ARITHMETIC PROPERTIES OF FREELY α-GENERATED LATTICES

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Introduction. In § 1 we give a characterization of a lattice L that is freely α-generated by a given partially ordered set P. In § 2 we obtain a representation of an element of such a lattice as a sum (product) of additively (multiplicatively) irreducible elements which, although not unique, has some of the desirable features of the canonical representation, in Whitman **(2)**, of an element of a free lattice. The usefulness of this representation is illustrated in § 3, where some further arithmetic properties of these lattices are derived.

We use $+$ and $\cdot$ for the binary operations of lattice addition and multiplication, and $\sum$ and Π for the corresponding operations on arbitrary sets and sequences of lattice elements. In other respects the terminology will be the same as in Crawley and Dean **(1)**. Briefly, the basic definitions are as follows:

A lattice L is said to be weakly $\aleph_\alpha$ complete if every non-empty subset of L of cardinality less than $\aleph_\alpha$ has a least upper bound and a greatest lower bound in L. By an α-sublattice of a weakly $\aleph_\alpha$ complete lattice L is meant a sublattice M of L with the property that all sums and products in L of non-empty subsets of M of cardinality less than $\aleph_\alpha$ belong to M. By the sublattice of L α-generated by a subset S of L is meant the smallest α-sublattice of L that contains S. In particular, L is said to be α-generated by S if the smallest α-sublattice of L that contains S is L itself. A weakly $\aleph_\alpha$ complete lattice L is said to be freely α-generated by a partially ordered set P if L contains P as a subpartially ordered set, L is α-generated by P, and every order-preserving map of P into a weakly $\aleph_\alpha$ complete lattice M can be extended to a homomorphism of L into M, preserving all sums and products of less than $\aleph_\alpha$ elements.

Suppose the weakly $\aleph_\alpha$ complete lattice L is α-generated by a subset P. Let $L_0 = P$, and for $\xi > 0$ let L_ξ be the set of all sums $\sum U$ and products ΠU with $U \subseteq \bigcup \{L_\eta | \eta < \xi\}$ and $0 < |U| < \aleph_\alpha$. For sufficiently large ξ we have $L_\xi = L$, and for each element a of L there therefore exists a smallest ordinal ξ with $a \in L_\xi$. This ordinal is called the rank of a—in symbols $\xi = r(a)$.

1. A characterization of freely α-generated lattices.

If $\aleph_\alpha$ is a singular cardinal then, as was observed in Crawley and Dean **(1)**, a weakly $\aleph_\alpha$ complete lattice L is also weakly $\aleph_{\alpha+1}$ complete, and L is (freely) α-generated by a subpartially ordered set P if and only if L is (freely) $(\alpha + 1)$-generated by P. We therefore consider only the case when $\aleph_\alpha$ is regular.

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THEOREM 1. *For any regular cardinal $\aleph_\alpha$, a weakly $\aleph_\alpha$ complete lattice L is freely α-generated by a partially ordered set P if and only if L contains P as a subpartially ordered set, L is α-generated by P, and the following conditions hold:*

(i) *For all $x \in P$ and $U \subseteq P$ with $0 < |U| < \aleph_\alpha$, if $\Pi U \leqslant x$, then $u \leqslant x$ for some $u \in U$.*

(ii) *For all $x \in P$ and $U \subseteq P$ with $0 < |U| < \aleph_\alpha$, if $x \leqslant \sum U$, then $x \leqslant u$ for some $u \in U$.*

(iii) *For all U, $V \subseteq L$ with $0 < |U| < \aleph_\alpha$ and $0 < |V| < \aleph_\alpha$, if $\Pi U \leqslant \sum V$, then either $u \leqslant \sum V$ for some $u \in U$ or $\Pi U \leqslant v$ for some $v \in V$.*

Proof. In Crawley and Dean **(1)** a lattice $F_\alpha(P)$ freely α-generated by P is constructed, and this lattice is easily seen to satisfy (i)–(iii). The given conditions are therefore necessary.

Observe that (i) implies the condition (i′) obtained from (i) by replacing the assumption $U \subseteq P$ by $U \subseteq L$. For, because of the regularity of $\aleph_\alpha$, every subset U of L with $0 < |U| < \aleph_\alpha$ is contained in an α-sublattice M of L that is α-generated by a subset Q of P with $|Q| < \aleph_\alpha$. If $x \in P$, and if $\bar{x}$ is the product of all elements $y \in Q$ with $y \not\leqslant x$, then $\bar{x} \not\leqslant x$ by (i), and any element u of Q satisfies the condition

$$u \leqslant x \text{ or } \bar{x} \leqslant u.$$

The set of all elements u of L for which this condition holds is an α-sublattice of L, and since it contains Q it must contain M. In particular, every element u of U has this property. If now $\Pi U \leqslant x$, then $\bar{x}$ cannot be contained in every member of U, for this would imply that $\bar{x} \leqslant x$. Consequently some member of U must be contained in x.

