Canad. J. Math. Vol. 77 (4), 2025 pp. 1377–1397
http://dx.doi.org/10.4153/S0008414X24000348

Uniqueness and Hyers–Ulam's stability for a fractional nonlinear partial integro-differential equation with variable coefficients and a mixed boundary condition

Chenkuan Li

Abstract. Introducing a pair-parameter matrix Mittag–Leffler function, we study the uniqueness and Hyers–Ulam stability to a new fractional nonlinear partial integro-differential equation with variable coefficients and a mixed boundary condition using Banach's contractive principle as well as Babenko's approach in a Banach space. These investigations have serious applications since uniqueness and stability analysis are essential topics in various research fields. The techniques used also work for different types of differential equations with initial or boundary conditions, as well as integral equations. Moreover, we present a Python code to compute approximate values of our newly established pair-parameter matrix Mittag–Leffler functions, which extend the multivariate Mittag–Leffler function. A few examples are given to show applications of the key results obtained.

1 Introduction

In this section, we are going to introduce some basic concepts on fractional calculus, a pair-parameter (β, γ) matrix Mittag–Leffler function, Babenko's approach dealing with a fractional differential equation with a nonlocal initial condition, as well as the current work on fractional partial differential equations.

Let $\omega \in [0,1]^n \subset \mathbb{R}^n$ and $\chi \in [0,1]$. Then we define for $\beta_1, \dots, \beta_n \geq 0$ [4],

$$I_1^{\beta_1} \dots I_n^{\beta_n} \Lambda(\chi, \omega) = \frac{1}{\Gamma(\beta_1) \dots \Gamma(\beta_n)}$$
$$\cdot \int_0^{\omega_1} \cdots \int_0^{\omega_n} (\omega_1 - \tau_1)^{\beta_1 - 1} \dots (\omega_n - \tau_n)^{\beta_n - 1} \Lambda(\chi, \tau_1, \dots, \tau_n) d\tau_n \dots d\tau_1,$$

where Λ is a continuous mapping from $[0,1] \times [0,1]^n$ to $\mathbb{R}$.

Received by the editors December 19, 2023; revised March 18, 2024; accepted April 9, 2024.
Published online on Cambridge Core April 16, 2024.
This research is supported by the Natural Sciences and Engineering Research Council of Canada (Grant No. 2019-03907).
AMS subject classification: **35A02, 35C15, 45E10, 26A33.**
Keywords: Fractional nonlinear partial integro-differential equation, Hyers–Ulam stability, Banach's contractive principle, pair-parameter Mittag–Leffler function, Babenko's approach.

In particular, we have

$$I_1^0 \dots I_n^0 \Lambda(\chi,\omega) = \Lambda(\chi,\omega)$$

from [5].

The partial Liouville–Caputo fractional derivative ${}_c\partial^\alpha/\partial\chi^\alpha$ of order $2 < \alpha \le 3$ with respect to χ is defined in [4] as

$$\left(\frac{{}_c\partial^\alpha}{\partial\chi^\alpha}\Lambda\right)(\chi,\omega) = \frac{1}{\Gamma(3-\alpha)}\int_0^\chi (\chi-\tau)^{2-\alpha}\Lambda_\chi'''(\tau,\omega)d\tau.$$

One of the most essential subjects of differential equations is the stability theory of Hyers–Ulam [9]. The idea of such stability for differential equations is the substitution of the equation with a given inequality that acts as a perturbation of the equation.

In this paper, we study the uniqueness and Hyers–Ulam stability for the following new fractional nonlinear partial integro-differential equation (FNPIDE) for $\alpha_{ij} \ge 0$ $(i = 1,2,\dots,n,\ j = 1,2,\dots,l \in \mathbb{N})$:

$$\begin{cases} \frac{{}_c\partial^\alpha}{\partial\chi^\alpha}\Lambda(\chi,\omega) + \sum_{j=1}^{l} a_j(\omega) I_1^{\alpha_{1j}} \dots I_n^{\alpha_{nj}}\Lambda(\chi,\omega) = \phi(\chi,\omega,\Lambda(\chi,\omega)), \\ \Lambda(0,\omega) = \phi_1(\omega),\ \ \Lambda(1,\omega) = \phi_2(\omega),\ \ \Lambda'(1,\omega) = \phi_3(\omega), \end{cases} \tag{1.1}$$

where $(\chi,\omega) \in [0,1]\times[0,1]^n$, $a_j, \phi_k \in C([0,1]^n)$ for $k = 1,2,3$, and $\phi : [0,1]\times[0,1]^n \times \mathbb{R} \to \mathbb{R}$ satisfies certain conditions to be given later.

In addition, the operator I_χ^α is the partial Riemann–Liouville fractional integral of order α with respect to χ, given by

$$(I_\chi^\alpha\Lambda)(\chi,\omega) = \frac{1}{\Gamma(\alpha)}\int_0^\chi (\chi-\tau)^{\alpha-1}\Lambda(\tau,\omega)d\tau, \quad \chi \in [0,1].$$

Our main techniques are to derive an equivalent integral equation of equation (1.1) by Babenko's approach and then to obtain the uniqueness and Hyers–Ulam stability using Banach's contractive principle and newly established pair-parameter Mittag–Leffler functions below.

Assume $\alpha_{ij} \ge 0, \alpha_i > 0$ for all $i = 1,\dots,n,\ j = 1,\dots,l$, and there is $1 \le i_0 \le n$ such that $\alpha_{i_0 j} > 0$ for all $j = 1,\dots,l$. We define

$$M = \begin{bmatrix} \alpha_{11}\dots & \alpha_{1l} & \alpha_1 \\ \alpha_{21}\dots & \alpha_{2l} & \alpha_2 \\ \dots & & \\ \alpha_{n1}\dots & \alpha_{nl} & \alpha_n \end{bmatrix}. \tag{1.2}$$

Definition 1.1 Let $\beta \ge 0,\ \gamma > 0$. A pair-parameter (β,γ) matrix Mittag–Leffler function is defined by

$$E_M^{(\beta,\gamma)}(\zeta_1,\dots,\zeta_l) = \sum_{k=0}^{\infty}\frac{1}{\Gamma(\beta k+\gamma)}\sum_{k_1+\dots+k_l=k}\binom{k}{k_1,\dots,k_l} \cdot \frac{\zeta_1^{k_1}\dots\zeta_l^{k_l}}{\Gamma(\alpha_{11}k_1+\dots+\alpha_{1l}k_l+\alpha_1)\dots\Gamma(\alpha_{n1}k_1+\dots+\alpha_{nl}k_l+\alpha_n)},$$

where $\zeta_i \in \mathbb{C}$ for $i = 1, 2, \ldots, l$, and

$$\binom{k}{k_1, \ldots, k_l} = \frac{k!}{k_1! \ldots k_l!}.$$

It follows that

$$E_M^{(0,1)}(\zeta_1, \ldots, \zeta_l) = E_M^{(0,2)}(\zeta_1, \ldots, \zeta_l) = E_M(\zeta_1, \ldots, \zeta_l),$$

where E_M is a matrix Mittag–Leffler function given in [6].

Since there exists a positive constant θ such that

$$\begin{aligned}
&\Gamma(\beta k + \gamma) \geq \theta, \\
&\Gamma(\alpha_{11} k_1 + \cdots + \alpha_{1l} k_l + \alpha_1) \geq \theta, \\
&\ldots, \\
&\Gamma(\alpha_{n1} k_1 + \cdots + \alpha_{nl} k_l + \alpha_n) \geq \theta,
\end{aligned}$$

we claim

$$\begin{aligned}
&\left| E_M^{(\beta,\gamma)}(\zeta_1, \ldots, \zeta_l) \right| \\
&\leq \frac{1}{\theta^n} \sum_{k=0}^{\infty} \sum_{k_1 + \cdots + k_l = k} \binom{k}{k_1, \ldots, k_l} \frac{|\zeta_1|^{k_1} \ldots |\zeta_l|^{k_l}}{\Gamma(\alpha_{i_0 1} k_1 + \cdots + \alpha_{i_0 l} k_l + \alpha_{i_0})} \\
&= \frac{1}{\theta^n} E_{(\alpha_{i_0 1}, \ldots, \alpha_{i_0 l}), \alpha_{i_0}}(|\zeta_l|, \ldots, |\zeta_l|) < +\infty,
\end{aligned}$$

which implies that $E_M^{(\beta,\gamma)}(\zeta_1, \ldots, \zeta_l)$ is well defined as the multivariate Mittag–Leffler function $E_{(\alpha_{i_0 1}, \ldots, \alpha_{i_0 l}), \alpha_{i_0}}(|\zeta_l|, \ldots, |\zeta_l|)$ converges [3]. Obviously,

$$\begin{aligned}
&E_P^{(0,1)}(\zeta_1, \ldots, \zeta_l) = E_P^{(0,2)}(\zeta_1, \ldots, \zeta_l) \\
&= \sum_{k=0}^{\infty} \sum_{k_1 + \cdots + k_l = k} \binom{k}{k_1, \ldots, k_l} \frac{\zeta_1^{k_1} \ldots \zeta_l^{k_l}}{\Gamma(\alpha_{i_0 1} k_1 + \cdots + \alpha_{i_0 l} k_l + \alpha_{i_0})} \\
&= E_{(\alpha_{i_0 1}, \ldots, \alpha_{i_0 l}), \alpha_{i_0}}(\zeta_l, \ldots, \zeta_l),
\end{aligned}$$

where

$$P = \begin{bmatrix} 0 \ldots & 0 & 1 \\ \ldots & & \\ \alpha_{i_0 1} \ldots & \alpha_{i_0 l} & \alpha_{i_0} \\ \ldots & & \\ 0 \ldots & 0 & 1 \end{bmatrix},$$

and

$$E_{P_0}^{(0,1)}(\zeta) = E_{P_0}^{(0,2)}(\zeta) = \sum_{k=0}^{\infty} \frac{\zeta^k}{\Gamma(\alpha_{i_0 1} k + \alpha_{i_0})} = E_{\alpha_{i_0 1}, \alpha_{i_0}}(\zeta),$$

