aligned}$$

is a Sp_{2g}-torsor that extends the de Rham cohomology $\mathcal{H}^1_{\mathrm{dR}}$ to $\mathrm{Sh}^{\mathrm{tor}}$.

(6) The logarithmic Hodge–de Rham spectral sequence

$$E_1^{i,j} = R^j(\bar{f}^{\mathrm{tor}})_*\bar{\Omega}^i_{\bar{A}^{\mathrm{tor}}/\mathrm{Sh}^{\mathrm{tor}}} \Rightarrow \mathcal{H}^i_{\log-\mathrm{dR}} := R^i(\bar{f}^{\mathrm{tor}})_*\bar{\Omega}^\bullet_{\bar{A}^{\mathrm{tor}}/\mathrm{Sh}^{\mathrm{tor}}}$$

degenerates at page 1, which proves the existence of a P-reduction of the Sp_{2g}-torsor $\mathcal{H}^1_{\log-\mathrm{dR}}$ extending the Hodge–de Rham filtration to $\mathrm{Sh}^{\mathrm{tor}}$.

The Hodge line bundle $\omega = \det \Omega^{\mathrm{tor}}$ is usually not ample on the Siegel variety $\mathrm{Sh}^{\mathrm{tor}}$ but it satisfies a weaker positivity result we explain. We recall the definition of the minimal compactification of the Siegel variety.

Definition 7.8 [FC90, Chapter V]. The minimal compactification $\mathrm{Sh}^{\min}$ of the Siegel variety Sh is defined as the scheme

$$\mathrm{Proj}(\oplus_{n\geq 0} H^0(\mathrm{Sh}^{\mathrm{tor}}, \omega^{\otimes n})),$$

where $\omega = \det \Omega^{\mathrm{tor}}$ is the Hodge line bundle.

Proposition 7.9 [Mor85, Chapter IX, Theorem 2.1, p. 208]. *The Hodge line bundle ω is semi-ample on $\mathrm{Sh}^{\mathrm{tor}}$, i.e. there exists an integer $m \geq 1$ such that $w^{\otimes m}$ is globally generated over*

$\mathrm{Sh}^{\mathrm{tor}}$. *In particular, the Hodge line bundle descends to an ample line bundle on the minimal compactification.*

Proposition 7.10 [FC90, Chapter V, Theorem 5.8]. *The toroidal compactification* $\mathrm{Sh}^{\mathrm{tor}}$ *is the normalization of the blow-up of* $\mathrm{Sh}^{\mathrm{min}}$

$$\nu : \mathrm{Sh}^{\mathrm{tor}} \to \mathrm{Sh}^{\mathrm{min}}$$

along a coherent sheaf of ideals $\mathcal{I}$ *of* $\mathcal{O}_{\mathrm{Sh}^{\mathrm{min}}}$.

In particular, the pullback $\nu^*\mathcal{I}$ is of the form $\mathcal{O}_{\mathrm{Sh}^{\mathrm{tor}}}(-D)$ where D is an effective Cartier divisor whose associated reduced Cartier divisor is the boundary D_{red}. It follows from the ampleness of ω on $\mathrm{Sh}^{\mathrm{min}}$ and the ν-ampleness of $\mathcal{O}_{\mathrm{Sh}^{\mathrm{tor}}}(-D)$ that there exists $\eta_0 > 0$ such that $\omega^{\otimes \eta}(-D)$ is ample for every $\eta \geq \eta_0$. In other words, we have the following result.

Corollary 7.11. *The Hodge line bundle* $\omega = \det \Omega^{\mathrm{tor}}$ *is* D*-ample on the toroidal compactification* $\mathrm{Sh}^{\mathrm{tor}}$.

Remark 7.12. The effective Cartier divisor D appearing in the corollary obviously depends on the choice of the $GL(X^*)$-equivariant polarization function on the decomposition in polyhedral cones Σ.

7.2 Automorphic vector bundles

We define the automorphic vector bundles over the Siegel variety. We choose an intermediary parabolic subgroup $P_0 \subset P$ of type $I_0 \subset I \subset \Delta$ and we denote by $P_{0,L} := P_0 \cap L \subset L$ the parabolic subgroup of L.

Definition 7.13. We define the flag bundle $\pi : Y_{I_0}^{\mathrm{tor}} \to \mathrm{Sh}^{\mathrm{tor}}$ of type I_0 as the flag bundle $\mathcal{F}_{P_{0,L}}(\Omega^{\mathrm{tor}})$ (as in Definition 5.4) of type $P_{0,L}$ of the L-torsor Ω^{tor}.

Definition 7.14. From Definitions 5.6 and 5.7, we have functors

$$\mathcal{W} : \mathrm{Rep}(L) \to \mathrm{Loc}(\mathcal{O}_{\mathrm{Sh}^{\mathrm{tor}}}),$$
$$\mathcal{L} : \mathrm{Rep}(P_{0,L}) \to \mathrm{Loc}(\mathcal{O}_{Y_{I_0}^{\mathrm{tor}}}),$$

and we call any vector bundle in the essential image of these functors an automorphic bundle. Moreover, if λ is a character of P_0, we denote by $\nabla(\lambda)$ the automorphic vector bundle $\mathcal{W}(\mathrm{Ind}_{P_{0,L}}^{L} \lambda)$ on $\mathrm{Sh}^{\mathrm{tor}}$ and $\mathcal{L}_\lambda$ the automorphic line bundle $\mathcal{L}(\lambda)$ on $Y_{I_0}^{\mathrm{tor}}$. With our conventions the module $\mathrm{Ind}_{P_{0,L}}^{L} \lambda$ is isomorphic to the costandard representation of highest weight $w_0 w_{0,L} \lambda$.

Corollary 7.15. *Let* λ *be a dominant character of* P_0. *We have an isomorphism of vector bundles*

$$R\pi_* \mathcal{L}_\lambda = \nabla(\lambda)[0].$$

Proof. The proof is a direct consequence of Proposition 5.10. □

Example 7.16. We have the following special cases.

(1) If $\lambda = (0, \ldots, 0, -1)$, then $\nabla(\lambda) = \Omega^{\mathrm{tor}}$;
(2) If $\lambda = (0, \ldots, 0, -n)$ with $n \geq 1$, then $\nabla(\lambda) = \mathrm{Sym}^n \Omega^{\mathrm{tor}}$.
(3) If $\lambda = (-1, \ldots, -1)$, then $\nabla(\lambda) = \Lambda^g \Omega^{\mathrm{tor}} = \omega$.

We recall the Kodaira–Spencer isomorphism.

https://doi.org/10.1112/S0010437X24007607

PROPOSITION 7.17 [FC90, Chapter 3, § 9]. *The Kodaira–Spencer map on the toroidal compactification of the Siegel variety*

$$\rho_{KS} : \mathrm{Sym}^2\, \Omega^{\mathrm{tor}} \xrightarrow{\sim} \Omega^1_{\mathrm{Sh}^{\mathrm{tor}}}(\log D)$$

is an isomorphism between the automorphic bundle $\nabla(0, \dots, 0, -2)$ and the sheaf of logarithmic 1-differentials $\Omega^1_{\mathrm{Sh}^{\mathrm{tor}}}(\log D)$. Taking the determinant yields an isomorphism of line bundles

$$\Omega^d_{\mathrm{Sh}^{\mathrm{tor}}}(\log D) \simeq \nabla(-2\rho^L),$$

where d is the dimension of $\mathrm{Sh}^{\mathrm{tor}}$ and

$$\rho^L = \frac{1}{2} \sum_{\alpha \in \Phi^+ \setminus \Phi^+_L} \alpha.$$

We recall a result on the D-ampleness of automorphic line bundles $\mathcal{L}_\lambda$ that admits generalized Hasse invariants.

