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ON REFLEXIVE COMPACT OPERATORS

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1. Introduction. Let A be a compact operator on a separable Hilbert space $\mathscr{H}$. The aim of this paper is to investigate the relationship between the weak closure of the algebra of polynomials in A (denoted by $U(A)$) and its invariant subspace lattice Lat A.

The operator A is *reflexive* if any operator which leaves invariant the members of Lat A must be in $U(A)$. The following question was mentioned in the closing chapter of [**7**]. If every invariant subspace of A is spanned by the eigenvalues that it contains, is A reflexive? The main result of this paper is a positive answer for compact operators. Some related questions are then discussed.

2. Preliminaries. For a linear manifold $\mathscr{M}$, $[\mathscr{M}]$ will denote its closure. $\mathscr{N}(A)$ will denote the null space of A, and $\mathscr{R}(A)$ its range. $A|\mathscr{M}$ will denote the restriction of A to $\mathscr{M}$.

For n a positive integer, $\mathscr{H}^{(n)}$ denotes the direct sum of n copies of $\mathscr{H}$ and $A^{(n)}$ is the direct sum of n copies of A acting on $\mathscr{H}^{(n)}$ in the standard fashion; i.e. if $\langle x_1, \ldots, x_n\rangle \in \mathscr{H}^{(n)}$,

$$A^{(n)} \langle x_1, \ldots, x_n\rangle = \langle Ax_1, \ldots, Ax_n\rangle.$$

"Subspace" will mean closed linear manifold.

The following well known lemma will be used ([**7**, Chap. 7]).

LEMMA 1. *If* Lat $A^{(n)} \subseteq$ Lat $B^{(n)}$ *for all positive integers* $n \geqq 1$, *then* $B \in U(A)$.

Definition. Spectral synthesis holds for A if every invariant subspace $\mathscr{M}$ of A is spanned by the root vectors corresponding to non-zero eigenvalues of A in $\mathscr{M}$. *Strict spectral synthesis* holds for A if spectral synthesis holds for A and A is injective.

We will proceed in two stages. First we will consider the case where A is injective and then we extend the result to the general case.

3. Reflexivity of compact injective operators. Throughout this section we will assume A is compact injective. Some concepts and results of Markus [**5**] will be used.

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Definition. The sequence $\{\phi_j | j = 1, 2, \ldots\}$ is *minimal* in $\mathscr{H}$ if $\phi_j \notin \bigvee \{\phi_k | k \neq j\}$ and *complete* if $\bigvee \{\phi_j | j = 1, 2, \ldots\} = \mathscr{H}$.

If $\{\phi_j | j = 1, 2, \ldots\}$ is minimal and complete, it has a unique biorthogonal sequence $\{\psi_j | j = 1, 2, \ldots\}$.

Definition. $\{\phi_j | j = 1, 2, \ldots\}$ is *strongly complete* if for any $f \in \mathscr{H}$, $f \in \bigvee \{\phi_j | (f, \psi_j) \neq 0\}$.

There are generalizations of these concepts to subspaces.

Definition. Let $\{\mathscr{N}_j | j = 1, 2, \ldots\}$ be a sequence of non-zero subspaces of $\mathscr{H}$, such that $\bigvee \{\mathscr{N}_j | j = 1, 2, \ldots\} = \mathscr{H}$. $\{\mathscr{N}_j | j = 1, 2, \ldots\}$ is *separated* if for any j, the subspaces $\mathscr{N}_j$ and $\mathscr{N}^j = \bigvee \{\mathscr{N}_k | k \neq j\}$ intersect only at $\{0\}$ and $\mathscr{N}_j \dotplus \mathscr{N}^j = \mathscr{H}$ (direct sum).

P_j will denote the projection on $\mathscr{N}_j$ along $\mathscr{N}^j$.

Definition. $\{\mathscr{N}_j | j = 1, 2, \ldots\}$ is *strongly complete* if for any $f \in \mathscr{H}$, $f \in \bigvee \{P_j f | j = 1, 2, \ldots\}$.

The next lemma is an immediate consequence of the above definitions.

LEMMA 2. *Let $\{\mathscr{N}_j | j = 1, 2, \ldots\}$ be a sequence of finite dimensional subspaces of $\mathscr{H}$ and for each j, let $\{\phi_k^{(j)} : 1 \leqq k \leqq n_j\}$ be a basis for $\mathscr{N}_j$. Then $\{\mathscr{N}_j | j = 1, 2, \ldots\}$ is strongly complete if and only if $\{\phi_k^{(j)} | 1 \leqq k \leqq n_j ; j = 1, 2, \ldots\}$ is strongly complete.*

The importance of the concept of strong completeness becomes clear from the following theorem proved in [**5**, Theorem 6.1].