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which is the well-known two-parameter Mittag–Leffler function, and

$$P_0 = \begin{bmatrix} 0\dots & 0 & 1 \\ \dots & & \\ \alpha_{i_0 1}\dots & 0 & \alpha_{i_0} \\ \dots & & \\ 0\dots & 0 & 1 \end{bmatrix}.$$

Babenko's approach (BA) [1] is a useful tool for dealing with various integral or differential equations (including PDEs) with initial or boundary problems. Let f be a continuous function on $[0,1]\times\mathbb{R}$ with

$$\|f\| = \sup_{(x,y)\in[0,1]\times\mathbb{R}} |f(x,y)| < +\infty.$$

To demonstrate this method in detail, we convert the following fractional differential equation with a nonlocal initial condition into an equivalent implicit integral equation:

$$\begin{cases} {}_cD^\alpha\Phi(x) + a\Phi(x) = f(x,\Phi(x)), & x\in[0,1], \\ \Phi(0) = \beta\int_0^1 \Phi(x)dx, \end{cases} \tag{1.3}$$

where $0<\alpha\le 1$, a and β are constants.

Evidently, we get by applying the operator I^α to equation (1.3)

$$I^\alpha({}_cD^\alpha\Phi(x)) + aI^\alpha\Phi(x) = I^\alpha f(x,\Phi(x)),$$

which infers that

$$\Phi(x) - \Phi(0) + aI^\alpha\Phi(x) = I^\alpha f(x,\Phi(x)),$$

and

$$(1+aI^\alpha)\,\Phi(x) = I^\alpha f(x,\Phi(x)) + \beta\int_0^1 \Phi(x)dx.$$

Treating the factor $(1+aI^\alpha)$ as a variable and using BA, we come to

$$\begin{aligned} \Phi(x) &= (1+aI^\alpha)^{-1} I^\alpha f(x,\Phi(x)) + \beta(1+aI^\alpha)^{-1}\int_0^1 \Phi(x)dx \\ &= \sum_{k=0}^\infty (-1)^k a^k I^{\alpha k+\alpha} f(x,\Phi(x)) + \beta\sum_{k=0}^\infty (-1)^k a^k I^{\alpha k}\int_0^1 \Phi(x)dx \\ &= \sum_{k=0}^\infty (-1)^k a^k \frac{1}{\Gamma(\alpha k+\alpha)}\int_0^x (x-s)^{\alpha k+\alpha-1} f(s,\Phi(s))ds \\ &\quad + \beta\sum_{k=0}^\infty (-1)^k a^k \frac{1}{\Gamma(\alpha k+1)} x^{\alpha k}\int_0^1 \Phi(x)dx \\ &= \int_0^x (x-s)^{\alpha-1} E_{\alpha,\alpha}(-a(x-s)^\alpha) f(s,\Phi(s))ds + \beta\int_0^1 \Phi(x)dx\, E_{\alpha,1}(-ax^\alpha), \end{aligned}$$

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by noting that

$$\left|\int_0^x (x-s)^{\alpha-1} E_{\alpha,\alpha}(-a(x-s)^\alpha) f(s,\Phi(s))ds\right| \leq \frac{1}{\alpha}\|f\| \sum_{k=0}^{\infty} \frac{|a|^k}{\Gamma(\alpha k+\alpha)} < +\infty,$$

and

$$\Phi(0) = \beta \int_0^1 \Phi(x)dx.$$

In summary, equation (1.3) is equivalent to the following integral equation:

(1.4)

$$\Phi(x) = \int_0^x (x-s)^{\alpha-1} E_{\alpha,\alpha}(-a(x-s)^\alpha) f(s,\Phi(s))ds + \beta \int_0^1 \Phi(x)dx\, E_{\alpha,1}(-ax^\alpha).$$

The above integral equation, in fact, plays an important role in studying the uniqueness of equation (1.3) in the Banach space $C[0,1]$ with the norm

$$\|\Phi\| = \max_{x\in[0,1]} |\Phi(x)| < +\infty.$$

We further assume there is a constant $\mathcal{L} > 0$ such that f satisfies the following Lipschitz condition:

$$|f(x,y_1) - f(x,y_2)| \leq \mathcal{L}|y_1 - y_2|,$$

and

$$\mathcal{B} = \frac{\mathcal{L}}{\alpha} E_{\alpha,\alpha}(|a|) + |\beta| E_{\alpha,1}(|a|) < 1.$$

Then equation (1.3) has a unique solution in $C[0,1]$.

To prove this, we define a nonlinear mapping M over $C[0,1]$ as

$$(M\Phi)(x) = \int_0^x (x-s)^{\alpha-1} E_{\alpha,\alpha}(-a(x-s)^\alpha) f(s,\Phi(s))ds + \beta \int_0^1 \Phi(x)dx\, E_{\alpha,1}(-ax^\alpha).$$

It follows from the above that $(M\Phi)(x) \in C[0,1]$. We are going to show that M is contractive. For $\Phi_1, \Phi_2 \in C[0,1]$, we have

$$\begin{aligned}(M\Phi_1)(x) - (M\Phi_2)(x) &= \int_0^x (x-s)^{\alpha-1} E_{\alpha,\alpha}(-a(x-s)^\alpha) [f(s,\Phi_1(s)) - f(s,\Phi_2(s))]ds \\ &\quad + \beta \int_0^1 [\Phi_1(x) - \Phi_2(x)]dx\, E_{\alpha,1}(-ax^\alpha).\end{aligned}$$

Hence,

$$\|M\Phi_1 - M\Phi_2\| \leq \left(\frac{\mathcal{L}}{\alpha} E_{\alpha,\alpha}(|a|) + |\beta| E_{\alpha,1}(|a|)\right) \|\Phi_1 - \Phi_2\| = \mathcal{B}\|\Phi_1 - \Phi_2\|.$$

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Since $\mathcal{B} < 1$, we claim that equation (1.3) has a unique solution in $C[0,1]$ by Banach's contractive principle (BCP).

We define $S([0,1] \times [0,1]^n)$ as the Banach space of all continuous mappings from $[0,1] \times [0,1]^n$ to $\mathbb{R}$ with the norm

$$\|\Lambda\| = \sup_{(\chi,\omega)\in[0,1]\times[0,1]^n} |\Lambda(\chi,\omega)|, \quad \text{for } \Lambda \in S([0,1]\times[0,1]^n).$$

Fractional partial differential equations (a generalization of classical PDEs of integer order) are used to model various phenomena in physics, engineering, and other fields. There are intensive studies on fractional PDEs using various approaches, such as integral transforms [8], analytical and numerical solutions [10], homotopy analysis technique [2, 11], variational iteration method [12] and so on. Very recently, Li et al. [7] investigated the uniqueness of solutions for the following fractional PDE with nonlocal initial value conditions for $2 < \alpha \le 3$, $0 < \alpha_1 \le 1$ and $\alpha_2 > 0$ based on BCP, BA and the multivariate Mittag–Leffler function for a constant η:

$$\begin{cases} \dfrac{{}_c\partial^{\alpha}}{\partial\chi^{\alpha}}\Lambda(\chi,\omega) + c_0(\omega)\dfrac{{}_c\partial^{\alpha_1}}{\partial\chi^{\alpha_1}}\Lambda(\chi,\omega) + c_1(\omega)\Lambda(\chi,\omega) + c_2(\omega)I_{\omega}^{\alpha_2}\Lambda(\chi,\omega) \\ = f(\chi,\omega,\Lambda(\chi,\omega)), \\ \Lambda(0,\omega) = \eta\displaystyle\int_0^1 \Lambda(\chi,\omega)d\chi, \ \ \dfrac{\partial}{\partial\chi}\Lambda(0,\omega) = \displaystyle\int_0^1 \psi(\chi)\Lambda(\chi,\omega)d\chi, \ \Lambda''_{\chi}(0,\omega) = 0, \end{cases}$$

where $(\chi,\omega) \in [0,1]\times[0,b]$, $\psi \in C[0,1]$ and $f : [0,1]\times[0,b]\times\mathbb{R} \to \mathbb{R}$ satisfies certain conditions.

We will first convert equation (1.1) into an equivalent implicit integral equation in a series by BA in Section 2, and then further study the uniqueness of solutions via BCP in the space $S([0,1]\times[0,1]^n)$. In Section 3, we derive the Hyers–Ulam stability based on the implicit integral equation and present several examples demonstrating applications of the key results obtained in Section 4. Finally, we summarize the entire work in Section 5.

2 Uniqueness

We begin converting equation (1.1) to an implicit integral equation then derive sufficient conditions for the uniqueness based on Banach's contractive principle.