DEFINITION 7.18. Let λ be a character of T. For every coroot such that $\langle \lambda, \alpha^\vee \rangle \neq 0$, we set

$$\mathrm{Orb}(\lambda, \alpha^\vee) = \left\{ \frac{|\langle \lambda, w\alpha^\vee \rangle|}{|\langle \lambda, \alpha^\vee \rangle|} \mid w \in W \right\}$$

and we say that λ is:

(1) orbitally p-close if $\max_{\alpha \in \Phi} \mathrm{Orb}(\lambda, \alpha^\vee) \leq p - 1$;

(2) $\mathcal{Z}_\emptyset$-ample if $\langle \lambda, \alpha^\vee \rangle > 0$ for all $\alpha \in I$ and $\langle \lambda, \alpha^\vee \rangle < 0$ for all $\alpha \in \Phi^+ \setminus \Phi^+_L$.

The following result is due to [BGKS].

PROPOSITION 7.19 [Ale24, Theorem 5.11]. *Let λ be a character of T. If λ is orbitally p-close and $\mathcal{Z}_B$-ample, then $\mathcal{L}_\lambda$ is D-ample on Y^{tor}.*

We can now state and prove one of our main results.

THEOREM 7.20. *Let λ be a dominant character of T:*

(1) *if λ is a positive parallel weight, i.e. $\lambda = k(1, \dots, 1)$ with $k < 0$; or*

(2) *if $2\lambda + 2\rho_L$ is orbitally p-close and $\mathcal{Z}_\emptyset$-ample;*

then the automorphic vector bundle $\nabla(\lambda)$ is (φ, D)-ample on $\mathrm{Sh}^{\mathrm{tor}}$.

Proof. This a direct consequence from Theorem 6.5 and Proposition 7.19. ∎

To illustrate our result when $g = 2$, we represent the weights $\lambda = (k_1, k_2)$ such that the automorphic bundle $\nabla(\lambda)$ is (φ, D)-ample on the Siegel threefold for different values of p in Figure 1.

8. Hyperbolicity of the Siegel variety

8.1 The supersingular pencil of Moret-Bailly

Recall that k is an algebraically closed field of characteristic p. Denote by Sh_g the Siegel variety of genus g and full level $N \geq 3$ (with $p \nmid N$) over k and $\mathrm{Sh}^{\mathrm{tor}}_g$ a smooth toroidal compactification with boundary a normal crossing divisor D_{red}. Recall that D denotes the effective divisor supported on

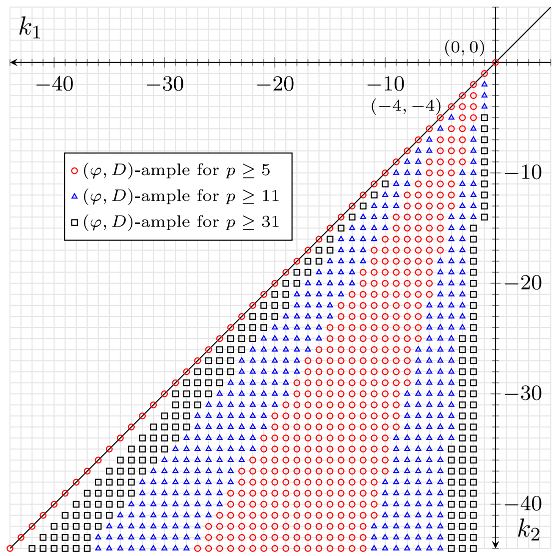


FIGURE 1 (colour online). The (φ, D)-ampleness of automorphic bundles $\nabla(\lambda)$ when $g = 2$.

the boundary that appears as the exceptional divisor of the blow-up from $\mathrm{Sh}_g^{\mathrm{tor}}$ to the minimal compactification of Sh_g. In [Mor81], Moret-Bailly constructed a non-isotrivial family $A \to \mathbb{P}_k^1$ of principally polarized supersingular abelian surfaces over the projective line with a full level N-structure. This family yields a closed immersion $\iota_2 : \mathbb{P}_k^1 \hookrightarrow \mathrm{Sh}_2$ whose image belongs to the supersingular locus of the Siegel threefold. In particular, we already know that $\mathrm{Sh}_g^{\mathrm{tor}}$ is not hyperbolic when $g = 2$. This family can be used to contradict the hyperbolicity of the Siegel variety for all $g \geq 2$: take an abelian variety A_0 of dimension $g - 2$ over k and consider the closed immersion $\iota := \iota_{A_0} \circ \iota_2$

$$\begin{tikzcd} \mathbb{P}_k^1 \arrow[r, "\iota_2"] \arrow[dr, "\iota"'] & \mathrm{Sh}_2^{\mathrm{tor}} \arrow[d, "\iota_{A_0}"] \\ & \mathrm{Sh}_g^{\mathrm{tor}} \end{tikzcd}$$

where ι_{A_0} sends an abelian surface A to the fibre product $A \times_k A_0$. It also shows that the logarithmic cotangent bundle $\Omega^1_{\mathrm{Sh}_g^{\mathrm{tor}}}(\log D_{\mathrm{red}})$ cannot be nef. Indeed, ι induces a surjective morphism

$$\iota^* \Omega^1_{\mathrm{Sh}_g^{\mathrm{tor}}}(\log D_{\mathrm{red}}) \to \Omega^1_{\mathbb{P}^1}$$

and if $\Omega^1_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}})$ was nef, it would imply that $\Omega^1_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(-1)$ is nef. In the rest of this subsection, we study more closely the non-positivity of certain automorphic bundles. Our goal is to show the following.

PROPOSITION 8.1. *Assume that* $g \in \{2, 3\}$. *Any automorphic bundle* $\nabla(k_1, \ldots, k_g)$ *on* $\mathrm{Sh}^{\mathrm{tor}}$ *where* $k_1 = 0$ *is not nef.*

Remark 8.2. In particular, we recover that the bundle $\Omega^1_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}}) = \nabla(0, \ldots, 0, -2)$ is not nef. We believe that this result generalizes to every $g \geq 2$.

Proof. Consider a dominant character λ of T and write $I_0 \subset I$ for the set of simple roots such that $\langle \lambda, \alpha^\vee \rangle = 0$. As a consequence, the line bundle $\mathcal{L}_\lambda$ on $Y_{I_0}^{\mathrm{tor}}$ is relatively π-ample which implies that we have a surjective map for some $n \geq 1$ large enough

$$\pi^* \pi_* \mathcal{L}_\lambda^{\otimes n} = \pi^* \nabla(n\lambda) \to \mathcal{L}_\lambda^{\otimes n}.$$

In particular, if $\nabla(\lambda)$ was nef, it would imply that $\nabla(\lambda)^{\otimes n}$, hence $\nabla(n\lambda)$ and $\mathcal{L}_\lambda$ would be nef. We are reduced to show the non-nefness of $\mathcal{L}_\lambda$ on $Y_{I_0}^{\mathrm{tor}}$, which can be tested on $Y_\emptyset^{\mathrm{tor}}$. We claim that we can always find an Ekedahl–Oort (EO) stratum $Y_{I_0,w}^{\mathrm{tor}}$ such that the following intersection product is negative

$$c_1(\mathcal{L}_\lambda)^{l(w)} \cdot [\overline{Y_{I_0}^{\mathrm{tor}}}] < 0.$$

These intersection computations are done in Appendix A. □

8.2 Understanding the failure of hyperbolicity in positive characteristic

We have seen that $\Omega^1_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}})$ cannot be nef as we can always see $\mathbb{P}^1$ as a closed curve in $\mathrm{Sh}^{\mathrm{tor}}$. Consider a partition λ with height $\mathrm{ht}(\lambda) \leq \dim \mathrm{Sh}^{\mathrm{tor}}$ and denote by S_λ the corresponding Schur functor

$$S_\lambda : \mathrm{Loc}(\mathcal{O}_{\mathrm{Sh}^{\mathrm{tor}}}) \to \mathrm{Loc}(\mathcal{O}_{\mathrm{Sh}^{\mathrm{tor}}})$$

as a strict polynomial functor on the category of locally free modules of finite rank over $\mathrm{Sh}^{\mathrm{tor}}$. We start with the following lemma.