THEOREM 1. *Suppose A is compact and its root vectors corresponding to non-zero eigenvalues are eigenvectors. Then A allows strict spectral synthesis if and only if the eigenspaces corresponding to non-zero eigenvalues are strongly complete.*

We may now proceed to the first stage of our program.

THEOREM 2. *Let A be compact and injective. If every invariant subspace of A is spanned by eigenvectors, then A is reflexive.*

Proof. As was pointed out in [**7**, Chap. 10], it suffices to show that every invariant subspace of $A^{(2)}$ is spanned by the eigenvectors it contains.

Since every invariant subspace of A is spanned by eigenvectors, so are the root spaces of A. Noting that the restriction of A to a root space has only one point in its spectrum, it is immediately seen that every root space of A is in fact an eigenspace.

Let $\{\mathscr{N}_j | j = 1, 2, \ldots\}$ denote the sequence of eigenspaces of A. By Theorem 1 and the above, $\{\mathscr{N}_j | j = 1, 2, \ldots\}$ is strongly complete.

Now consider $\{\mathscr{N}_j^{(2)} | j = 1, 2, \ldots\}$. This is the sequence of (finite-dimensional) eigenspaces of $A^{(2)}$. Let $\{\phi_k^{(j)} | 1 \leqq k \leqq n_j\}$ be a basis for $\mathscr{N}_j$ and

$$\psi_{k_1}^{(j)} = \langle \phi_k^{(j)}, 0 \rangle, \quad \phi_{k_2}^{(j)} = \langle 0, \phi_k^{(j)} \rangle.$$

Since $\{\mathcal{N}_j | j = 1, 2, \ldots\}$ is strongly complete, so is $\{\phi_k^{(j)} | 1 \leqq k \leqq n_j;\ j = 1, 2, \ldots\}$. It follows easily from the definition that so is $\{\psi_{ki}^{(j)} | 1 \leqq i \leqq 2;\ 1 \leqq k \leqq n_j;\ j = 1, 2, \ldots\}$ in $\mathcal{H}^{(2)}$. Thus by Lemma 4, $\{\mathcal{N}_j^{(2)}\}$ is strongly complete. Applying Theorem 1, we see that $A^{(2)}$ allows spectral synthesis and the proof is complete.

4. The general case. A will be assumed to be *compact* though not necessarily injective.

THEOREM 3. *If every invariant subspace of A is spanned by eigenvectors of A, then A is reflexive.*

A portion of the proof will be given in a series of lemmas. All assume the hypothesis of Theorem 3.

LEMMA 3. *For each $\lambda \neq 0$, in the spectrum of A, let $\mathcal{N}_\lambda$ denote the eigenspaces of A corresponding to λ and let $E = \bigvee \{\mathcal{N}_\lambda | \lambda \in \sigma(A)$ and $\lambda \neq 0\}$. Then the compression of A to $E^\perp$ is zero and $E = [R(A)]$.*

Proof. Assume $||A|| = 1$. Suppose $x \in E^\perp$ and $\epsilon > 0$. Since every invariant subspace of A is spanned by eigenvectors of A, so is $\mathcal{H}$. Thus there exists a sequence $\{x_i\}_{i=1}^n$ of eigenvectors of A such that

$$\left\| \sum_{i=1}^{n} \alpha_i x_i - x \right\| < \epsilon$$

for some constants $\{\alpha_i\}_{i=1}^n$. Let $x_1, \ldots, x_k$ correspond to non-zero eigenvalues and $x_{k+1}, \ldots, x_n$ to zero.

Let P be the projection on $E^\perp$. Since $x_1, \ldots, x_k \in E$, $PAP\ x_i = 0$ for $1 \leqq i \leqq k$, and $Ax_i = 0$ for $k + 1 \leqq i \leqq n$ implies $PAPx_i = 0$ for $k + 1 \leqq i \leqq n$. Thus

$$||PAPx|| = \left\| PAP\left(x - \sum_{i=1}^{n} \alpha_i x_i\right) \right\| \leqq ||PAP||\epsilon \leqq \epsilon.$$

Thus $PAP = 0$.

Since for $x \in \mathcal{N}_\lambda$, $Ax = \lambda x$, it follows that $E \subset [R(A)]$. Thus $\mathcal{N}(A^*) \subset E^\perp$. By the above $E^\perp \subset \mathcal{N}(A^*)$ thus giving $E = [R(A)]$.

