This is a ``preproof'' accepted article for *Canadian Mathematical Bulletin*
This version may be subject to change during the production process.
DOI: 10.4153/S0008439525100878

Canad. Math. Bull. Vol. **00** (0), 2020 pp. 1–20
http://dx.doi.org/10.4153/xxxx

The Schur-Agler class in infinitely many variables

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Abstract. **We define the Schur-Agler class in infinite variables to consist of functions whose restrictions to finite dimensional polydisks belong to the Schur-Agler class. We show that a natural generalization of an Agler decomposition holds and the functions possess transfer function realizations that allow us to extend the functions to the unit ball of ℓ^∞. We also give a Pick interpolation type theorem which displays a subtle difference with finitely many variables. Finally, we make a brief connection to Dirichlet series derived from the Schur-Agler class in infinite variables via the Bohr correspondence.**

1 Introduction

This article is about establishing basic properties of the Schur-Agler class in infinitely many variables. To back up a bit, the *Schur class* in N variables, $\mathcal{S}_N$, will refer to the set of analytic functions $f : \mathbb{D}^N \to \overline{\mathbb{D}}$ where $\mathbb{D}^N$ is the N dimensional unit polydisk

$$\mathbb{D}^N = \{z = (z_1, \dots, z_N) \in \mathbb{C}^N : \forall j, |z_j| < 1\}.$$

The Schur class, $\mathcal{S}_\infty$, in infinitely many variables will refer to holomorphic functions (meaning complex Fréchet differentiable) on $Ball(\ell^\infty)$ that are bounded by one in supremum norm. (We review standard notations in Section 2.)

A remarkable result, attributed to Hilbert, is that if we are given Schur class functions $f_N \in \mathcal{S}_N$ for each N, and if for $N > M$ we have

$$f_N(z_1, \dots, z_M, 0, \dots, 0) \equiv f_M(z_1, \dots, z_M),$$

then there exists a holomorphic function $f : Ball(c_0) \to \overline{\mathbb{D}}$ such that $f(z_1, \dots, z_N, 0, \dots) = f_N(z_1, \dots, z_N)$ and such that f is continuous in the norm topology on $Ball(c_0)$. See [15], Theorem 2.21 for details.

Even more, f has a homogeneous expansion

$$f(z) = \sum_{m=0}^{\infty} P_m(z)$$

where each P_m is an m-homogeneous form on c_0 (see [15], Proposition 2.28). Davie-Gamelin [13] proved that f and its homogeneous expansion extends further to $Ball(\ell^\infty)$. This extension (called the Aron-Berner extension) is somewhat elaborate as

2020 Mathematics Subject Classification: 47A48, 47A13, 47A57, 46E50, 30B50, 32A38, 46G20, 32E30.
Keywords: Schur class, Schur-Agler class, Agler class, transfer function realization, Hilbert polydisk, Dirichlet series, Bohr correspondence, von Neumann's inequality, infinite polydisk, holomorphy in infinite dimensions, Pick interpolation.

it requires passing to the symmetric m-multilinear form associated to each P_m, extending to ℓ^∞ and then proving that the extended homogeneous expansion converges in $Ball(\ell^\infty)$.

Remark 1.1 An important subtlety in all of this theory is that, although we can form a Taylor series $\sum_\alpha f_\alpha z^\alpha$ associated to f that converges absolutely to f on the Hilbert multidisk $\mathbb{D}_2^\infty = Ball(\ell^\infty) \cap \ell^2$, in general there will be points in $Ball(c_0)$ where the Taylor series does not converge absolutely. See [15], Proposition 4.6 and Theorem 10.1.

Remark 1.2 An important motivation in recent decades for the study of $\mathcal{S}_\infty$ is through its application to Dirichlet series. In particular, the space $H^\infty(Ball(c_0))$ of bounded holomorphic functions on $Ball(c_0)$ is isometrically isomorphic to the space $\mathscr{H}^\infty$ of Dirchlet series that converge and are bounded on the right half plane $\{z \in \mathbb{C} : \Re z > 0\}$. The isomorphism, called the the Bohr correspondence, is given by

$$F \in H^\infty(Ball(c_0)) \mapsto f(s) = F(p_1^{-s}, p_2^{-s}, \dots) \in \mathscr{H}^\infty$$

where $p_1 = 2, p_2 = 3, \dots$ are the prime numbers. The norm in both cases refers to the supremum norm. The space $\mathscr{H}^\infty$ and its isomorphism with $H^\infty(Ball(c_0))$ appears naturally in the study of dilation completeness problems on $L^2(0, 1)$ as presented in Hedenmalm-Lindqvist-Seip [17]. See also [19], [22], [12], [20], [21],[15]. ⋄