Theorem 2.1 *Suppose $a_j, \phi_1, \phi_2, \phi_3 \in C([0,1]^n)$ for $j = 1,2,\ldots,j$, ϕ is a continuous function on $[0,1]\times[0,1]^n\times\mathbb{R}$ with*

$$\sup_{(\chi,\omega,y)\in[0,1]\times[0,1]^n\times\mathbb{R}} |\phi| < +\infty,$$

$\alpha_{ij} \ge 0$ for all $i = 1,\ldots,n$, $j = 1,\ldots,l$, and there is $1 \le i_0 \le n$ such that $\alpha_{i_0 j} > 0$ for all $j = 1,\ldots,l$. Furthermore, we assume that

$$\mathcal{M}_j = \begin{bmatrix} \alpha_{11}\ldots & \alpha_{1l} & \alpha_{1j}+1 \\ \alpha_{21}\ldots & \alpha_{2l} & \alpha_{2j}+1 \\ \ldots & & \\ \alpha_{n1}\ldots & \alpha_{nl} & \alpha_{nj}+1 \end{bmatrix},$$

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and

$$Q = 1 - \left(\frac{1}{4} + \frac{1}{\alpha}\right)\frac{1}{\Gamma(\alpha)}\sum_{j=1}^{l} A_j E^{(\alpha,1)}_{\mathcal{M}_j}(A_1,\ldots,A_l) > 0.$$

Then equation (1.1) is equivalent to the following implicit integral equation:

$$\begin{aligned}\Lambda = \sum_{k=1}^{\infty}(-1)^k \sum_{k_1+\cdots+k_l=k}\binom{k}{k_1,\ldots,k_l}&\left(a_1(\omega)I^{\alpha}_{\chi}I^{\alpha_{11}}_1\ldots I^{\alpha_{n1}}_n\right)^{k_1}\cdots\left(a_l(\omega)I^{\alpha}_{\chi}I^{\alpha_{1l}}_1\ldots I^{\alpha_{nl}}_n\right)^{k_l}\\ &\cdot\big(\phi_1(\omega)(1-2\chi+\chi^2)+\phi_2(\omega)(2\chi-\chi^2)+\phi_3(\omega)(\chi^2-\chi)+I^{\alpha-1}_{\chi=1}(\chi\phi-\chi^2\phi)\\ &+I^{\alpha}_{\chi=1}(\chi^2\phi-2\chi\phi)+I^{\alpha}_{\chi}\phi+\sum_{j=1}^{l}a_j(\omega)I^{\alpha_{1j}}_1\ldots I^{\alpha_{nj}}_n I^{\alpha-1}_{\chi=1}(\chi\Lambda-\chi^2\Lambda)\\ &+\sum_{j=1}^{l}a_j(\omega)I^{\alpha_{1j}}_1\ldots I^{\alpha_{nj}}_n I^{\alpha}_{\chi=1}(\chi^2\Lambda-2\chi\Lambda)\big).\end{aligned} \tag{2.1}$$

In addition, Λ is a uniformly bounded function satisfying

$$\begin{aligned}\|\Lambda\| \le \frac{1}{Q}E^{(\alpha,1)}_{\mathcal{M}_0}(A_1,\ldots,A_l)\cdot&\left(\max_{\omega\in[0,1]^n}|\phi_1(\omega)|+\max_{\omega\in[0,1]^n}|\phi_2(\omega)|+\frac{1}{4}\max_{\omega\in[0,1]^n}|\phi_3(\omega)|\right)\\ &+\frac{1}{Q}\left(\frac{\frac{1}{4}+\frac{1}{\alpha}}{\Gamma(\alpha)}E^{(\alpha,1)}_{\mathcal{M}_0}(A_1,\ldots,A_l)+E^{(\alpha,\alpha+1)}_{\mathcal{M}_0}(A_1,\ldots,A_l)\right)\sup_{(\chi,\omega,y)\in[0,1]\times[0,1]^n\times\mathbb{R}}|\phi|\\ &< +\infty,\end{aligned}$$

where

$$\mathcal{M}_0 = \begin{bmatrix}\alpha_{11}\ldots & \alpha_{1l} & 1\\ \alpha_{21}\ldots & \alpha_{2l} & 1\\ \ldots & & \\ \alpha_{n1}\ldots & \alpha_{nl} & 1\end{bmatrix}.$$

Proof It follows from [7] that

$$I^{\alpha}_{\chi}\left(\frac{{}_c\partial^{\alpha}}{\partial\chi^{\alpha}}\Lambda\right)(\chi,\omega) = \Lambda(\chi,\omega)-\Lambda(0,\omega)-\Lambda'_{\chi}(0,\omega)\chi-\Lambda''_{\chi}(0,\omega)\frac{\chi^2}{2},$$

where $0 < \alpha \le 3$.

Applying the integral operator I^{α}_{χ} to equation (1.1) and using the condition $\Lambda(0,\omega) = \phi_1(\omega)$, we get

$$\begin{aligned}&\Lambda(\chi,\omega)-\phi_1(\omega)-\Lambda'_{\chi}(0,\omega)\chi-\Lambda''_{\chi}(0,\omega)\frac{\chi^2}{2}\\ &+\sum_{j=1}^{l}a_j(\omega)I^{\alpha}_{\chi}I^{\alpha_{1j}}_1\ldots I^{\alpha_{nj}}_n\Lambda(\chi,\omega) = I^{\alpha}_{\chi}\phi(\chi,\omega,\Lambda(\chi,\omega)).\end{aligned} \tag{2.2}$$

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Setting $\chi = 1$, we come to

$$\Lambda(1,\omega) - \phi_1(\omega) - \Lambda'_\chi(0,\omega) - \Lambda''_\chi(0,\omega)\frac{1}{2} + \sum_{j=1}^{l} a_j(\omega) I^{\alpha}_{\chi=1} I_1^{\alpha_{1j}} \dots I_n^{\alpha_{nj}} \Lambda(\chi,\omega) = I^{\alpha}_{\chi=1}\phi(\chi,\omega,\Lambda(\chi,\omega)). \tag{2.3}$$

Differentiating equation (2.2) with respect to χ, we deduce that for $\chi = 1$,

$$\phi_3(\omega) - \Lambda'_\chi(0,\omega) - \Lambda''_\chi(0,\omega) + \sum_{j=1}^{l} a_j(\omega) I^{\alpha-1}_{\chi=1} I_1^{\alpha_{1j}} \dots I_n^{\alpha_{nj}} \Lambda(\chi,\omega) = I^{\alpha-1}_{\chi=1}\phi(\chi,\omega,\Lambda(\chi,\omega)), \tag{2.4}$$

by the given initial condition.

From equations (2.3) and (2.4), we derive that

$$\frac{1}{2}\Lambda''_\chi(0,\omega) = \phi_1(\omega) - \phi_2(\omega) + \phi_3(\omega) + \sum_{j=1}^{l} a_j(\omega) I_1^{\alpha_{1j}} \dots I_n^{\alpha_{nj}} \left(I^{\alpha}_{\chi=1} - I^{\alpha-1}_{\chi=1}\right)\Lambda(\chi,\omega) + \left(I^{\alpha}_{\chi=1} - I^{\alpha-1}_{\chi=1}\right)\phi(\chi,\omega,\Lambda(\chi,\omega)),$$

and

$$\Lambda'_\chi(0,\omega) = 2\phi_2(\omega) - 2\phi_1(\omega) - \phi_3(\omega) + \sum_{j=1}^{l} a_j(\omega) I_1^{\alpha_{1j}} \dots I_n^{\alpha_{nj}} \left(I^{\alpha-1}_{\chi=1} - 2I^{\alpha}_{\chi=1}\right)\Lambda(\chi,\omega) + \left(I^{\alpha-1}_{\chi=1} - 2I^{\alpha}_{\chi=1}\right)\phi(\chi,\omega,\Lambda(\chi,\omega)).$$

Hence,

$$\begin{aligned}\left(1 + \sum_{j=1}^{l} a_j(\omega) I^{\alpha}_{\chi} I_1^{\alpha_{1j}} \dots I_n^{\alpha_{nj}}\right)\Lambda(\chi,\omega) &= \phi_1(\omega)(1-2\chi+\chi^2) + \phi_2(\omega)(2\chi-\chi^2) + \phi_3(\omega)(\chi^2-\chi) + I^{\alpha-1}_{\chi=1}(\chi\phi - \chi^2\phi)\\ &\quad + I^{\alpha}_{\chi=1}(\chi^2\phi - 2\chi\phi) + I^{\alpha}_{\chi}\phi + \sum_{j=1}^{l} a_j(\omega) I_1^{\alpha_{1j}} \dots I_n^{\alpha_{nj}} I^{\alpha-1}_{\chi=1}(\chi\Lambda - \chi^2\Lambda)\\ &\quad + \sum_{j=1}^{l} a_j(\omega) I_1^{\alpha_{1j}} \dots I_n^{\alpha_{nj}} I^{\alpha}_{\chi=1}(\chi^2\Lambda - 2\chi\Lambda).\end{aligned}$$

Using BA, we deduce that

$$\Lambda(\chi,\omega) = \left(1 + \sum_{j=1}^{l} a_j(\omega) I^{\alpha}_{\chi} I_1^{\alpha_{1j}} \dots I_n^{\alpha_{nj}}\right)^{-1} \cdot \big(\phi_1(\omega)(1-2\chi+\chi^2) + \phi_2(\omega)(2\chi-\chi^2) + \phi_3(\omega)(\chi^2-\chi) + I^{\alpha-1}_{\chi=1}(\chi\phi - \chi^2\phi)$$