Dually, (ii) implies the condition (ii′) obtained from it by replacing the assumption $U \subseteq P$ by $U \subseteq L$.

Now suppose (i)–(iii) hold. There exists a weakly $\aleph_\alpha$ complete lattice L' that is freely α-generated by P, and the identity map of P can be extended to a homomorphism f of L' onto L. The proof will be complete if it is shown that f is an isomorphism, and to do this it suffices to show that, for all a, $b \in L'$,

(i) $$f(a) \leqslant f(b) \text{ implies that } a \leqslant b.$$

This will be proved by induction on the ordered pairs $\langle r(a), r(b)\rangle$ under the lexicographic ordering.

Any member of L is either in P or else is a sum or a product of less than $\aleph_\alpha$ elements, each of which is of lower rank than the given element. If a, $b \in P$, then (1) obviously holds. If $a \in P$ and $b = \sum V$, then

$$f(a) \leqslant \sum_{v \in V} f(v),$$

and it follows by (ii′) that $f(a) \leqslant f(v)$ for some $v \in V$, so that the given case reduces to an earlier one. Similarly, if $a = \Pi U$ and $b \in P$, the present

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case reduces to an earlier one by (i′), and if a is a product and b is a sum, the reduction is obtained by applying (iii). Finally, if either a is a sum or b is a product, then the present case reduces to earlier ones without the use of any special properties of L.

COROLLARY 2. (*Crawley and Dean* (**1**, *Theorem* 4).) *Suppose* $\aleph_\alpha$ *is a regular cardinal,* M *is a weakly* $\aleph_\alpha$ *complete lattice that is freely* α*-generated by a partially ordered set* Q, P *is a subset of* M, *and* L *is the sublattice of* M *that is* α*-generated by* P. *Then* L *is freely* α*-generated by* P *if and only if the conditions* (i) *and* (ii) *of Theorem* 1 *hold.*

2. Canonical forms. Throughout the remainder of this paper it will be assumed that $\aleph_\alpha$ is a regular cardinal, and that L is a weakly $\aleph_\alpha$ complete lattice that is freely α-generated by a partially ordered set P.

We consider only canonical sum-representations; the corresponding results for products follow by duality. An element $a \in L$ is said to be additively reducible if there exists $U \subseteq L$ such that $0 < |U| < \aleph_\alpha$, $a = \sum U$ and $u < a$ for all $u \in U$.

THEOREM 3. *If* $a \in L$ *is additively reducible, then there exists a subset* U *of* L *with the following properties:*

(i) $0 < |U| < \aleph_\alpha$ *and* $a = \sum U$.

(ii) $r(u) < r(a)$ *for all* $u \in U$.

(iii) *For each* $u \in U$, *if* $u \notin P$ *then there exists* $V \subseteq L$ *such that* $0 < |V| < \aleph_\alpha$, $u = \prod V$ *and, for all* $v \in V$, $r(v) < r(u)$ *and* $v \nleq a$.

Proof. First observe that without the condition $v \nleq a$ in (iii) the theorem would be trivial. In fact, because of 1(iii), every additively reducible element a is multiplicatively irreducible. Consequently a must be the sum of less than $\aleph_\alpha$ elements, each of which is of lower rank than a. From this we infer by transfinite induction that for any additively reducible element a there exists a set U such that (i) and (ii) hold, and such that every member of U is additively irreducible. But this last condition is clearly equivalent to the weakened form of (iii).

From now on we consider a fixed element a that is assumed to be additively reducible. For $U \subseteq L$ let $C(U)$ be the set of all elements $b \in L$ such that either $b \in U$ or else there exists a subset V of U with the properties that

$$|V| < \aleph_\alpha,\ r(v) < r(b) \text{ for all } v \in V,\ b \leqslant \sum V.$$

For U, $U' \subseteq L$ define $U \prec U'$ to mean that $U \subseteq C(U')$. Let $\mathfrak{K}_0$ be the family of all subsets U of L with the properties that each member of U is additively irreducible, $r(u) < r(a)$ for all $u \in U$, and $a = \sum U$. Finally, let $\mathfrak{K}_1$ be the family of all $U \in \mathfrak{K}_0$ with the property that, for all $U' \in \mathfrak{K}_0$, the conditions $U \prec U'$ and $U' \subseteq U$ jointly imply that $U = U'$.