LEMMA 4. *Suppose* Lat $A \subseteq$ Lat B. *Then B commutes with A.*

Proof. Since $\mathcal{H}$ is spanned by eigenvectors of A, it is enough to show that $ABx = BAx$ for any eigenvector x of A. But Lat $A \subseteq$ Lat B implies $Bx = \lambda x$ and the rest follows immediately.

LEMMA 5. *If $\mathcal{M} \in$ Lat $A^{(n)}$, then $[A^{(n)}\mathcal{M}] \in$ Lat $B^{(n)}$ and is spanned by the eigenvectors corresponding to the non-zero eigenvalues it contains.*

Proof. Let E be the subspace defined in Lemma 3, and note that if $E = [R(A)]$, $E^{(n)} = [R(A^{(n)})]$ (since $R(A)^{(n)} = R(A^{(n)})$). Now A restricted to E is injective and Lat $A|E \subseteq$ Lat $B|E$. Thus by Theorem 2, $B|E \in U(A|E)$. Since $E^{(n)} \subseteq \text{Lat}\, A^{(n)} \cap \text{Lat}\, B^{(n)}$, it follows that Lat $(A^{(n)}|E^{(n)}) \subseteq$ Lat $(B^{(n)}|E^{(n)})$. Since $[A^{(n)}\mathscr{M}] \subseteq E^{(n)}$ it follows that $[A^{(n)}\mathscr{M}] \in$ Lat $B^{(n)}$. Also since $A^{(n)}\mathscr{M} \subseteq E^{(n)}$ and $A|E$ is injective the argument of Theorem 1 shows that $[A^{(n)}\mathscr{M}]$ is spanned by the eigenvectors it contains. But this is identical to the eigenvectors corresponding to the non-zero eigenvalues which are in $\mathscr{M}$.

Proof of Theorem. Suppose $\mathscr{M} \in$ Lat $A^{(n)}$. By Lemma 2.3 of [**6**] and the fact that every invariant subspace of A is spanned by egienvectors, it follows that $\mathscr{M}$ has a decomposition of the form

$$\mathscr{N}(A^{(n)}|\mathscr{M}) \oplus \mathscr{L}$$

where the eigenvectors of the compression of A^* to $\mathscr{L}$ corresponding to non-zero eigenvalues span $\mathscr{L}$. If Q is the projection on $\mathscr{N}(A^{(n)}|\mathscr{M})^{\perp}$ it follows that $\mathscr{L} \in$ Lat $QA^{(n)}Q$ and $[QA^{(n)}Q\mathscr{L}] = \mathscr{L}$.

It is easily seen that $\mathscr{N}(A^{(n)}\mathscr{M})$ is invariant under $B^{(n)}$. For if, $\langle x_1, \ldots, x_n \rangle$ is in $\mathscr{N}(A^{(n)}|\mathscr{M})$, $Ax_i = 0$ for $1 \leqq i \leqq n$. Thus Lat $A \subset$ Lat B implies that the one dimensional subspaces spanned by x_i and $x_i + x_j$ for $1 \leqq i, j \leqq n$ are all invariant under B. Thus there exists λ such that $Bx_i = \lambda x_i$ for $1 \leqq i \leqq n$. It follows that $B^{(n)}\langle x_1, \ldots, x_n\rangle = \langle \lambda x_1, \ldots, \lambda x_n \rangle$ which is in $\mathscr{N}(A^{(n)}|\mathscr{M})$.

Thus it suffices to show that $QB^{(n)}Q\mathscr{L} \subset \mathscr{L}$. Note that $[A^{(n)}\mathscr{M}] = [A^{(n)}\mathscr{L}] \in$ Lat $B^{(n)}$ by Lemma 5. Thus if $x \in \mathscr{L}$, $B^{(n)}A^{(n)}x \in [A^{(n)}\mathscr{M}] \subset \mathscr{M}$ and $QB^{(n)}A^{(n)}x \in Q\mathscr{M} = \mathscr{L}$.

But

$$\begin{aligned} QB^{(n)}A^{(n)}x &= QB^{(n)}A^{(n)}Qx \\ &= QB^{(n)}QA^{(n)}Qx \end{aligned}$$

since $\mathscr{N}(A^{(n)}|\mathscr{M}) \in$ Lat $A^{(n)} \cap$ Lat $B^{(n)}$ and $A^{(n)}B^{(n)} = B^{(n)}A^{(n)}$. Thus

$$(QB^{(n)}Q)(QA^{(n)}Q)\mathscr{L} \subset \mathscr{L}.$$

Since $[QA^{(n)}Q\mathscr{L}] = \mathscr{L}$ it follows that $QB^{(n)}Q\mathscr{L} \subset \mathscr{L}$ and the proof is complete.