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$$+ I^{\alpha}_{\chi=1}(\chi^2\phi - 2\chi\phi) + I^{\alpha}_{\chi}\phi + \sum_{j=1}^{l} a_j(\omega) I_1^{\alpha_{1j}} \dots I_n^{\alpha_{nj}} I^{\alpha-1}_{\chi=1}(\chi\Lambda - \chi^2\Lambda)$$

$$+ \sum_{j=1}^{l} a_j(\omega) I_1^{\alpha_{1j}} \dots I_n^{\alpha_{nj}} I^{\alpha}_{\chi=1}(\chi^2\Lambda - 2\chi\Lambda))$$

$$= \sum_{k=1}^{\infty}(-1)^k \left(\sum_{j=1}^{l} a_j(\omega) I^{\alpha}_{\chi} I_1^{\alpha_{1j}} \dots I_n^{\alpha_{nj}}\right)^k$$

$$\cdot (\phi_1(\omega)(1-2\chi+\chi^2) + \phi_2(\omega)(2\chi-\chi^2) + \phi_3(\omega)(\chi^2-\chi) + I^{\alpha-1}_{\chi=1}(\chi\phi - \chi^2\phi)$$

$$+ I^{\alpha}_{\chi=1}(\chi^2\phi - 2\chi\phi) + I^{\alpha}_{\chi}\phi + \sum_{j=1}^{l} a_j(\omega) I_1^{\alpha_{1j}} \dots I_n^{\alpha_{nj}} I^{\alpha-1}_{\chi=1}(\chi\Lambda - \chi^2\Lambda)$$

$$+ \sum_{j=1}^{l} a_j(\omega) I_1^{\alpha_{1j}} \dots I_n^{\alpha_{nj}} I^{\alpha}_{\chi=1}(\chi^2\Lambda - 2\chi\Lambda))$$

$$= \sum_{k=1}^{\infty}(-1)^k \sum_{k_1+\cdots+k_l=k} \binom{k}{k_1,\dots,k_l} \left(a_1(\omega) I^{\alpha}_{\chi} I_1^{\alpha_{11}} \dots I_n^{\alpha_{n1}}\right)^{k_1} \cdots \left(a_l(\omega) I^{\alpha}_{\chi} I_1^{\alpha_{1l}} \dots I_n^{\alpha_{nl}}\right)^{k_l}$$

$$\cdot (\phi_1(\omega)(1-2\chi+\chi^2) + \phi_2(\omega)(2\chi-\chi^2) + \phi_3(\omega)(\chi^2-\chi) + I^{\alpha-1}_{\chi=1}(\chi\phi - \chi^2\phi)$$

$$+ I^{\alpha}_{\chi=1}(\chi^2\phi - 2\chi\phi) + I^{\alpha}_{\chi}\phi + \sum_{j=1}^{l} a_j(\omega) I_1^{\alpha_{1j}} \dots I_n^{\alpha_{nj}} I^{\alpha-1}_{\chi=1}(\chi\Lambda - \chi^2\Lambda)$$

$$+ \sum_{j=1}^{l} a_j(\omega) I_1^{\alpha_{1j}} \dots I_n^{\alpha_{nj}} I^{\alpha}_{\chi=1}(\chi^2\Lambda - 2\chi\Lambda)) = T_1 + \cdots + T_8,$$

where

$$T_1 = \sum_{k=1}^{\infty}(-1)^k \frac{\chi^{\alpha k}}{\Gamma(\alpha k+1)} \left(1 - \frac{2\chi}{1+\alpha k} + \frac{2\chi^2}{(2+\alpha k)(1+\alpha k)}\right) \sum_{k_1+\cdots+k_l=k} \binom{k}{k_1,\dots,k_l}$$
$$\cdot \left(a_1(\omega) I_1^{\alpha_{11}} \dots I_n^{\alpha_{n1}}\right)^{k_1} \cdots \left(a_l(\omega) I_1^{\alpha_{1l}} \dots I_n^{\alpha_{nl}}\right)^{k_l} \phi_1(\omega),$$

$$T_2 = 2\sum_{k=1}^{\infty}(-1)^k \frac{\chi^{\alpha k+1}}{\Gamma(\alpha k+2)} \left(1 - \frac{\chi}{\alpha k+2}\right) \sum_{k_1+\cdots+k_l=k} \binom{k}{k_1,\dots,k_l}$$
$$\cdot \left(a_1(\omega) I_1^{\alpha_{11}} \dots I_n^{\alpha_{n1}}\right)^{k_1} \cdots \left(a_l(\omega) I_1^{\alpha_{1l}} \dots I_n^{\alpha_{nl}}\right)^{k_l} \phi_2(\omega),$$

$$T_3 = \sum_{k=1}^{\infty}(-1)^k \frac{\chi^{\alpha k+1}}{\Gamma(\alpha k+2)} \left(\frac{2\chi}{\alpha k+2} - 1\right) \sum_{k_1+\cdots+k_l=k} \binom{k}{k_1,\dots,k_l}$$
$$\cdot \left(a_1(\omega) I_1^{\alpha_{11}} \dots I_n^{\alpha_{n1}}\right)^{k_1} \cdots \left(a_l(\omega) I_1^{\alpha_{1l}} \dots I_n^{\alpha_{nl}}\right)^{k_l} \phi_3(\omega),$$

$$T_4 = \sum_{k=1}^{\infty}(-1)^k \sum_{k_1+\cdots+k_l=k} \binom{k}{k_1,\dots,k_l} \left(a_1(\omega) I_1^{\alpha_{11}} \dots I_n^{\alpha_{n1}}\right)^{k_1} \cdots \left(a_l(\omega) I_1^{\alpha_{1l}} \dots I_n^{\alpha_{nl}}\right)^{k_l}$$
$$\cdot I^{\alpha k}_{\chi} I^{\alpha-1}_{\chi=1}(\chi - \chi^2)\phi,$$

$$T_5 = \sum_{k=1}^{\infty}(-1)^k \sum_{k_1+\cdots+k_l=k} \binom{k}{k_1,\dots,k_l} \left(a_1(\omega) I_1^{\alpha_{11}} \dots I_n^{\alpha_{n1}}\right)^{k_1} \cdots \left(a_l(\omega) I_1^{\alpha_{1l}} \dots I_n^{\alpha_{nl}}\right)^{k_l}$$
$$\cdot I^{\alpha k}_{\chi} I^{\alpha}_{\chi=1}(\chi^2 - 2\chi)\phi,$$

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$$T_6 = \sum_{k=1}^{\infty}(-1)^k \sum_{k_1+\cdots+k_l=k}\binom{k}{k_1,\ldots,k_l}\left(a_1(\omega)I_1^{\alpha_{11}}\ldots I_n^{\alpha_{n1}}\right)^{k_1}\cdots\left(a_l(\omega)I_1^{\alpha_{1l}}\ldots I_n^{\alpha_{nl}}\right)^{k_l} \cdot I_\chi^{\alpha k+\alpha}\phi,$$

$$T_7 = \sum_{j=1}^{l} a_j(\omega)\sum_{k=1}^{\infty}(-1)^k \sum_{k_1+\cdots+k_l=k}\binom{k}{k_1,\ldots,k_l} \cdot\left(a_1(\omega)I_1^{\alpha_{11}}\ldots I_n^{\alpha_{n1}}\right)^{k_1}\cdots\left(a_l(\omega)I_l^{\alpha_{1l}}\ldots I_n^{\alpha_{nl}}\right)^{k_l} I_\chi^{\alpha k}I_1^{\alpha_{1j}}\ldots I_n^{\alpha_{nj}}I_\chi^{\alpha-1}(\chi-\chi^2)\Lambda,$$

and finally,

$$T_8 = \sum_{j=1}^{l} a_j(\omega)\sum_{k=1}^{\infty}(-1)^k \sum_{k_1+\cdots+k_l=k}\binom{k}{k_1,\ldots,k_l} \cdot\left(a_1(\omega)I_1^{\alpha_{11}}\ldots I_n^{\alpha_{n1}}\right)^{k_1}\cdots\left(a_l(\omega)I_l^{\alpha_{1l}}\ldots I_n^{\alpha_{nl}}\right)^{k_l} I_\chi^{\alpha k}I_1^{\alpha_{1j}}\ldots I_n^{\alpha_{nj}}I_{\chi=1}^{\alpha}(\chi^2-2\chi)\Lambda.$$

Let

$$\max_{\omega\in[0,1]^n}|a_j(\omega)| = A_j,\ \ j=1,2,\ldots,l,$$
$$\max_{\chi\in[0,1]}|1-2\chi+\chi^2| = 1,\quad \max_{\chi\in[0,1]}|2\chi-\chi^2| = 1,\quad \max_{\chi\in[0,1]}|\chi^2-\chi| = \frac{1}{4}.$$

Thus,

$$\|\Lambda\| \le \sum_{k=0}^{\infty}\frac{1}{\Gamma(\alpha k+1)}\sum_{k_1+\cdots+k_l=k}\binom{k}{k_1,\ldots,k_l} \cdot\frac{A_1^{k_1}\ldots A_l^{k_l}}{\Gamma(\alpha_{11}k_1+\cdots+\alpha_{1l}k_l+1)\ldots\Gamma(\alpha_{n1}k_1+\cdots+\alpha_{nl}k_l+1)} \cdot\left(\max_{\omega\in[0,1]^n}|\phi_1(\omega)| + \max_{\omega\in[0,1]^n}|\phi_2(\omega)| + \frac{1}{4}\max_{\omega\in[0,1]^n}|\phi_3(\omega)|\right) + T_{21}+T_{22}+T_{23},$$

where

$$T_{21} = \frac{\frac{1}{4}+\frac{1}{\alpha}}{\Gamma(\alpha)}\sum_{k=0}^{\infty}\frac{1}{\Gamma(\alpha k+1)}\sum_{k_1+\cdots+k_l=k}\binom{k}{k_1,\ldots,k_l} \cdot\frac{A_1^{k_1}\ldots A_l^{k_l}}{\Gamma(\alpha_{11}k_1+\cdots+\alpha_{1l}k_l+1)\ldots\Gamma(\alpha_{n1}k_1+\cdots+\alpha_{nl}k_l+1)}\sup_{(\chi,\omega,y)\in[0,1]\times[0,1]^n\times\mathbb{R}}|\phi|,$$