Lemma 8.3. *If $S_\lambda \Omega^1_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}})$ is (φ, D)-ample and $\iota : V \hookrightarrow \mathrm{Sh}^{\mathrm{tor}}$ is any subvariety such that:*

(1) *V is smooth;*

(2) *$\iota^{-1} D_{\mathrm{red}}$ is a normal crossing divisor;*

(3) *$\dim V \geq ht(\lambda)$;*

then the logarithmic canonical bundle $\omega_V(\iota^{-1} D_{\mathrm{red}})$ is $(\varphi, \iota^{-1} D)$-ample. In particular, it is nef and big with exceptional locus contained in the boundary and V is of log general type with respect to D.

Proof. The surjective morphism

$$\iota^* \Omega^1_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}}) \to \Omega^1_V(\log \iota^{-1} D_{\mathrm{red}})$$

induces a surjective morphism

$$\iota^* S_\lambda \Omega^1_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}}) \to S_\lambda \Omega^1_V(\log \iota^{-1} D_{\mathrm{red}})$$

and by Proposition 4.32 and 4.29, we deduce that $S_\lambda \Omega^1_V(\log \iota^{-1} D_{\mathrm{red}})$ is $(\varphi, \iota^{-1} D)$-ample. Since $\mathrm{ht}(\lambda) \leq \dim V$, the bundle

$$\det S_\lambda \Omega^1_V(\log \iota^{-1} D_{\mathrm{red}}) = (\omega_V(\iota^{-1} D_{\mathrm{red}}))^{\otimes(|\lambda| \dim \nabla(\lambda))/g}$$

is non-zero and $(\varphi, \iota^{-1} D)$-ample. We conclude with Proposition 4.31. □

With this fundamental lemma in mind, the aim is to find partitions λ that ensure the (φ, D)-ampleness of $S_\lambda \Omega^1_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}})$, which is isomorphic to $S_\lambda \mathrm{Sym}^2 \Omega^{\mathrm{tor}}$ by the Kodaira–Spencer isomorphism (Proposition 7.17). Recall that under the assumption $p \geq 2|\lambda| - 1$, the plethysm $S_\lambda \circ \mathrm{Sym}^2$ is filtered by Schur functors S_η by Proposition 3.16. This allows us to state the following lemma.

Lemma 8.4. *Let λ be a partition and assume that $p \geq 2|\lambda| - 1$. If $S_\lambda \circ \mathrm{Sym}^2$ is filtered by Schur functors S_η such that $S_\eta \Omega^{\mathrm{tor}}$ is (φ, D)-ample or zero, then $S_\lambda \Omega^1_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}})$ is (φ, D)-ample.*

Proof. Since $\mathrm{Sh}^{\mathrm{tor}}$ is smooth, this is a direct consequence of Proposition 4.28. □

Remark 8.5. If η has more than g parts, then $S_\eta \Omega^{\mathrm{tor}} = 0$. Otherwise, $S_\eta \Omega^{\mathrm{tor}} = \nabla(w_0 w_{0,L} \eta)$.

By Theorem 7.20, we are reduced to find a partition λ such that all the partition η with at most g parts appearing in the plethysm $S_\lambda \circ \mathrm{Sym}^2$ are such that $2\rho_L + 2w_0 w_{0,L}\eta$ is orbitally p-close and $\mathcal{Z}_\emptyset$-ample. See Appendix B for some explicit plethysm computations in the cases $g = 2, 3, 4$ which helped us to build some intuition about the general case.

8.2.1 *The general case.* Consider the Siegel variety $\mathrm{Sh}^{\mathrm{tor}}$ of genus g over k. In this section, we prove the following result.

THEOREM 8.6. *Assume that* $p \geq g^2 + 3g + 1$. *For all* $k \geq g(g-1)/2 + 1$, *the bundle* $\Omega^k_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}})$ *is* (φ, D)*-ample.*

COROLLARY 8.7. *Assume that* $p \geq g^2 + 3g + 1$. *Any subvariety* $\iota : V \hookrightarrow \mathrm{Sh}^{\mathrm{tor}}$ *of codimension* $\leq g - 1$ *satisfying:*

(1) V *is smooth;*
(2) $\iota^{-1} D_{\mathrm{red}}$ *is a normal crossing divisor;*

is of log general type with respect to D.

Recall the following conjecture.

CONJECTURE 8.8 (Green–Griffiths–Lang). Let X be an irreducible projective complex variety. Denote by $\mathrm{Exc}(X)$ the Zariski closure of the union of the images of all non-constant holomorphic maps $\mathbb{C} \to X$. Then X is of general type if and only if $\mathrm{Exc}(X) \neq X$.

Remark 8.9. The Green–Griffiths–Lang conjecture fails in positive characteristic. Specifically, in characteristic $p > 0$, there exist unirational surfaces of general type. These surfaces are dominated by the projective plane $\mathbb{P}^2$ via rational maps, yet they possess a big canonical bundle, classifying them as surfaces of general type. This phenomenon contradicts the expectation from the conjecture that varieties of general type should exhibit hyperbolic behavior and not admit non-constant rational curves.

Motivated by the Green–Griffiths–Lang conjecture, we can formulate the following.

CONJECTURE 8.10. *For* p *large enough, there is a closed subscheme* $E \subset \mathrm{Sh}^{\mathrm{tor}}$ *such that for any subvariety* $\iota : V \to \mathrm{Sh}^{\mathrm{tor}}$ *satisfying:*

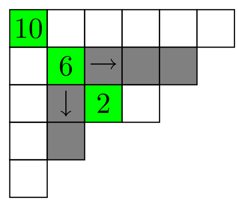
(1) V *is smooth;*
(2) $\iota^{-1} D_{\mathrm{red}}$ *is a normal crossing divisor;*

V *is of log general type if and only if* $V \not\subseteq E$.

Theorem 8.6 indicates that such an exceptional locus $E \subset \mathrm{Sh}^{\mathrm{tor}}$ should have codimension $> g - 1$. We believe it has exactly codimension g.

Proof of Theorem 8.6. The strategy is to study a ∇-filtration of the kth-exterior power of the bundle $\Omega^1_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}})$ and check that all the graded pieces are (φ, D)-ample automorphic vector bundles when $p \geq g^2 + 3g + 1$ and $k \geq g(g+1)/2 - (g-1)$. By the Kodaira–Spencer isomorphism of Proposition 7.17, the bundle $\Lambda^k \Omega^1_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}})$ is isomorphic to $\mathcal{W}(\Lambda^k \mathrm{Sym}^2 \mathrm{std}_{\mathrm{GL}_g})$. By Proposition 3.16, the GL_g-module $\Lambda^k \mathrm{Sym}^2 \mathrm{std}_{\mathrm{GL}_g}$ has a ∇-filtration when $p > k$ and it implies that $\Lambda^k\Omega^1_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}})$ is filtered by automorphic bundles $\nabla(w_0w_{0,L}\lambda)$ where the λ are the highest weights of the ∇-filtration of $\Lambda^k \operatorname{Sym}^2 \mathrm{std}_{\mathrm{GL}_g}$. As explained in Example 3.1, determining the Schur functors appearing in a plethysm $S_\lambda \circ S_\mu$ is often a hard task; however, the plethysm $\Lambda^k \circ \operatorname{Sym}^2$ belongs to the one of the few cases where a general formula is known. We start with a notation.