The relation $\prec$ obviously quasi-orders the family of all subsets of L. It

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will be shown that $\prec$ partially orders $\mathfrak{K}_1$, that $\mathfrak{K}_1$ contains a member U with $|U| < \aleph_\alpha$, such that U is maximal in $\mathfrak{K}_1$ with respect to the relation $\prec$, and that every such member U of $\mathfrak{K}_1$ satisfies the condition (iii) (and of course also (i) and (ii)). The proofs of these assertions will be based on the statements I–VI below.

STATEMENT I. *There exists a set* $U \in \mathfrak{K}_0$ *with* $|U| < \aleph_\alpha$.

Proof. This is precisely the observation, made above, that a is the sum of less than $\aleph_\alpha$ additively irreducible elements, each of which is of lower rank than a.

STATEMENT II. *If* λ *is a limit ordinal,* $U_\xi \in \mathfrak{K}_0$ *for all* $\xi < \lambda$, $U_\xi \prec U_\eta$ *whenever* $\xi < \eta < \lambda$, *and* $U_\lambda = \bigcup_{\xi<\lambda} \bigcap_{\xi<\eta<\lambda} U_\eta$, *then* $U_\lambda \in \mathfrak{K}_0$.

Proof. We shall show that if $b \in U_\xi$ where $\xi < \lambda$, then $b \in C(U_\lambda)$. Assume that the corresponding statement holds for all b' with $r(b') < r(b)$. If $b \notin U_\lambda$, then there exists an ordinal η such that $\xi < \eta < \lambda$ and $b \notin U_\eta$. Consequently $b \leqslant \sum W$ where $W \subseteq U_\eta$, $|W| < \aleph_\alpha$, and $r(w) < r(b)$ for all $w \in W$. By the inductive hypothesis, $W \subseteq C(U_\lambda)$, and therefore $b \in C(U_\lambda)$. Thus our assertion holds for every element b, whence it follows that $U_\xi \prec U_\lambda$ for all $\xi < \lambda$.

Clearly, for each $u \in U_\lambda$, u is additively irreducible and $r(u) < r(a)$. Also, a is an upper bound for U_λ, and since $U_0 \subseteq C(U_\lambda)$ it follows that $a = \sum U_\lambda$. Thus $U_\lambda \in \mathfrak{K}_0$.

STATEMENT III. *For each* $U \in \mathfrak{K}_0$ *there exists a subset* U' *of* U *such that* $U \prec U'$ *and* $U' \in \mathfrak{K}_1$.

Proof. There exists a maximal strictly decreasing sequence of sets $U_\xi \in \mathfrak{K}_0$, $\xi < \lambda$, such that $U_0 = U$ and $U_\xi \prec U_\eta$ whenever $\xi < \eta < \lambda$. By Statement II, the type λ of this sequence cannot be a limit ordinal, for we could then take

$$U_\lambda = \bigcup_{\xi<\lambda} \bigcap_{\xi<\eta<\lambda} U_\eta = \bigcup_{\xi<\lambda} U_\xi$$

and obtain a longer sequence with the same properties. Thus $\lambda = \eta + 1$ for some ordinal η, and we infer from the maximality of the sequence that $U' = U_\eta$ is a member of $\mathfrak{K}_1$.

STATEMENT IV. *If* $U \in \mathfrak{K}_1$, $U' \subseteq L$ *and* $U \prec U' \prec U$, *then* $U \subseteq U'$.

Proof. Indeed, if $b \in U$ and $b \notin U'$, then there exists $V \subseteq U'$ such that $|V| < \aleph_\alpha$, $r(v) < r(b)$ for all $v \in V$, and $b \leqslant \sum V$. For each $v \in V$ there exists $W_v \subseteq U$ such that $|W_v| < \aleph_\alpha$, $r(w) < r(v)$ for all $w \in W$ and $v \leqslant \sum W_v$. It readily follows that $b \in C(U - \{b\})$, which is impossible because $U \in \mathfrak{K}_1$.

Observe that this implies in particular that the relation $\prec$ partially orders $\mathfrak{K}_1$.

STATEMENT V. *For each* $U' \in \mathfrak{K}_0$ *there exists a maximal member* U *of* $\mathfrak{K}_1$, *with respect to the relation* $\prec$, *such that* $U' \prec U$.

Proof. By Statement III we may assume that $U' \in \mathfrak{K}_1$. From Statements II and III we infer that every transfinite sequence of members of $\mathfrak{K}_1$ that is increasing with respect to the relation $\prec$ has an upper bound in $\mathfrak{K}_1$. Hence the conclusion follows by Zorn's Lemma.