$$T_{22} = \sum_{k=0}^{\infty}\frac{1}{\Gamma(\alpha k+\alpha+1)}\sum_{k_1+\cdots+k_l=k}\binom{k}{k_1,\ldots,k_l} \cdot\frac{A_1^{k_1}\ldots A_l^{k_l}}{\Gamma(\alpha_{11}k_1+\cdots+\alpha_{1l}k_l+1)\ldots\Gamma(\alpha_{n1}k_1+\cdots+\alpha_{nl}k_l+1)}\sup_{(\chi,\omega,y)\in[0,1]\times[0,1]^n\times\mathbb{R}}|\phi|,$$

and

$$T_{23} = \frac{\|\Lambda\|}{4\,\Gamma(\alpha)} \sum_{j=1}^{l} A_j \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k+1)} \sum_{k_1+\cdots+k_l=k} \binom{k}{k_1,\ldots,k_l}$$
$$\cdot \frac{A_1^{k_1}\ldots A_l^{k_l}}{\Gamma(\alpha_{11}k_1+\cdots+\alpha_{1l}k_l+\alpha_{1j}+1)\ldots\Gamma(\alpha_{n1}k_1+\cdots+\alpha_{nl}k_l+\alpha_{nj}+1)}$$
$$+ \frac{\|\Lambda\|}{\Gamma(\alpha+1)} \sum_{j=1}^{l} A_j \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k+1)} \sum_{k_1+\cdots+k_l=k} \binom{k}{k_1,\ldots,k_l}$$
$$\cdot \frac{A_1^{k_1}\ldots A_l^{k_l}}{\Gamma(\alpha_{11}k_1+\cdots+\alpha_{1l}k_l+\alpha_{1j}+1)\ldots\Gamma(\alpha_{n1}k_1+\cdots+\alpha_{nl}k_l+\alpha_{nj}+1)}$$
$$= \frac{\|\Lambda\|}{4\,\Gamma(\alpha)} \sum_{j=1}^{l} A_j E_{\mathcal{M}_j}^{(\alpha,1)}(A_1,\ldots,A_l) + \frac{\|\Lambda\|}{\alpha\Gamma(\alpha)} \sum_{j=1}^{l} A_j E_{\mathcal{M}_j}^{(\alpha,1)}(A_1,\ldots,A_l)$$
$$= \|\Lambda\| \left(\frac{1}{4}+\frac{1}{\alpha}\right) \frac{1}{\Gamma(\alpha)} \sum_{j=1}^{l} A_j E_{\mathcal{M}_j}^{(\alpha,1)}(A_1,\ldots,A_l).$$

Using our assumption

$$Q = 1 - \left(\frac{1}{4}+\frac{1}{\alpha}\right) \frac{1}{\Gamma(\alpha)} \sum_{j=1}^{l} A_j E_{\mathcal{M}_j}^{(\alpha,1)}(A_1,\ldots,A_l) > 0,$$

we claim that

$$\|\Lambda\| \le \frac{1}{Q} E_{\mathcal{M}_0}^{(\alpha,1)}(A_1,\ldots,A_l) \cdot \left(\max_{\omega\in[0,1]^n} |\phi_1(\omega)| + \max_{\omega\in[0,1]^n} |\phi_2(\omega)| + \frac{1}{4} \max_{\omega\in[0,1]^n} |\phi_3(\omega)| \right)$$
$$+ \frac{1}{Q} \left(\frac{\frac{1}{4}+\frac{1}{\alpha}}{\Gamma(\alpha)} E_{\mathcal{M}_0}^{(\alpha,1)}(A_1,\ldots,A_l) + E_{\mathcal{M}_0}^{(\alpha,\alpha+1)}(A_1,\ldots,A_l) \right) \sup_{(\chi,\omega,y)\in[0,1]\times[0,1]^n\times\mathbb{R}} |\phi|$$
$$< +\infty,$$

which indicates that Λ is a uniformly bounded function. This completes the proof of Theorem 2.1. ■

Theorem 2.2 *Suppose $a_j, \phi_1, \phi_2, \phi_3 \in C([0,1]^n)$ for $j = 1, 2, \ldots, j$, ϕ is a continuous and bounded function on $[0,1]\times[0,1]^n\times\mathbb{R}$, satisfying the Lipschitz condition for a positive constant $\mathcal{C}$*

$$|\phi(\chi,\omega,y_1) - \phi(\chi,\omega,y_2)| \le \mathcal{C}|y_1 - y_2|, \quad y_1, y_2 \in \mathbb{R},$$

$\alpha_{ij} \ge 0$ for all $i = 1,\ldots,n$, $j = 1,\ldots,l$, and there is $1 \le i_0 \le n$ such that $\alpha_{i_0 j} > 0$ for all $j = 1,\ldots,l$. Furthermore, we assume that

$$\mathcal{M}_j = \begin{bmatrix} \alpha_{11}\ldots & \alpha_{1l} & \alpha_{1j}+1 \\ \alpha_{21}\ldots & \alpha_{2l} & \alpha_{2j}+1 \\ \ldots & & \\ \alpha_{n1}\ldots & \alpha_{nl} & \alpha_{nj}+1 \end{bmatrix},$$

https://doi.org/10.4153/S0008414X24000348 Published online by Cambridge University Press

and

$$q = \left(\frac{1}{4} + \frac{1}{\alpha}\right)\frac{1}{\Gamma(\alpha)}\sum_{j=1}^{l} A_j E_{\mathcal{M}_j}^{(\alpha,1)}(A_1, \ldots, A_l)$$
$$+ \mathcal{C}\left(\frac{\frac{1}{4}+\frac{1}{\alpha}}{\Gamma(\alpha)} E_{\mathcal{M}_0}^{(\alpha,1)}(A_1,\ldots,A_l) + E_{\mathcal{M}_0}^{(\alpha,\alpha+1)}(A_1,\ldots,A_l)\right) < 1,$$

where

$$\mathcal{M}_0 = \begin{bmatrix} \alpha_{11}\ldots & \alpha_{1l} & 1 \\ \alpha_{21}\ldots & \alpha_{2l} & 1 \\ \ldots & & \\ \alpha_{n1}\ldots & \alpha_{nl} & 1 \end{bmatrix}.$$

Then equation (1.1) has a unique uniformly bounded solution in the space $S([0,1]\times[0,1]^n)$.

Proof We define a nonlinear mapping $\mathcal{F}$ over $S([0,1]\times[0,1]^n)$ as

$$\begin{aligned}
&(\mathcal{F}\Lambda)(\chi,\omega) \\
&= \sum_{k=0}^{\infty}(-1)^k \sum_{k_1+\cdots+k_l=k}\binom{k}{k_1,\ldots,k_l}\left(a_1(\omega)I_\chi^\alpha I_1^{\alpha_{11}}\ldots I_n^{\alpha_{n1}}\right)^{k_1}\cdots\left(a_l(\omega)I_\chi^\alpha I_1^{\alpha_{1l}}\ldots I_n^{\alpha_{nl}}\right)^{k_l} \\
&\quad\cdot\big(\phi_1(\omega)(1-2\chi+\chi^2) + \phi_2(\omega)(2\chi-\chi^2) + \phi_3(\omega)(\chi^2-\chi) + I_{\chi=1}^{\alpha-1}(\chi\phi-\chi^2\phi) \\
&\quad + I_{\chi=1}^{\alpha}(\chi^2\phi - 2\chi\phi) + I_\chi^\alpha\phi + \sum_{j=1}^{l} a_j(\omega)I_1^{\alpha_{1j}}\ldots I_n^{\alpha_{nj}} I_{\chi=1}^{\alpha-1}(\chi\Lambda - \chi^2\Lambda) \\
&\quad + \sum_{j=1}^{l} a_j(\omega)I_1^{\alpha_{1j}}\ldots I_n^{\alpha_{nj}} I_{\chi=1}^{\alpha}(\chi^2\Lambda - 2\chi\Lambda)\big).
\end{aligned}$$

It follows from the proof of Theorem 2.1 that $(\mathcal{F}\Lambda)\in S([0,1]\times[0,1]^n)$. We shall show that $\mathcal{F}$ is contractive. Indeed, for $\Lambda_1,\ \Lambda_2 \in S([0,1]\times[0,1]^n)$, we have

$$\begin{aligned}
&(\mathcal{F}\Lambda_1)(\chi,\omega) - (\mathcal{F}\Lambda_2)(\chi,\omega) \\
&= \sum_{k=1}^{\infty}(-1)^k \sum_{k_1+\cdots+k_l=k}\binom{k}{k_1,\ldots,k_l}\left(a_1(\omega)I_\chi^\alpha I_1^{\alpha_{11}}\ldots I_n^{\alpha_{n1}}\right)^{k_1}\cdots\left(a_l(\omega)I_\chi^\alpha I_1^{\alpha_{1l}}\ldots I_n^{\alpha_{nl}}\right)^{k_l} \\
&\quad\cdot\big(I_{\chi=1}^{\alpha-1}(\chi\phi(\chi,\omega,\Lambda_1) - \chi^2\phi(\chi,\omega,\Lambda_1)) - I_{\chi=1}^{\alpha-1}(\chi\phi(\chi,\omega,\Lambda_2) - \chi^2\phi(\chi,\omega,\Lambda_2)) \\
&\quad + I_{\chi=1}^{\alpha}(\chi^2\phi(\chi,\omega,\Lambda_1) - 2\chi\phi(\chi,\omega,\Lambda_1)) - I_{\chi=1}^{\alpha}(\chi^2\phi(\chi,\omega,\Lambda_2) - 2\chi\phi(\chi,\omega,\Lambda_2)) \\
&\quad + I_\chi^\alpha\phi(\chi,\omega,\Lambda_1) - I_\chi^\alpha\phi(\chi,\omega,\Lambda_2) + \sum_{j=1}^{l} a_j(\omega)I_1^{\alpha_{1j}}\ldots I_n^{\alpha_{nj}} I_{\chi=1}^{\alpha-1}(\chi\Lambda_1 - \chi^2\Lambda_1) \\
&\quad - \sum_{j=1}^{l} a_j(\omega)I_1^{\alpha_{1j}}\ldots I_n^{\alpha_{nj}} I_{\chi=1}^{\alpha-1}(\chi\Lambda_2 - \chi^2\Lambda_2) + \sum_{j=1}^{l} a_j(\omega)I_1^{\alpha_{1j}}\ldots I_n^{\alpha_{nj}} I_{\chi=1}^{\alpha}(\chi^2\Lambda_1 - 2\chi\Lambda_1) \\
&\quad - \sum_{j=1}^{l} a_j(\omega)I_1^{\alpha_{1j}}\ldots I_n^{\alpha_{nj}} I_{\chi=1}^{\alpha}(\chi^2\Lambda_2 - 2\chi\Lambda_2)\big)
\end{aligned}$$