Notation 8.11. Let k be a positive integer and λ a partition of k in r distinct parts. We denote by $2[\lambda]$ the partition of $2k$ whose main-diagonal hook lengths are $2\lambda_1, \ldots, 2\lambda_r$, and whose ith-part has length $\lambda_i + i$. For example, we have

$2[(5,3,1)] =$  $= (6,5,4,2,1)$

where the diagonal hook have lengths $10, 6, 2$.

Lemma 8.12 [Wil09, Lemma 7]. *Assume that $p > k$. Then the polynomial functor $\Lambda^k \circ \operatorname{Sym}^2$ has a filtration where the graded pieces are the Schur functors $S_{2[\lambda]}$ where λ range over the set of partitions of k in distinct parts.*

Example 8.13. Consider the case $k = 5$. The partitions of 5 in distinct parts are (5), $(4,1)$ and $(3,2)$. The plethysm $\Lambda^5 \circ \operatorname{Sym}^2$ is then filtered by the Schur functors $S_{2[(5)]} = S_{(6,1^4)}$, $S_{2[(4,1)]} = S_{(5,3,1^2)}$ and $S_{2[(3,2)]} = S_{(4,4,2)}$.

Since we evaluate this plethysm at the Hodge bundle Ω^{tor} which has rank g, we can discard the partitions $2[\lambda]$ of height strictly greater than g (for such partitions, the evaluation vanishes). Since the height of $2[\lambda]$ is λ_1, we want to study the (φ, D)-ampleness of the automorphic bundles

$$S_{2[\lambda]}\Omega^{\mathrm{tor}} = \mathcal{W}(\nabla(2[\lambda])) = \nabla(w_0w_{0,L}2[\lambda]),$$

where λ is partition of k in distinct parts with $\lambda_1 \leq g$. By Theorem 7.20, we know it is the case when $2w_0w_{0,L}2[\lambda] + 2\rho_L$ is $\mathcal{Z}_\emptyset$-ample and orbitally p-close. Even if the second condition is always satisfied for p large enough, the first condition may not be satisfied as explained in Appendix B.2.. In Proposition 8.1, we have seen that automorphic bundles of the form $\nabla(\eta)$ where $\eta = (\eta_1 \geq \cdots \geq \eta_g)$ is a dominant character such that $\eta_1 = 0$ are not nef, hence not (φ, D)-ample. Conversely, we will see that any automorphic bundle $\nabla(\eta)$, where η is a dominant character such that $\eta_1 \leq -1$, is (φ, D)-ample if p is greater than a specific bound which depends on η. We start with the following lemma.

Lemma 8.14. *Consider two GL_g-dominant character $\lambda = (\lambda_1 \geq \cdots \geq \lambda_g \geq 0)$ and $\mu = (\mu \geq \cdots \geq \mu \geq 0)$. The GL_g-module $\nabla(\lambda) \otimes \nabla(\mu)$ is filtered by costandard modules $\nabla(\eta)$ such that $\eta_g \geq \lambda_g + \mu_g$ and $\eta_1 \leq \lambda_1 + \mu_1$.*

Proof of Lemma. See Proposition 2.13 for the existence of the ∇-filtration. The tensor product of two polynomial representation of GL_g is still a polynomial representation. Apply it to $\nabla(\lambda - (\lambda_g^g)) \otimes \nabla(\mu - (\mu_g^g))$ where $(\lambda_g^g) = (\lambda_g, \ldots, \lambda_g)$ and $(\mu_g^g) = (\mu_g, \ldots, \mu_g)$ to get the first inequality. The second inequality follows from the fact that $\lambda + \mu$ is the highest weight of $\nabla(\lambda) \otimes \nabla(\mu)$. □

Proposition 8.15. *Let $\eta = (\eta_1 \geq \cdots \geq \eta_g)$ be a dominant character such that $\eta_1 \leq -1$. Then the automorphic bundle $\nabla(\eta)$ is (φ, D)-ample if $p \geq (g+1)|\eta_g| + g$.*

https://doi.org/10.1112/S0010437X24007607

Proof of the proposition. By Proposition 4.31, it is enough to show that $\nabla(\eta)^{\otimes n}$ is (φ, D)-ample for some $n \geq 1$. By Lemma 8.14, the bundle $\nabla(\eta)^{\otimes n}$ is filtered by automorphic bundles of the form $\nabla(\delta)$ where $\delta_1 \leq n\eta_1$ and $\delta_g \geq n\eta_g$. To apply Theorem 7.20, we need to see that each $2\delta + 2\rho_L$ is $\mathcal{Z}_\emptyset$-ample and orbitally p-close. We first focus on the $\mathcal{Z}_\emptyset$-ampleness of $\gamma := 2\delta + 2\rho_L$. In other words, we need to check that

$$\begin{aligned}\gamma = 2\delta + 2\rho_L &= (2\delta_1, \dots, 2\delta_g) + (g-1, g-3, \dots, -(g-1)) \\ &= (2\delta_1 + g - 1, \dots, 2\delta_g - g + 1)\end{aligned}$$

is such that $0 > 2\delta_1 + g - 1 > 2\delta_2 + g - 3 > \cdots > 2\delta_g - g + 1$. The first inequality being the only one non-trivial, it is enough to have $n > (g-1)/2$ as it implies

$$\begin{aligned}2\delta_1 + g - 1 &\leq 2n\eta_1 + g - 1 \\ &\leq -2n + g - 1 \\ &< 0.\end{aligned}$$

For the orbitally p-closeness of $\gamma = 2\delta + 2\rho_L$, we have the following bound

$$\begin{aligned}\max_{\alpha\in\Phi, w\in W, \langle\gamma,\alpha^\vee\rangle\neq 0}\left|\frac{\langle\gamma, w\alpha^\vee\rangle}{\langle\gamma, \alpha^\vee\rangle}\right| &\leq \max_{1\leq i\leq j\leq g}\frac{|\gamma_j| + |\gamma_i|}{2} \\ &\leq \frac{2|\gamma_g|}{2} \\ &\leq |2\delta_g - (g-1)| \\ &\leq 2|\delta_g| + (g-1) \\ &\leq 2n|\eta_g| + (g-1) \quad \text{by Lemma 8.14}\end{aligned}$$

and we deduce that it is enough to have $2n|\eta_g| + g \leq p$. Combining it with the restriction $n = \lfloor (g-1)/2 \rfloor + 1 \leq (g+1)/2$ which ensure the $\mathcal{Z}_\emptyset$-ampleness of γ, we get

$$p \geq (g+1)|\eta_g| + g. \qquad \square$$

With Proposition 8.15 in mind, recall that we want to prove that the bundle

$$\nabla(w_0 w_{0,L} 2[\lambda])$$

is (φ, D)-ample when λ is a partition of k in distinct parts such that $\mathrm{ht}(2[\lambda]) = \lambda_1 \leq g$. If there exists such a partition λ with $\lambda_1 \leq g-1$, we will not be able to apply Proposition 8.15 to $w_0 w_{0,L} 2[\lambda]$ as the first term will be 0. To avoid these partitions, we prove the following lemma.