5. Reflexivity relative to $(A)'$. It is clear that if A has root vectors of multiplicity greater than 1, then A is in general not reflexive. This is true even in the finite dimensional case. However, in the finite dimensional case, $U(A) =$ Alg Lat $A \cap (A)'$. This was shown to be the case for certain classes of compact operators in [**1**; **2**]. Here we prove a more general result.

LEMMA 6. *Let A be a compact operator and $\mathscr{N}_\lambda$ the root spaces of A corresponding to an eigenvalue $\lambda \neq 0$ of A.*

Then:

(i) *If $B \in$ (Alg Lat A) $\cap$ $(A)'$, there exists a polynomial p such that $Bx = p(A)x$ for all $x \in \mathscr{N}_\lambda$.*

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(ii) *If* $B \in (A)'$ *and* A *has a cyclic vector, there exists a polynomial* p *such that* $Bx = p(A)x$ *for all* $x \in \mathcal{N}_\lambda$.
(iii) *If* $B \in (A)''$ *then there exists a polynomial* p *such that* $Bx = p(A)x$ *for all* $x \in \mathcal{N}_\lambda$.

Proof. These follow from the fact that $\mathcal{N}_\lambda$ is finite dimensional, $\mathcal{N}_\lambda \in \text{Lat } B$ in all three cases and the corresponding finite dimensional theorems.

THEOREM 4. *Let* A *be a compact injective operator and* $\{\mathcal{N}_j\}$ *the sequence of root spaces of* A. *Suppose* $\{\mathcal{N}_j\}$ *is strongly complete. Then:*
(i) $U(A) = (\text{Alg Lat } A) \cap (A)'$.
(ii) *If* A *has a cyclic vector,* $U(A) = (A)'$.
(iii) $U(A) = (A)''$.

Proof. By the argument used in the proof of Theorem 2, $\{\mathcal{N}_j^{(k)}\}$ is strongly complete for each integer k. Thus by [**5**, Corollary 6.1], spectral synthesis holds for $A^{(k)}$.

Suppose $\mathcal{M} \in \text{Lat } A^{(k)}$. Then $\mathcal{M}$ is spanned by root vectors that it contains. Let $\langle x_1, \ldots, x_k \rangle \in \mathcal{M}$ be such a root vector, corresponding to the eigenvalue λ. Then $x_i \in \mathcal{N}_\lambda$ for $1 \leqq i \leqq k$. Suppose $B \in (\text{Alg Lat } A) \cap (A)'$. By Lemma 6, there exists a polynomial p such that $Bx_i = p(A)x_i$. Thus $B^{(k)}\langle x_1, \ldots, x_k \rangle = \langle p(A)x_1, \ldots, p(A)x_k \rangle \in \mathcal{M}$. (i) now follows from Lemma 2. The proofs for (ii) and (iii) are similar.

6. C_p operators. Let A be compact, $H = (A^*A)^{1/2}$. The eigenvalues of H are the s-numbers of A. We enumerate them in decreasing order taking account their multiplicities and denote them by $\{s_j(A)\}$. A is in C_p if $\{s_j(A)\} \in l^p$ $(1 \leqq p \leqq \infty)$. The operators in C_1 are called *nuclear*.

Definition. A is *dissipative* if $(1/2i)(A - A^*)$ is non-negative.

THEOREM 5. *Let* A *be a nuclear dissipative operator. Then:*
(i) $U(A) = (\text{Alg Lat } A) \cap (A)'$.
(ii) *If* A *has a cyclic vector then* $U(A) = (A)'$.
(iii) $U(A) = (A)''$.

Proof. By the argument given in Theorem 3, it is enough to show spectral synthesis for $A^{(n)}$, $n \geqq 1$. Since if A is nuclear and dissipative so is $A^{(n)}$, it suffices to verify spectral synthesis for A. By [**4**, p. 231] the root vectors of A span $\mathcal{H}$. Let $\mathcal{M} \in \text{Lat } A$ and P be the projection on $\mathcal{M}$. Then PAP is nuclear since C_1 is an ideal. Also, by [**4**, p. 225] PAP is dissipative. By [**4**, p. 231] the root vectors of PAP span $\mathcal{M}$ and the proof is complete.

7. Remarks. 1) It was shown in [**3**] that if $U(A)$ is generated by compact operators and if A is invertible, then $A^{-1} \in U(A)$. This motivates the following question: Is a commutative algebra generated by compact operators closed under inverses?

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2) The problem of characterizing all compact reflexive operators seems quite difficult. The main difficulties arise (as expected) in the quasi-nilpotent case.

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