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$$= \sum_{k=1}^{\infty}(-1)^k \sum_{k_1+\cdots+k_l=k} \binom{k}{k_1,\ldots,k_l} \left(a_1(\omega) I_\chi^\alpha I_1^{\alpha_{11}} \ldots I_n^{\alpha_{n1}}\right)^{k_1} \cdots \left(a_l(\omega) I_\chi^\alpha I_1^{\alpha_{1l}} \ldots I_n^{\alpha_{nl}}\right)^{k_l}$$

$$\cdot \big(\sum_{j=1}^{l} a_j(\omega) I_1^{\alpha_{1j}} \ldots I_n^{\alpha_{nj}} I_{\chi=1}^{\alpha-1}(\chi-\chi^2)(\Lambda_1-\Lambda_2) + \sum_{j=1}^{l} a_j(\omega) I_1^{\alpha_{1j}} \ldots I_n^{\alpha_{nj}} I_{\chi=1}^{\alpha}$$

$$\cdot (\chi^2-2\chi)(\Lambda_1-\Lambda_2) + I_{\chi=1}^{\alpha-1}(\chi-\chi^2)(\phi(\chi,\omega,\Lambda_1)-\phi(\chi,\omega,\Lambda_2))$$

$$+ I_{\chi=1}^{\alpha}(\chi^2-2\chi)(\phi(\chi,\omega,\Lambda_1)-\phi(\chi,\omega,\Lambda_2)) + I_\chi^\alpha(\phi(\chi,\omega,\Lambda_1)-\phi(\chi,\omega,\Lambda_2))\big).$$

Therefore,

$$\begin{aligned} &\|\mathcal{F}\Lambda_1 - \mathcal{F}\Lambda_2\| \\ &\quad \le \left(\frac{1}{4}+\frac{1}{\alpha}\right)\frac{1}{\Gamma(\alpha)}\sum_{j=1}^{l} A_j E_{\mathcal{M}_j}^{(\alpha,1)}(A_1,\ldots,A_l)\|\Lambda_1-\Lambda_2\| \\ &\qquad + \mathcal{C}\left(\frac{\frac{1}{4}+\frac{1}{\alpha}}{\Gamma(\alpha)} E_{\mathcal{M}_0}^{(\alpha,1)}(A_1,\ldots,A_l) + E_{\mathcal{M}_0}^{(\alpha,\alpha+1)}(A_1,\ldots,A_l)\right)\|\Lambda_1-\Lambda_2\| \\ &\quad = q\|\Lambda_1-\Lambda_2\|. \end{aligned}$$

Since $q<1$, equation (1.1) has a unique uniformly bounded solution in $S([0,1]\times[0,1]^n)$ by BCP. The proof is completed. ∎

3 The Hyers–Ulam stability

In this section, we are going to derive the Hyers–Ulam stability of equation (1.1) using the implicit integral equation from Section 2.

Definition 3.1 We say that the FNPIDE (1.1) is Hyers–Ulam stable if there exists a constant $\mathcal{K}>0$ such that for all $\varepsilon>0$ and a continuously differentiable function Λ satisfying the three boundary conditions and the inequality

$$\left\| \frac{{}_c\partial^\alpha}{\partial\chi^\alpha}\Lambda(\chi,\omega) + \sum_{j=1}^{l} a_j(\omega) I_1^{\alpha_{1j}} \ldots I_n^{\alpha_{nj}}\Lambda(\chi,\omega) - \phi(\chi,\omega,\Lambda(\chi,\omega)) \right\| < \varepsilon,$$

then there exists a solution Λ_0 of equation (1.1) such that

$$\|\Lambda(\chi,\omega)-\Lambda_0(\chi,\omega)\| < \mathcal{K}\varepsilon,$$

where $\mathcal{K}$ is a Hyers–Ulam stability constant.

Theorem 3.1 *Suppose $a_j, \phi_1, \phi_2, \phi_3 \in C([0,1]^n)$ for $j=1,2,\ldots,j$, ϕ is a continuous function on $[0,1]\times[0,1]^n\times\mathbb{R}$ satisfying the Lipschitz condition for a positive constant $\mathcal{C}$*

$$|\phi(\chi,\omega,y_1)-\phi(\chi,\omega,y_2)| \le \mathcal{C}|y_1-y_2|, \quad y_1,y_2\in\mathbb{R},$$

https://doi.org/10.4153/S0008414X24000348 Published online by Cambridge University Press

$\alpha_{ij} \geq 0$ for all $i = 1, \ldots, n,\ j = 1, \ldots, l$, and there is $1 \leq i_0 \leq n$ such that $\alpha_{i_0 j} > 0$ for all $j = 1, \ldots, l$. Furthermore, we assume that

$$\mathcal{M}_j = \begin{bmatrix} \alpha_{11} \ldots & \alpha_{1l} & \alpha_{1j}+1 \\ \alpha_{21} \ldots & \alpha_{2l} & \alpha_{2j}+1 \\ \ldots & & \\ \alpha_{n1} \ldots & \alpha_{nl} & \alpha_{nj}+1 \end{bmatrix},$$

and

$$q = \left(\frac{1}{4} + \frac{1}{\alpha}\right) \frac{1}{\Gamma(\alpha)} \sum_{j=1}^{l} A_j E_{\mathcal{M}_j}^{(\alpha,1)}(A_1, \ldots, A_l)$$
$$+ \mathcal{C}\left(\frac{\frac{1}{4} + \frac{1}{\alpha}}{\Gamma(\alpha)} E_{\mathcal{M}_0}^{(\alpha,1)}(A_1, \ldots, A_l) + E_{\mathcal{M}_0}^{(\alpha,\alpha+1)}(A_1, \ldots, A_l) \right) < 1,$$

where

$$\mathcal{M}_0 = \begin{bmatrix} \alpha_{11} \ldots & \alpha_{1l} & 1 \\ \alpha_{21} \ldots & \alpha_{2l} & 1 \\ \ldots & & \\ \alpha_{n1} \ldots & \alpha_{nl} & 1 \end{bmatrix}.$$

Then equation (1.1) is Hyers–Ulam stable in the space $S([0,1] \times [0,1]^n)$.

Proof Let

$$\Lambda_1(\chi, \omega) = \frac{{}_c\partial^\alpha}{\partial \chi^\alpha} \Lambda(\chi, \omega) + \sum_{j=1}^{l} a_j(\omega) I_1^{\alpha_{1j}} \ldots I_n^{\alpha_{nj}} \Lambda(\chi, \omega) - \phi(\chi, \omega, \Lambda(\chi, \omega)).$$

Then

$$\frac{{}_c\partial^\alpha}{\partial \chi^\alpha} \Lambda(\chi, \omega) + \sum_{j=1}^{l} a_j(\omega) I_1^{\alpha_{1j}} \ldots I_n^{\alpha_{nj}} \Lambda(\chi, \omega) = \phi(\chi, \omega, \Lambda(\chi, \omega)) + \Lambda_1(\chi, \omega),$$

and from our assumption

$$\|\Lambda_1\| < \varepsilon.$$

It follows from the proof of Theorem 2.1 that

$$\Lambda(\chi, \omega)$$
$$= \sum_{k=0}^{\infty} (-1)^k \sum_{k_1 + \cdots + k_l = k} \binom{k}{k_1, \ldots, k_l} \left(a_1(\omega) I_\chi^\alpha I_1^{\alpha_{11}} \ldots I_n^{\alpha_{n1}}\right)^{k_1} \cdots \left(a_l(\omega) I_\chi^\alpha I_1^{\alpha_{1l}} \ldots I_n^{\alpha_{nl}}\right)^{k_l}$$
$$\cdot \Big(\phi_1(\omega)(1 - 2\chi + \chi^2) + \phi_2(\omega)(2\chi - \chi^2) + \phi_3(\omega)(\chi^2 - \chi) + I_{\chi=1}^{\alpha-1}(\chi\phi - \chi^2\phi)$$
$$+ I_{\chi=1}^{\alpha}(\chi^2\phi - 2\chi\phi) + I_\chi^\alpha \phi + \sum_{j=1}^{l} a_j(\omega) I_1^{\alpha_{1j}} \ldots I_n^{\alpha_{nj}} I_{\chi=1}^{\alpha-1}(\chi\Lambda - \chi^2\Lambda)$$
$$+ \sum_{j=1}^{l} a_j(\omega) I_1^{\alpha_{1j}} \ldots I_n^{\alpha_{nj}} I_{\chi=1}^{\alpha}(\chi^2\Lambda - 2\chi\Lambda)$$
$$+ I_{\chi=1}^{\alpha-1}(\chi\Lambda_1 - \chi^2\Lambda_1) + I_{\chi=1}^{\alpha}(\chi^2\Lambda_1 - 2\chi\Lambda_1) + I_\chi^\alpha \Lambda_1\Big),$$