Lemma 8.16. *Assume that $p > k$. All the automorphic bundles $\nabla(\eta)$ appearing as graded pieces of the ∇-filtration of $\Lambda^k \operatorname{Sym}^2 \Omega^{\mathrm{tor}}$ satisfy $\eta_1 \leq -1$ if and only if $k \geq g(g-1)/2 + 1$.*

Proof of lemma. Assume that $k \geq g(g-1)/2 + 1$. We need to check that there exists no partition λ of k in distinct parts such that $\mathrm{ht}(2[\lambda]) = \lambda_1 \leq g-1$. Consider a partition λ of k in r-distinct parts. We have

$$\frac{g(g-1)}{2} + 1 \leq k = \lambda_1 + \lambda_2 + \cdots + \lambda_r \leq \frac{\lambda_1(\lambda_1+1)}{2},$$

which is possible only if $\lambda_1 \geq g$. Conversely, if $k \leq g(g-1)/2$, it is not hard to find a partition λ of k in distinct parts such that $\lambda_1 \leq g-1$. $\square$

Since $(2[\lambda])_1 = \lambda_1 + 1$, we conclude with Proposition 8.15 which says that each automorphic bundle $\nabla(w_0 w_{0,L} 2[\lambda])$ where $\lambda_1 = g$ is (φ, D)-ample when

$$p \geq g^2 + 3g + 1 = (g+1)\underbrace{|(w_0 w_{0,L} 2[\lambda])_g|}_{=g+1} + g. \qquad \square$$

A. Intersection computations on EO strata

We use the results of [WZ18] to do some computation on the Chow $\mathbb{Q}$-algebra of the partial flag bundle $\mathbb{P}(\Omega^{\mathrm{tor}})$. Recall that I denotes the type of the parabolic subgroup $P = P_{-\mu}$ of Sp_{2g}. Let I_0 denotes a subset of I and consider the morphisms

$$\zeta : \mathrm{Sh}^{\mathrm{tor}} \to \mathrm{Sp}_{2g}\text{-Zip}^{\mu}$$

and

$$\zeta_{I_0} : Y_{I_0}^{\mathrm{tor}} \to \mathrm{Sp}_{2g}\text{-ZipFlag}^{\mu, I_0}$$

as defined in [GK19a] and [GK19b]. For all $w \in {}^{I}W$, we denote by $\mathrm{Sh}_w^{\mathrm{tor}} := \zeta^{-1}([w])$ the EO stratum of the Siegel variety where $[w] \subset \mathrm{Sp}_{2g}\text{-Zip}^{\mu}$ is the corresponding substack. More generally,[15] for all $w \in {}^{I_0}W$, we denote by $Y_{I_0,w}^{\mathrm{tor}} := \zeta_{I_0}^{-1}([w])$ the EO stratum of the partial flag bundle of type $I_0 \subset I$ where $[w] \subset \mathrm{Sp}_{2g}\text{-ZipFlag}^{\mu, I_0}$ is the corresponding substack. The morphism ζ_{I_0} induces a pullback map on the corresponding Chow $\mathbb{Q}$-algebra

$$A^{\bullet}(\mathrm{Sp}_{2g}\text{-ZipFlag}^{\mu, I_0}) \xrightarrow{\zeta_{I_0}^*} A^{\bullet}(Y_{I_0}^{\mathrm{tor}})$$

and we call the image of $\zeta_{I_0}^*$ the tautological ring $\mathcal{T}_{I_0}$ of $Y_{I_0}^{\mathrm{tor}}$. Clearly, the Chow $\mathbb{Q}$-algebra of $\mathrm{Sp}_{2g}\text{-ZipFlag}^{\mu, I_0}$ is generated by the cycle classes of the EO strata $\overline{[w]}$ for $w \in {}^{I_0}W$ but we would like another description relying on Chern classes of automorphic bundles. We have a morphism of $\mathbb{Q}$-vector spaces

$$\begin{aligned} c_1 : X^*(T) &\longrightarrow A^1(\mathrm{Sp}_{2g}\text{-ZipFlag}^{\mu,\emptyset}) \\ \lambda &\longmapsto c_1(\mathcal{L}_\lambda) \end{aligned}$$

which induces a morphism of $\mathbb{Q}$-algebras $S \to A^{\bullet}(\mathrm{Sp}_{2g}\text{-ZipFlag}^{\mu,\emptyset})$ where $S = \mathrm{Sym}\, X^*(T)$ is the symmetric algebra of the characters of T. By [WZ18, Theorem 3], this map is surjective with kernel generated by the W-invariant elements of degree > 0. We deduce a description of the Chow $\mathbb{Q}$-algebra of $\mathrm{Sp}_{2g}\text{-ZipFlag}^{\mu,\emptyset}$ as

$$\begin{tikzcd} S \arrow[r] \arrow[d] & A^{\bullet}(\mathrm{Sp}_{2g}\text{-ZipFlag}^{\mu,\emptyset}) \\ S/\mathcal{I}S \arrow[ur] & \end{tikzcd}$$

where $\mathcal{I}$ is the augmentation ideal of the W-invariant elements of S. This ideal admits an explicit description as the augmentation ideal of a polynomial algebra

$$\mathcal{I} = \mathbb{Q}[f_1, \ldots, f_g]_{\geq 1}, \quad f_i = x_1^{2i} + \cdots + x_g^{2i}.$$

In particular, the tautological ring $\mathcal{T}_{\emptyset}$ is generated as a $\mathbb{Q}$-algebra by the cycle classes of the closed EO strata $\overline{Y_{\emptyset,w}^{\mathrm{tor}}}$ and by the Chern classes of the automorphic line bundles $c_1(\mathcal{L}_\lambda)$. The goal is now to express $[\overline{Y_{\emptyset,w}^{\mathrm{tor}}}]$ as an element of $S/\mathcal{I}S$ and to compute intersection products of the form

$$c_1(\mathcal{L}_\lambda)^{l(w)} \cdot [\overline{Y_{\emptyset,w}^{\mathrm{tor}}}].$$

[15] If $I_0 = I$, then $Y_{I_0}^{\mathrm{tor}} = \mathrm{Sh}^{\mathrm{tor}}$.

https://doi.org/10.1112/S0010437X24007607

Following the strategy of [WZ18], we have implemented on Sage an algorithm which computes $[\overline{Y^{\mathrm{tor}}_{\emptyset,w}}]$ as an element of $S/\mathcal{I}S$. In order to be more explicit, we choose a system of positive roots in a way to obtain

$$I = \{e_i - e_{i+1} \mid i = 1, \dots g-1\} \subset \Delta = \{e_i - e_{i+1} \mid i = 1, \dots g-1\} \cup \{2e_g\}.$$

The Weyl group $W = S_g \ltimes (\mathbb{Z}/2\,\mathbb{Z})^g$ contains $2^g g!$ elements we can write as a product of the simple reflections $s_1, s_2, \dots, s_g$ associated to $e_1 - e_2, \dots, e_{g-1} - e_g, 2e_g$.