STATEMENT VI. *If U is a maximal member of $\mathfrak{K}_1$ with respect to the relation $\prec$, then* (iii) *holds.*

Proof. If an element $u = b$ violates the condition (iii), then b must be multiplicatively reducible and, in fact, $b = \Pi V$ where $|V| < \aleph_\alpha$, $r(v) < r(b)$ for all $v \in V$, and $b' \leqslant a$ for some $b' \in V$. It readily follows that the set $U' = (U - \{b\}) \cup \}b'\}$ belongs to $\mathfrak{K}_0$, and that $U \prec U'$. By Statement V there exists $U'' \in \mathfrak{K}_1$ with $U' \prec U''$, and therefore $U \prec U''$. By the maximality of U this implies that $U = U''$. But then $U \prec U' \prec U$, so that $U \subseteq U'$ by Statement IV. This is a contradiction because $b \notin U'$.

We are now ready to complete the proof of the theorem. By Statement I we can find $U' \in \mathfrak{K}_0$ with $|U'| < \aleph_\alpha$, and by Statement V we can find a maximal member U of $\mathfrak{K}_1$ such that $U' \prec U$. By Statement VI and the definition of $\mathfrak{K}_1$, this set U has all the required properties, except possibly $|U| < \aleph_\alpha$. However, there exists $V \subseteq U$ such that $U' \subseteq C(V)$ and $|V| < \aleph_\alpha$, and it is easy to check that all of the conditions (i)–(iii) are satisfied with U replaced by V.

It follows from (iii) that each member of U is additively irreducible. On the other hand, the given representation of a may be redundant; in fact a need not have an irredundant representation as a sum of additively irreducible elements. What makes the given representation useful is the following minimal property:

COROLLARY 4. *If a and U satisfy the conditions* (i)–(iii) *of Theorem* 3, *and if $U' \subseteq L$, $0 < |U'| < \aleph_\alpha$ and $a = \sum U'$, then for each $u \in U$ there exists $u' \in U'$ with $u \leqslant u'$.*

Proof. We have $u \leqslant \sum U'$, and for $u \in P$ the conclusion follows from 1(ii).* If $u \notin P$, then we choose V according to 3(iii). Thus $u = \Pi V \leqslant \sum U' = a$, and no member of V is contained in a, whence it follows by 1(iii) that u must be contained in some member of U'.

3. Distributivity and relative complements.

The results in this section are simple consequences of Theorem 3 and Corollary 4.

THEOREM 5. *Suppose $a \in L$, I is a set with $0 < |I| < \aleph_\alpha$, and for each $i \in I$, $U_i \subseteq L$, $0 < |U_i| < \aleph_\alpha$, and $a = \sum U_i$. Then*

$$a = \sum_{f \in F} \prod_{i \in I} f(i)$$

where F is the set of all functions f on I such that $f(i) \in U_i$ for all $i \in I$.

*For, as was observed in the proof of Theorem 1, the property 1(ii) implies the condition (ii′) formulated there.

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Proof. Disregarding the trivial case in which a is additively irreducible, let U be as in Theorem 3. Then, by Corollary 4, each member u of U is contained in some member of U_i, for all $i \in I$, and we can therefore find a function $f_u \in F$ such that

$$u \leqslant \prod_{i \in I} f_u(i).$$

Consequently

$$a = \sum_{u \in U} \prod_{i \in I} f_u(i),$$

whence the desired conclusion readily follows.

THEOREM 6. *If $a, b \in L$, and $b < a$, then there exists $v_0 \in L$ such that for all $v \in L$, $a = b + v$ if and only if $v_0 \leqslant v \leqslant a$.*

Proof. If a is additively irreducible, then the conclusion holds with $v_0 = a$. Assuming that a is additively reducible, choose U according to Theorem 3, and let v_0 be the sum of all $u \in U$ with $u \nleqslant b$. Then $v_0 \leqslant a$, and each member of U is contained either in b or in v_0, so that $b + v_0 = a$. Clearly $v_0 \leqslant v \leqslant a$ implies that $a = b + v$. Conversely, if $a = b + v$, then each member of U is contained either in b or in v, and therefore $v_0 \leqslant v$.

COROLLARY 7. *If $a, b, c \in L$, then there exist $v_0, v_1 \in L$ such that, for all $v \in L$, the conditions $a = b + v$ and $c = bv$ are jointly equivalent to $v_0 \leqslant v \leqslant v_1$.*

Proof. If the equations $a = b + v$ and $c = bv$ have no common solution, we choose v_0, v_1 with $v_0 > v_1$. Assuming that the two equations do have a common solution, and excluding certain trivial cases by further assuming that $c<b<a$, we use Theorem 6 and its dual to infer that there exist $w_0, w_1 \in L$ such that, for all $v \in L$, $a = b + v$ if and only if $w_0 \leqslant v \leqslant a$, and $c = bv$ if and only if $c \leqslant v \leqslant w_1$. The elements $v_0 = w_0 + c$ and $v_1 = w_1 a$ therefore have the required properties.

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