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and

$$\begin{aligned}
&\Lambda_0(\chi,\omega)\\
&=\sum_{k=0}^{\infty}(-1)^k\sum_{k_1+\cdots+k_l=k}\binom{k}{k_1,\ldots,k_l}\left(a_1(\omega)I_\chi^\alpha I_1^{\alpha_{11}}\ldots I_n^{\alpha_{n1}}\right)^{k_1}\cdots\left(a_l(\omega)I_\chi^\alpha I_1^{\alpha_{1l}}\ldots I_n^{\alpha_{nl}}\right)^{k_l}\\
&\quad\cdot(\phi_1(\omega)(1-2\chi+\chi^2)+\phi_2(\omega)(2\chi-\chi^2)+\phi_3(\omega)(\chi^2-\chi)+I_{\chi=1}^{\alpha-1}(\chi\phi-\chi^2\phi)\\
&\quad+I_{\chi=1}^{\alpha}(\chi^2\phi-2\chi\phi)+I_\chi^\alpha\phi+\sum_{j=1}^{l}a_j(\omega)I_1^{\alpha_{1j}}\ldots I_n^{\alpha_{nj}}I_{\chi=1}^{\alpha-1}(\chi\Lambda_0-\chi^2\Lambda_0)\\
&\quad+\sum_{j=1}^{l}a_j(\omega)I_1^{\alpha_{1j}}\ldots I_n^{\alpha_{nj}}I_{\chi=1}^{\alpha}(\chi^2\Lambda_0-2\chi\Lambda_0)),
\end{aligned}$$

by noting that ϕ is a continuous and

$$\begin{aligned}
&\sup_{(\chi,\omega)\in[0,1]\times[0,1]^n}|\phi(\chi,\omega,\Lambda(\chi,\omega))|\\
&=\sup_{(\chi,\omega)\in[0,1]\times[0,1]^n}|\phi(\chi,\omega,\Lambda(\chi,\omega))-\phi(\chi,\omega,0)+\phi(\chi,\omega,0)|\\
&\le\mathcal{C}\|\Lambda\|+\sup_{(\chi,\omega)\in[0,1]\times[0,1]^n}|\phi(\chi,\omega,0)|<+\infty,
\end{aligned}$$

if $\Lambda\in S([0,1]\times[0,1]^n)$.

Hence,

$$\begin{aligned}
&|\Lambda(\chi,\omega)-\Lambda_0(\chi,\omega)|\\
&\le\sum_{k=0}^{\infty}\sum_{k_1+\cdots+k_l=k}\binom{k}{k_1,\ldots,k_l}A_1^{k_1}\ldots A_l^{k_l}I_\chi^{\alpha k}I_1^{\alpha_{11}k_1+\cdots+\alpha_{1l}k_l}\\
&\quad\ldots I_n^{\alpha_{n1}k_1+\cdots+\alpha_{nl}k_l}\cdot(I_{\chi=1}^{\alpha-1}|(\chi-\chi^2)(\phi(\chi,\omega,\Lambda)-\phi(\chi,\omega,\Lambda_0))|\\
&\quad+I_{\chi=1}^{\alpha}|(\chi^2-2\chi)(\phi(\chi,\omega,\Lambda)-\phi(\chi,\omega,\Lambda_0))|+I_\chi^\alpha|(\phi(\chi,\omega,\Lambda)-\phi(\chi,\omega,\Lambda_0))|\\
&\quad+\sum_{j=1}^{l}a_j(\omega)I_1^{\alpha_{1j}}\ldots I_n^{\alpha_{nj}}I_{\chi=1}^{\alpha-1}|(\chi-\chi^2)(\Lambda-\Lambda_0)|\\
&\quad+\sum_{j=1}^{l}a_j(\omega)I_1^{\alpha_{1j}}\ldots I_n^{\alpha_{nj}}I_{\chi=1}^{\alpha}|(\chi^2-2\chi)(\Lambda-\Lambda_0)|\\
&\quad+I_{\chi=1}^{\alpha-1}|(\chi\Lambda_1-\chi^2\Lambda_1)|+I_{\chi=1}^{\alpha}|(\chi^2\Lambda_1-2\chi\Lambda_1)|+I_\chi^\alpha|\Lambda_1|),
\end{aligned}$$

which implies that

$$\begin{aligned}
&\|\Lambda-\Lambda_0\|\\
&\le\sum_{k=0}^{\infty}\sum_{k_1+\cdots+k_l=k}\binom{k}{k_1,\ldots,k_l}A_1^{k_1}\ldots A_l^{k_l}I_\chi^{\alpha k}I_1^{\alpha_{11}k_1+\cdots+\alpha_{1l}k_l}\\
&\quad\ldots I_n^{\alpha_{n1}k_1+\cdots+\alpha_{nl}k_l}\\
&\quad\cdot(I_{\chi=1}^{\alpha-1}|(\chi\Lambda_1-\chi^2\Lambda_1)|+I_{\chi=1}^{\alpha}|(\chi^2\Lambda_1-2\chi\Lambda_1)|+I_\chi^\alpha|\Lambda_1|)
\end{aligned}$$

$$+\left(\frac{1}{4}+\frac{1}{\alpha}\right)\frac{1}{\Gamma(\alpha)}\sum_{j=1}^{l}A_jE_{\mathcal{M}_j}^{(\alpha,1)}(A_1,\ldots,A_l)\|\Lambda-\Lambda_0\|$$

$$+\mathcal{C}\left(\frac{\frac{1}{4}+\frac{1}{\alpha}}{\Gamma(\alpha)}E_{\mathcal{M}_0}^{(\alpha,1)}(A_1,\ldots,A_l)+E_{\mathcal{M}_0}^{(\alpha,\alpha+1)}(A_1,\ldots,A_l)\right)\|\Lambda-\Lambda_0\|$$

$$=q\|\Lambda_1-\Lambda_0\|+\left(\frac{\frac{1}{4}+\frac{1}{\alpha}}{\Gamma(\alpha)}E_{\mathcal{M}_0}^{(\alpha,1)}(A_1,\ldots,A_l)+E_{\mathcal{M}_0}^{(\alpha,\alpha+1)}(A_1,\ldots,A_l)\right)\|\Lambda_1\|.$$

Finally, we have

$$\begin{aligned}&\|\Lambda(\chi,\omega)-\Lambda_0(\chi,\omega)\|\\&\quad\le\frac{1}{1-q}\left(\frac{\frac{1}{4}+\frac{1}{\alpha}}{\Gamma(\alpha)}E_{\mathcal{M}_0}^{(\alpha,1)}(A_1,\ldots,A_l)+E_{\mathcal{M}_0}^{(\alpha,\alpha+1)}(A_1,\ldots,A_l)\right)\|\Lambda_1\|\\&\quad<\mathcal{K}\varepsilon,\end{aligned}$$

where

$$\mathcal{K}=\frac{1}{1-q}\left(\frac{\frac{1}{4}+\frac{1}{\alpha}}{\Gamma(\alpha)}E_{\mathcal{M}_0}^{(\alpha,1)}(A_1,\ldots,A_l)+E_{\mathcal{M}_0}^{(\alpha,\alpha+1)}(A_1,\ldots,A_l)\right).$$

This completes the proof. ∎

Remark 3.2 We should point out that Theorem 3.1 does not require the condition that ϕ is a bounded function. Moreover, Λ_0 is not a uniformly bounded function in general, which is different from Theorem 2.2. Since $\Omega=[0,1]\times[0,1]^n$ is bounded and closed (compact) the Hyers–Ulam stability is guaranteed by noting the fact that all continuous functions reach their maximum and minimum over Ω. The Hyers–Ulam stability constant $\mathcal{K}$ obtained above is the best possible in our approach. There is a possible lower bound on the Hyers–Ulam stability constant but it would be tough and difficult to find it.

4 Examples

We will present two examples demonstrating applications of key theorems obtained from previous sections.

Example 4.1 The following fractional differential equation with a nonlocal initial condition:

$$\begin{cases}{}_cD^{0.5}\Phi(x)+2\Phi(x)=\dfrac{1}{513}\sin(x\Phi(x)), & x\in[0,1],\\ \Phi(0)=\dfrac{1}{1349}\displaystyle\int_0^1\Phi(x)dx,\end{cases}\tag{4.1}$$

has a unique solution in $C[0,1]$.

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Proof Clearly,

$$f(x,y)=\frac{1}{513}\sin(xy)$$

is bounded and

$$|f(x,y_1)-f(x,y_2)|\le\frac{1}{513}|xy_1-xy_2|\le\frac{1}{513}|y_1-y_2|,$$

if $x\in[0,1]$. It remains to find the value

$$\begin{aligned}\mathcal{B}&=\frac{\mathcal{L}}{\alpha}E_{\alpha,\alpha}(|a|)+|\beta|E_{\alpha,1}(|a|)=\frac{2}{513}E_{0.5,0.5}(2)+\frac{1}{1349}E_{0.5,1}(2)\\&\approx 0.851641+0.0807568<1.\end{aligned}$$

Hence, equation (4.1) has a unique solution in the Banach space $C[0,1]$. ∎

Example 4.2 The following FNPIDE with a mixed boundary condition:

$$\begin{cases}\frac{{}_c\partial^{2.5}}{\partial\chi^{2.5}}\Lambda(\chi,\omega)+\sum_{j=1}^{4}a_j(\omega)I_1^{\alpha_{1j}}\dots I_4^{\alpha_{4j}}\Lambda(\chi,\omega)\\=\frac{1}{59}\cos(\chi\omega+\Lambda(\chi,\omega))+\frac{1}{\chi^2+\omega^2+2},\\\Lambda(0,\omega)=\omega^2+1,\quad\Lambda(1,\omega)=\frac{1}{9}\omega,\ \ \Lambda'(1,\omega)=\frac{1}{6}\omega^3,\end{cases}\tag{4.2}$$

where

$$a_1(\omega)=\frac{\omega}{3},\quad a_2(\omega)=\frac{1}{2}\omega^2,\quad a_3(\omega)=\frac{|\omega|}{3},\quad a_4(\omega)=\frac{1}{9},$$

and

$$(\alpha_{ij})_{1\le i,j\le 4}=\begin{bmatrix}1.1&1.3&0.7&1.4\\1.3&2.3&3.1&2\\0.7&1.6&2.1&1.2\\2&3.1&4.1&2.2\end{bmatrix},$$

has a unique uniformly bounded solution and the Hyers–Ulam stability in the space $S([0,1]\times[0,1]^4)$.