A.1. The case $g = 2$

We represent the Weyl group of Sp_4 with a diagram

$$\begin{tikzcd}
 & w_0 = s_2s_1s_2s_1 \arrow[dl] \arrow[dr] & \\
w_1 = s_2s_1s_2 \arrow[d] \arrow[drr] & & w_1' = s_1s_2s_1 \arrow[d] \arrow[dll] \\
w_2 = s_2s_1 \arrow[d] \arrow[drr] & & w_2' = s_1s_2 \arrow[d] \arrow[dll] \\
w_3 = s_2 \arrow[dr] & & w_3' = s_1 \arrow[dl] \\
 & e &
\end{tikzcd}$$

where an arrow is drawn from w to w' if $w' \leq w$ and $l(w') = l(w) - 1$. Consider the line bundle $\mathcal{L}_\lambda$ on $Y^{\mathrm{tor}}_{\emptyset} = \mathbb{P}(\Omega^{\mathrm{tor}})$ where $\lambda = (k_1, k_2)$ and recall that $\mathcal{L}_{\lambda_\Omega} = \mathcal{L}_{(0,-1)} = \mathcal{O}(1)$. In the graded algebra

$$\mathcal{T}_\emptyset = \mathbb{Q}[x_1, x_2]/(x_1^2 + x_2^2, x_1^2 x_2^2),$$

we have the following formulas:

$$\begin{cases}
[\overline{Y^{\mathrm{tor}}_{\emptyset,w_0}}] = 1, \\
[\overline{Y^{\mathrm{tor}}_{\emptyset,w_1}}] = x_1 - p x_2, \\
[\overline{Y^{\mathrm{tor}}_{\emptyset,w_1'}}] = -(p-1)(x_1 + x_2), \\
[\overline{Y^{\mathrm{tor}}_{\emptyset,w_2}}] = -(p-1)(p x_1 + x_2) x_1, \\
[\overline{Y^{\mathrm{tor}}_{\emptyset,w_2'}}] = (p-1)(p x_2 - x_1) x_1, \\
[\overline{Y^{\mathrm{tor}}_{\emptyset,w_3}}] = (p^2-1)(p x_2 - x_1) x_1^2, \\
[\overline{Y^{\mathrm{tor}}_{\emptyset,w_3'}}] = (p^2+1)(p-1)(x_1^3 + x_2^3), \\
[\overline{Y^{\mathrm{tor}}_{\emptyset,e}}] = (p^4-1) x_1 x_2^3.
\end{cases}$$

Since the cycle $x_1 x_2^3$ has positive degree and since we are only concerned with the sign of the intersection products, we make the identification $x_1 x_2^3 = 1$ and we get the following intersection products:

$$\begin{cases} c_1(\mathcal{L}_\lambda)^4 \cdot [\overline{Y^{\mathrm{tor}}_{\emptyset,w_0}}] = (k_1x_1+k_2x_2)^4 = 4(k_1k_2^3-k_1^3k_2), \\ c_1(\mathcal{L}_\lambda)^3 \cdot [\overline{Y^{\mathrm{tor}}_{\emptyset,w_1}}] = (pk_1^3-3pk_1k_2^2-3k_1^2k_2+k_2^3), \\ c_1(\mathcal{L}_\lambda)^3 \cdot [\overline{Y^{\mathrm{tor}}_{\emptyset,w_1'}}] = (p-1)(k_1^3+3k_1^2k_2-3k_1k_2^2-k_2^3), \\ c_1(\mathcal{L}_\lambda)^2 \cdot [\overline{Y^{\mathrm{tor}}_{\emptyset,w_2}}] = (p-1)(k_1^2-k_2^2+2pk_1k_2), \\ c_1(\mathcal{L}_\lambda)^2 \cdot [\overline{Y^{\mathrm{tor}}_{\emptyset,w_2'}}] = (p-1)(p(k_2^2-k_1^2)+2k_1k_2), \\ c_1(\mathcal{L}_\lambda) \cdot [\overline{Y^{\mathrm{tor}}_{\emptyset,w_3}}] = (p^2-1)(k_2-pk_1), \\ c_1(\mathcal{L}_\lambda) \cdot [\overline{Y^{\mathrm{tor}}_{\emptyset,w_3'}}] = (p-1)(p^2+1)(k_1-k_2). \end{cases}$$

If $k_1=0$ and $k_2<0$, then $c_1(\mathcal{L}_\lambda)\cdot[\overline{Y^{\mathrm{tor}}_{\emptyset,w_3}}]=(p^2-1)k_2<0$, so $\mathcal{L}_\lambda$ is not nef.

A.2. The case $g=3$

The degree 9 part of the graded algebra $\mathcal{T}_\emptyset=\mathbb{Q}[x_1,x_2,x_3]/\mathcal{I}$ is a $\mathbb{Q}$-vector space of dimension 1 generated by $x_1^5x_2^3x_3$. We have

$$[\overline{Y^{\mathrm{tor}}_{\emptyset,e}}] = \underbrace{(p^9-p^8+p^7+2p^4-p^3+p^2-p+1)}_{>0}\, x_1^5x_2^3x_3$$

and since this polynomial in p is always positive, we may identify $x_1^5x_2^3x_3$ with 1. We have then

$$c_1(\mathcal{L}_{\lambda_\Omega})\cdot\overline{[Y^{\mathrm{tor}}_{\emptyset,s_3}]} = -p(p^5(p-1)-1)<0,$$

which shows that Ω^{tor}, hence $\Omega^1_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}})=\mathrm{Sym}^2\,\Omega^{\mathrm{tor}}$, is not nef.

B. Plethysm computations

The plethysm computations are accessible at github.com/ThibaultAlexandre/positivity-of-automorphic-bundles.

B.1. The case $g=2$

The Hodge bundle Ω^{tor} is locally free of rank 2 and $\Omega^1_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}})=\mathrm{Sym}^2\,\Omega^{\mathrm{tor}}$ is locally free of rank 3. Recall there is no need to assume that $p\geq 2|\lambda|-1$ when taking the highest exterior power. Under the assumption $p\geq 2\times 2-1=3$, we have

$$\begin{cases} \Lambda^3\,\mathrm{Sym}^2\,\Omega^{\mathrm{tor}} = \nabla(-3,-3), \\ \Lambda^2\,\mathrm{Sym}^2\,\Omega^{\mathrm{tor}} = \nabla(-1,-3), \\ \mathrm{Sym}^2\,\Omega^{\mathrm{tor}} = \nabla(0,-2). \end{cases}$$

Clearly, the line bundle $\nabla(-3,-3)$ is D-ample for any $p>0$ and $\nabla(0,-2)$ is never nef (hence, never (φ,D)-ample) by Proposition 8.1. For $\nabla(-1,-3)$, we need to check whether $(-1,-7)$ is orbitally p-close and $\mathcal{Z}_\emptyset$-ample. This condition is satisfied as soon as $p\geq 11$. By Lemma 8.3, this shows that any (good) subsurface of the Siegel threefold is of log general type when $p\geq 11$. Putting some extra effort, one can show that this result holds with $p=7$ as well. When $p\geq 2\times 4-1=7$, the bundle

$$S_{(2,2)}\circ\mathrm{Sym}^2\,\Omega^{\mathrm{tor}}$$

https://doi.org/10.1112/S0010437X24007607

is filtered by the automorphic vector bundles $\nabla(-2,-6)$ and $\nabla(-4,-4)$ which are (φ, D)-ample when $p \geq 7$. We get the following result.

Proposition B.1. *Assume that $g = 2$.*

(1) *If $p \geq 11$, then $\Lambda^2 \Omega^1_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}})$ is (φ, D)-ample.*
(2) *If $p = 7$, then $S_{(2,2)} \Omega^1_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}})$ is (φ, D)-ample.*

Corollary B.2. *Assume that $g = 2$ and $p \geq 7$. If $\iota : S \hookrightarrow \mathrm{Sh}^{\mathrm{tor}}$ is a subvariety of dimension ≥ 2 such that:*

(1) *S is smooth;*
(2) *$\iota^{-1} D_{\mathrm{red}}$ is a normal crossing divisor;*

then S is of log general type with respect to D.

B.2. The case $g = 3$

The Hodge bundle Ω^{tor} is locally free of rank 3 and $\Omega^1_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}}) = \mathrm{Sym}^2 \Omega^{\mathrm{tor}}$ is locally free of rank 6. Under the assumption $p \geq 2 \times 5 - 1 = 9$, we have

$$\begin{cases} \Lambda^6 \operatorname{Sym}^2 \Omega^{\mathrm{tor}} = \nabla(-4,-4,-4), \\ \Lambda^5 \operatorname{Sym}^2 \Omega^{\mathrm{tor}} = \nabla(-2,-4,-4), \\ \Lambda^4 \operatorname{Sym}^2 \Omega^{\mathrm{tor}} = \nabla(-1,-3,-4), \\ \Lambda^3 \operatorname{Sym}^2 \Omega^{\mathrm{tor}} \text{ is filtered by } \nabla(-1,-1,-4), \nabla(0,-3,-3), \\ \Lambda^2 \operatorname{Sym}^2 \Omega^{\mathrm{tor}} = \nabla(0,-1,-3), \\ \operatorname{Sym}^2 \Omega^{\mathrm{tor}} = \nabla(0,0,-2). \end{cases}$$

We deduce that $\Lambda^i \operatorname{Sym}^2 \Omega^{\mathrm{tor}}$ is (φ, D)-ample for $i = 5, 6$ when $p \geq 11$. For $i = 4$, we do not know if the bundle $\nabla(-1,-3,-4)$ is (φ, D)-ample since

$$2(-1,-3,-4) + 2\rho = (0,-6,-10)$$

is not $\mathcal{Z}_\emptyset$-ample. This incites us to consider the plethysm

$$S_{(2,2,2,2)} \circ \operatorname{Sym}^2 \Omega^{\mathrm{tor}},$$

which is filtered by $\nabla(-2,-6,-8)$, $\nabla(-3,-6,-7)$, $\nabla(-4,-4,-8)$ and $\nabla(-4,-6,-6)$ when $p \geq 2 \times 8 - 1 = 15$. These automorphic bundles are (φ, D)-ample when $p \geq 17$ by Theorem 7.20. It implies that $S_{(2,2,2,2)} \circ \operatorname{Sym}^2 \Omega^{\mathrm{tor}}$ is (φ, D)-ample when $p \geq 17$. We get the following result.

Proposition B.3. *Assume that $g = 3$.*

(1) *If $p \geq 11$, then $\Lambda^5 \Omega^1_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}})$ is (φ, D)-ample.*
(2) *If $p \geq 17$, then $S_{(2,2,2,2)} \Omega^1_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}})$ is (φ, D)-ample.*

Corollary B.4. *Assume that $g = 3$ and $p \geq 17$. If $\iota : V \hookrightarrow \mathrm{Sh}^{\mathrm{tor}}$ is a subvariety of dimension ≥ 4 such that:*

(1) *V is smooth;*
(2) *$\iota^{-1} D_{\mathrm{red}}$ is a normal crossing divisor;*

then V is of log general type with respect to D.

B.3. The case $g = 4$

The Hodge bundle Ω^{tor} is locally free of rank 4 and $\Omega^1_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}}) = \mathrm{Sym}^2\,\Omega^{\mathrm{tor}}$ is locally free of rank 10. Under the assumption $p \geq 2 \times 9 - 1 = 17$, we have

$$\begin{cases} \Lambda^{10}\,\mathrm{Sym}^2\,\Omega^{\mathrm{tor}} = \nabla(-5,-5,-5,-5), \\ \Lambda^{9}\,\mathrm{Sym}^2\,\Omega^{\mathrm{tor}} = \nabla(-3,-5,-5,-5), \\ \Lambda^{8}\,\mathrm{Sym}^2\,\Omega^{\mathrm{tor}} = \nabla(-2,-4,-5,-5), \\ \Lambda^{7}\,\mathrm{Sym}^2\,\Omega^{\mathrm{tor}} \text{ is filtered by } \nabla(-1,-4,-4,-5), \nabla(2,-2,-5,-5), \\ \Lambda^{6}\,\mathrm{Sym}^2\,\Omega^{\mathrm{tor}} \text{ is filtered by } \nabla(-1,-2,-4,-5), \nabla(0,-4,-4,-4), \\ \Lambda^{5}\,\mathrm{Sym}^2\,\Omega^{\mathrm{tor}} \text{ is filtered by } \nabla(-1,-1,-3,-5), \nabla(0,-2,-4,-4), \\ \Lambda^{4}\,\mathrm{Sym}^2\,\Omega^{\mathrm{tor}} \text{ is filtered by } \nabla(-1,-1,-1,-5), \nabla(0,-1,-3,-4), \\ \Lambda^{3}\,\mathrm{Sym}^2\,\Omega^{\mathrm{tor}} \text{ is filtered by } \nabla(-0,-1,-1,-4), \nabla(0,-0,-3,-3), \\ \Lambda^{2}\,\mathrm{Sym}^2\,\Omega^{\mathrm{tor}} = \nabla(-0,-0,-1,-3), \\ \mathrm{Sym}^2\,\Omega^{\mathrm{tor}} = \nabla(-0,-0,-0,-2). \end{cases}$$

We deduce that $\Lambda^i\,\mathrm{Sym}^2\,\Omega^{\mathrm{tor}}$ is (φ, D)-ample for $i = 8, 9, 10$ when $p \geq 17$. It does not work for $i = 7$ since the character

$$2(-1,-4,-4,-5) + 2\rho = (1,-7,-9,-13)$$

is not $\mathcal{Z}_{\emptyset}$-ample. Under the assumption $p \geq 2 \times 14 - 1 = 27$, the plethysm

$$S_{(2^7)} \circ \mathrm{Sym}^2\,\Omega^{\mathrm{tor}}$$

is filtered by the following list of automorphic bundles:

$$\begin{cases} \nabla(-2,-8,-8,-10), \\ \nabla(-3,-6,-9,-10), \\ \nabla(-3,-7,-9,-9), \\ \nabla(-3,-8,-8,-9), \\ \nabla(-4,-4,-10,-10), \\ \nabla(-4,-6,-8,-10), \\ \nabla(-4,-7,-8,-9), \\ \nabla(-4,-8,-8,-8), \\ \nabla(-5,-5,-9,-9), \\ \nabla(-5,-6,-8,-9), \\ \nabla(-5,-7,-7,-9), \\ \nabla(-6,-6,-6,-10), \\ \nabla(-6,-6,-8,-8), \\ \nabla(-7,-7,-7,-7), \end{cases}$$

which are all (φ, D)-ample when $p \geq 31$.

Proposition B.5. *Assume that $g = 4$.*

(1) *If $p \geq 17$, then $\Lambda^i \Omega^1_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}})$ is (φ, D)-ample for $i \geq 8$.*
(2) *If $p \geq 31$, then $S_{(2^7)} \Omega^1_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}})$ is (φ, D)-ample.*

https://doi.org/10.1112/S0010437X24007607

Corollary B.6. *Assume that $g = 4$ and $p \geq 31$. If $\iota : V \hookrightarrow \mathrm{Sh}^{\mathrm{tor}}$ is a subvariety of dimension ≥ 7 such that:*

(1) *V is smooth;*
(2) *$\iota^{-1} D_{\mathrm{red}}$ is a normal crossing divisor;*

then V is of log general type with respect to D.