Proof Clearly, a_j for $j=1,2,3,4$, $\phi_1,\phi_2,\phi_3\in C([0,1]^4)$ and

$$\phi(\chi,\omega,\Lambda)=\frac{1}{59}\cos(\chi\omega+\Lambda(\chi,\omega))+\frac{1}{\chi^2+\omega^2+2}$$

is a continuous and bounded function on $[0,1]\times[0,1]^4\times\mathbb{R}$, satisfying the Lipschitz condition with $\mathcal{C}=1/59$:

$$|\phi(\chi,\omega,y_1)-\phi(\chi,\omega,y_2)|\le\frac{1}{59}|\cos(\chi\omega+y_1)-\cos(\chi\omega+y_2)|\le\frac{1}{59}|y_1-y_2|.$$

Furthermore,

$$A_1=1/3,\quad A_2=1/2,\quad A_3=1/3,\quad A_4=1/9.$$

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We need to compute the value

$$q = \left(\frac{1}{4} + \frac{1}{\alpha}\right) \frac{1}{\Gamma(\alpha)} \sum_{j=1}^{l} A_j E_{\mathcal{M}_j}^{(\alpha,1)}(A_1, \ldots, A_l)$$
$$+ \mathcal{C} \left(\frac{\frac{1}{4} + \frac{1}{\alpha}}{\Gamma(\alpha)} E_{\mathcal{M}_0}^{(\alpha,1)}(A_1, \ldots, A_l) + E_{\mathcal{M}_0}^{(\alpha,\alpha+1)}(A_1, \ldots, A_l) \right)$$
$$= \left(\frac{1}{4} + \frac{1}{2.5}\right) \frac{1}{\Gamma(2.5)} \sum_{j=1}^{4} A_j E_{\mathcal{M}_j}^{(2.5,1)}(1/3, 1/2, 1/3, 1/9)$$
$$+ \frac{1}{59} \left(\frac{\frac{1}{4} + \frac{1}{2.5}}{\Gamma(2.5)} E_{\mathcal{M}_0}^{(2.5,1)}(1/3, 1/2, 1/3, 1/9) + E_{\mathcal{M}_0}^{(2.5,3.5)}(1/3, 1/2, 1/3, 1/9) \right),$$

where

$$M_1 = \begin{bmatrix} 1.1 & 1.3 & 0.7 & 1.4 & 2.1 \\ 1.3 & 2.3 & 3.1 & 2 & 2.3 \\ 0.7 & 1.6 & 2.1 & 1.2 & 1.7 \\ 2 & 3.1 & 4.1 & 2.2 & 3 \end{bmatrix},$$

$$M_2 = \begin{bmatrix} 1.1 & 1.3 & 0.7 & 1.4 & 2.3 \\ 1.3 & 2.3 & 3.1 & 2 & 3.3 \\ 0.7 & 1.6 & 2.1 & 1.2 & 2.6 \\ 2 & 3.1 & 4.1 & 2.2 & 4.1 \end{bmatrix},$$

$$M_3 = \begin{bmatrix} 1.1 & 1.3 & 0.7 & 1.4 & 1.7 \\ 1.3 & 2.3 & 3.1 & 2 & 4.1 \\ 0.7 & 1.6 & 2.1 & 1.2 & 3.1 \\ 2 & 3.1 & 4.1 & 2.2 & 5.1 \end{bmatrix},$$

$$M_4 = \begin{bmatrix} 1.1 & 1.3 & 0.7 & 1.4 & 2.4 \\ 1.3 & 2.3 & 3.1 & 2 & 3 \\ 0.7 & 1.6 & 2.1 & 1.2 & 2.2 \\ 2 & 3.1 & 4.1 & 2.2 & 3.2 \end{bmatrix},$$

and finally

$$M_0 = \begin{bmatrix} 1.1 & 1.3 & 0.7 & 1.4 & 1 \\ 1.3 & 2.3 & 3.1 & 2 & 1 \\ 0.7 & 1.6 & 2.1 & 1.2 & 1 \\ 2 & 3.1 & 4.1 & 2.2 & 1 \end{bmatrix}.$$

Using the following Python codes to get

$$q = 6.49406088226196 * 10^{-293} < 1.$$

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Hence, equation (4.2) has a unique uniformly bounded solution in the space $S([0,1] \times [0,1]^4)$ by Theorem 2.2, and it is Hyers–Ulam stable by Theorem 3.1. ■

```
# beginning codes for Example 8
import math
from sympy import gamma

def partition(n, m):
    if m == 1:
        yield (n,)
    else:
        for i in range(n+1):
            for j in partition(n-i, m-1):
                yield (i,) + j



def ME(M, z, alpha, beta): # (alpha, beta)-Matrix
                           #Mittag-Leffler function
    m = len(M)
    zl = len(z)
    result = 0
    for l in range(0, 20): #approximate value
        for l_partition in partition(l, zl):
            if all(map(lambda x: x >= 0, l_partition)):
                combination = 1
                for i in range(zl):
                   combination *= math.factorial(l_partition[i])
                combination = math.factorial(l) / combination
                gamproduct = 1
                for i in range(m):
                    gaminput = sum([M[i][j] * l_partition[j]
                    for j in range(zl)]) + M[i][zl]
                    gamproduct *= gamma(gaminput)
                numerator = 1
                for i in range(zl):
                    numerator *= z[i] ** l_partition[i]
                result += (numerator / gamproduct) * combination
                result *= (1/gamma(alpha * l + beta)) * result
    return result

#The following is our calculation of q value
alpha = 2.5
beta = 1
M1 = [[1.1, 1.3, 0.7, 1.4, 2.1], [1.3, 2.3, 3.1, 2, 2.3],
[0.7, 1.6, 2.1, 1.2, 1.7], [2, 3.1, 4.1, 2.2, 3]]
M2 = [[1.1, 1.3, 0.7, 1.4, 2.3], [1.3, 2.3, 3.1, 2, 3.3],
[0.7, 1.6, 2.1, 1.2, 2.6], [2, 3.1, 4.1, 2.2, 4.1]]
M3 = [[1.1, 1.3, 0.7, 1.4, 1.7], [1.3, 2.3, 3.1, 2, 4.1],
[0.7, 1.6, 2.1, 1.2, 3.1], [2, 3.1, 4.1, 2.2, 5.1]]
M4 = [[1.1, 1.3, 0.7, 1.4, 2.4], [1.3, 2.3, 3.1, 2, 3],
[0.7, 1.6, 2.1, 1.2, 2.2], [2, 3.1, 4.1, 2.2, 3.2]]
M0 = [[1.1, 1.3, 0.7, 1.4, 1], [1.3, 2.3, 3.1, 2, 1],
[0.7, 1.6, 2.1, 1.2, 1], [2, 3.1, 4.1, 2.2, 1]]
z = [1/3, 1/2, 1/3, 1/9]

result1 = ME(M1, z, alpha, beta)
result2= ME(M2, z, alpha, beta)
result3 = ME(M3, z, alpha, beta)
```

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```
result4 = ME(M4, z, alpha, beta)
result5 = ME(M0, z, alpha, beta)
result6 = ME(M0, z, alpha, alpha + 1)
result = (1/4 + 1/2.5) * (1/gamma(2.5))*(1/3 * result1 +
1/2 * result2 + 1/3 * result3 + 1/9 * result4)
+  1/59 * ((1/4 + 1/2.5)/gamma(2.5)) * result5
+ 1/59 * result6
print("The q value is", result)
#end codes
```

Remark 4.3 We have used the Python language to find approximates values of our newly established pair-parameter matrix Mittag–Leffler functions to study the uniqueness of solutions to equation (1.1). Slightly changing the codes we can compute values of the multivariate Mittag–Leffler functions. As far as we know from current research related to computation of the Mittag–Leffler functions, this approach is efficient and simple.

5 Conclusion

We have studied the uniqueness and Hyers–Ulam stability to the new equation (1.1) based on the pair-parameter matrix Mittag–Leffler functions, Banach's contractive principle as well as Babenko's approach. A few examples were provided to demonstrate applications of main results derived. The methods used in the current work are also suitable for different types of differential equations with various initial or boundary conditions, as well as integral equations with variable coefficients, which cannot be handled by any existing integral transforms.

Acknowledgments The author is thankful to the reviewers and editor for giving valuable comments and suggestions.

Competing interests The author declares no competing interests.

Data availability statement No data were used to support this study.

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Department of Mathematics and Computer Science, Brandon University, Brandon, MB, Canada
e-mail: lic@brandonu.ca

https://doi.org/10.4153/S0008414X24000348 Published online by Cambridge University Press