Acknowledgements

I would like to thank Benoit Stroh for his invaluable support over the years and for his careful reading of an early version of this paper. I am very grateful to Yohan Brunebarbe for explaining to me his results on the hyperbolicity of the Siegel variety over the complex numbers and to Arnaud Eteve for many helpful conversations about Schur functors. I also thank more broadly Sebastian Bartling, Diego Berger and Thibaut Ménès for their careful feedback. I would also like to acknowledge the anonymous referee for their careful reading of this paper and for noticing that Theorem 1 should hold for non-smooth subvarieties as well.

Conflicts of interest

None.

Journal information

Compositio Mathematica is owned by the Foundation Compositio Mathematica and published by the London Mathematical Society in partnership with Cambridge University Press. All surplus income from the publication of *Compositio Mathematica* is returned to mathematics and higher education through the charitable activities of the Foundation, the London Mathematical Society and Cambridge University Press.

References

Abr94 D. Abramovich, *Subvarieties of semiabelian varieties*, Compos. Math. **90** (1994), 37–52.

ABW82 K. Akin, D. A. Buchsbaum and J. Weyman, *Schur functors and Schur complexes*, Adv. Math. **44** (1982), 207–278.

Ale24 T. Alexandre, *Vanishing results for the coherent cohomology of automorphic vector bundles over the Siegel variety in positive characteristic*, Algebra Number Theory **19** (2025), 143–193.

AMRT10 A. Ash, D. Mumford, M. Rapoport and Y.-S. Tai, *Smooth compactifications of locally symmetric varieties*, second edition, Cambridge Mathematical Library (Cambridge University Press, Cambridge, 2010). With the collaboration of P. Scholze.

Bar71 C. M. Barton, *Tensor products of ample vector bundles in characteristic p*, Amer. J. Math. **93** (1971), 429–438.

Bof91 G. Boffi, *On some plethysms*, Adv. Math. **89** (1991), 107–126.

Bru18 Y. Brunebarbe, *Symmetric differentials and variations of Hodge structures*, J. Reine Angew. Math. **743** (2018), 133–161.

BGKS Y. Brunebarbe, W. Goldring, J.-S. Koskivirta and B. Stroh, *Ample automophic bundles on zip-schemes*, in preparation.

DI87 P. Deligne and L. Illusie, *Relèvements modulo p^2 et décomposition du complexe de de Rham*, Invent. Math. **89** (1987), 247–270.

Fal83 G. Faltings, *Endlichkeitssätze für abelsche Varietäten über Zahlkörpern*, Invent. Math. **73** (1983), 349–366.

FC90 G. Faltings and C.-L. Chai, *Degeneration of abelian varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 22 (Springer, Berlin, 1990). With an appendix by D. Mumford.

FS97 E. M. Friedlander and A. Suslin, *Cohomology of finite group schemes over a field*, Invent. Math. **127** (1997), 209–270.

Gie71 D. Gieseker, *p-ample bundles and their Chern classes*, Nagoya Math. J. **43** (1971), 91–116.

GK19a W. Goldring and J.-S. Koskivirta, *Strata Hasse invariants, Hecke algebras and Galois representations*, Invent. Math. **217** (2019), 887–984.

GK19b W. Goldring and J.-S. Koskivirta, *Stratifications of flag spaces and functoriality*, Int. Math. Res. Not. IMRN (2019), 3646–3682.

Har66 R. Hartshorne, *Ample vector bundles*, Publ. Math. Inst. Hautes Études Sci. (1966), 63–94.

Jan03 J. C. Jantzen, *Representations of algebraic groups*, second edition, Mathematical Surveys and Monographs, vol. 107 (American Mathematical Society, Providence, RI, 2003).

Kee99 S. Keel, *Basepoint freeness for nef and big line bundles in positive characteristic*, Ann. Math. (1999), 253–286.

KKMS73 G. Kempf, F. Faye Knudsen, D. Mumford and B. Saint-Donat, *Toroidal embeddings. I*, Lecture Notes in Mathematics, vol. 339 (Springer, Berlin, New York, 1973).

Kle69 S. L. Kleiman, *Ample vector bundles on algebraic surfaces*, Proc. Amer. Math. Soc. **21** (1969), 673–676.

Kun69 E. Kunz, *Characterizations of regular local rings of characteristic p*, Amer. J. Math. **91** (1969), 772–784.

Lan12 K.-W. Lan, *Toroidal compactifications of PEL-type Kuga families*, Algebra Number Theory **6** (2012), 885–966.

Lan86 S. Lang, *Hyperbolic and Diophantine analysis*, Bull. Amer. Math. Soc. (N.S.) **14** (1986), 159–205.

Laz04a R. Lazarsfeld, *Positivity in algebraic geometry I. Classical setting: line bundles and linear series*. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series / A Series of Modern Surveys in Mathematics], vol. 48 (Springer, Berlin, 2004).

Laz04b R. Lazarsfeld, *Positivity in algebraic geometry II. Positivity for vector bundles, and multiplier ideals*. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series / A Series of Modern Surveys in Mathematics], vol. 49 (Springer, Berlin, 2004).

Luo87 Z. Luo, *Kodaira dimension of algebraic function fields*, Amer. J. Math. **109** (1987), 669–693.

Luo88 Z. Luo, *An invariant approach to the theory of logarithmic kodaira dimension of algebraic varieties*, Bull. Amer. Math. Soc. **19** (1988), 319–323.

https://doi.org/10.1112/S0010437X24007607

Mat90 O. Mathieu, *Filtrations of G-modules*, Ann. Sci. Éc. Norm. Supér. (4) **23** (1990), 625–644.

Mor81 L. Moret-Bailly, *Familles de courbes et de variétés abéliennes sur $\mathbb{P}^1$. II. Exemples,* in *Seminar on Pencils of Curves of Genus at Least Two, Astérisque*, **86** (1981), 125–140.

Mor85 L. Moret-Bailly, *Pinceaux de variétés abéliennes*, Astérisque (1985), 266.

Mou97 C. Mourougane, *Images directes de fibrés en droites adjoints*, Publ. Res. Inst. Math. Sci. **33** (1997), 893–916.

Ols16 M. Olsson, *Algebraic spaces and stacks*, American Mathematical Society Colloquium Publications, vol. 62 (American Mathematical Society, Providence, RI, 2016).

Ric16 S. Riche, *Geometric Representation Theory in positive characteristic*, Habilitation à diriger des recherches, Université Blaise Pascal (Clermont Ferrand 2) (2016).

Ryd15 D. Rydh, *Noetherian approximation of algebraic spaces and stacks*, J. Algebra **422** (2015), 105–147.

Sta21 The Stacks Project Authors, *The Stacks Project* (2021), https://stacks.math.columbia.edu.

Tou13 A. Touzé, *Ringel duality and derivatives of non-additive functors*, J. Pure Appl. Algebra **217** (2013), 1642–1673.

WZ18 T. Wedhorn and P. Ziegler, Tautological rings of shimura varieties and cycle classes of Ekedahl–Oort strata, Algebra Number Theory **17** (2023), 923–980.

Wil09 M. Wildon, *Multiplicity-free representations of symmetric groups*, J. Pure Appl. Algebra **213** (2009), 1464–1477.

Zuo00 K. Zuo, *On the negativity of kernels of Kodaira–Spencer maps on Hodge bundles and applications*, Asian J. Math. **4** (2000), 279–301. Kodaira's issue.

Thibault Alexandre alexandre.thx@gmail.com
Sorbonne Université, Campus Pierre et Marie Curie, 4 place Jussieu, 75252 Paris Cedex 5, France