Returning to the main topic, the Schur-Agler class in N dimensions, $\mathcal{A}_N$, consists of $f \in \mathcal{S}_N$ such that for any N-tuple $T = (T_1, \dots, T_N)$ of strictly contractive commuting operators on a Hilbert space we have

$$\|f(T)\| \le 1$$

where $f(T)$ is defined using absolutely convergent power series. We will say *Agler class* for short. An inequality of von Neumann [24] proves that the Agler class in one variable coincides with the Schur class in one variable; $\mathcal{A}_1 = \mathcal{S}_1$. Andô's dilation theorem [9] proves that the same relation holds in two variables; namely; $\mathcal{A}_2 = \mathcal{S}_2$. Counterexamples first constructed by Varopoulos [23] show that $\mathcal{A}_N \neq \mathcal{S}_N$ for $N > 2$. A basic motivation for studying the Agler class is that it can provide insights into the more classical spaces $\mathcal{S}_1, \mathcal{S}_2$— see Agler-McCarthy-Young [4], [5], [7]. On the other hand, the Agler class is interesting more broadly: (1) for studying the operator theoretic problem of understanding the failure of von Neumann's inequality in 3 or more variables and (2) for providing a large source of interesting and easily constructible examples of functions within $\mathcal{S}_N$. Some recent papers on the Agler class are in [10], [11],[14], [18].

Functions in the Agler class have a variety of useful properties. First, they possess an *Agler decomposition* and an associated interpolation theorem. An Agler decomposition is a formula of the form

$$1 - f(z)\overline{f(w)} = \sum_{j=1}^{N} (1 - \bar{w}_j z_j) K_j(z, w) \tag{1.1}$$

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where $K_1, \ldots, K_N$ are positive semi-definite kernels on $\mathbb{D}^N \times \mathbb{D}^N$. The Agler-Pick interpolation theorem can be stated in the following form. Some standard references are [1], [2], [3].

Theorem 1.3 (**Agler**) *Given a finite subset $X \subset \mathbb{D}^N$ and a function $f : X \to \overline{\mathbb{D}}$ the following are equivalent:*

(1) There exists $\tilde{f} \in \mathcal{A}_N$ with $\tilde{f}\big|_X = f$.
(2) There exist positive semi-definite functions $K_1, \ldots, K_N$ on $X \times X$ such that for $z, w \in X$

$$1 - f(z)\overline{f(w)} = \sum_{j=1}^{N} (1 - z_j \bar{w}_j) K_j(z, w).$$

(3) For every N-tuple $T = (T_1, \ldots, T_N)$ of commuting, contractive, simultaneously diagonalizable matrices whose joint eigenspaces have dimension at most 1 and satisfy $\sigma(T) \subset X$, we have $\|f(T)\| \leq 1$.

Item (1) is a way of phrasing an interpolation problem as an extension problem. Item (2) is a restriction of an Agler decomposition. (See Section 2 for the definition of positive semi-definite function.) Item (3) says that the function f needs to satisfy a particular type of matrix von Neumann inequality. Notice that in this case $f(T)$ can be defined by using the diagonalization of T, and the dimension of the space that the T_j act on is at most $\#X$. Item (3) is not stated explicitly in the literature, at least not in this form, but it is a known component of the Agler-Pick interpolation theorem. (For the skeptical reader, the approach in this paper proves a generalization to infinite variables and does not directly rely on the finite variable theorem so one could pull a proof of the equivalence of (3) by simplifying certain proofs below.) We think item (3) is important to emphasize since it (conceptually) gives a way to *check* if interpolation is possible while item (2) is a useful *conclusion* when you know interpolation is possible. Item (3) is also the source of a subtlety in infinite variables.

A second key property of Agler class functions is that they possess a contractive transfer function realization formula, which means the following. There exists a contractive operator V acting on $\mathbb{C} \oplus \bigoplus_{j=1}^{N} \mathcal{H}_j$ where $\mathcal{H}_1, \ldots, \mathcal{H}_N$ are Hilbert spaces such that when we write V in block form $V = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ we have

$$f(z) = A + B\Delta(z)(1 - D\Delta(z))^{-1}C \tag{1.2}$$

where $\Delta(z) = \sum_{j=1}^{N} z_j P_j$ and each P_j represents projection onto $\mathcal{H}_j$ within the direct sum $\bigoplus_{k=1}^{N} \mathcal{H}_k$. It turns out that membership in the Agler class can be tested using *generic* matrices. Namely, analytic $f : \mathbb{D}^N \to \overline{\mathbb{D}}$ belongs to $\mathcal{A}_N$ if for every N-tuple T of commuting contractive simultaneously diagonalizable matrices with joint eigenspaces having dimension 1 we have $\|f(T)\| \leq 1$. With this reduction, defining $f(T)$ only requires the evaluation of f on the joint eigenvalues of T and not any regularity or absolute summability. This can be derived from item (3) in Theorem 1.3 or see [6] (Theorem 6.1 therein) where something more general is proven.

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Our goal is to prove that analogues of the Agler decomposition (1.1), the Agler-Pick interpolation theorem (Theorem 1.3), and the transfer function realization (1.2) hold in infinitely many variables. We also wish to make connections to Dirichlet series as in Remark 1.2. Remark 1.1 suggests that we cannot define $f(T)$ in the infinite variable setting using absolutely convergent series.

We would like to remark that this paper is not the first mention of the Agler class in infinitely many variables—see [8], Section 7. However one of our larger goals is to start with the simplest definition possible and deduce basic properties of the Agler class (such as its functional calculus) as well as to point out some subtleties of the theory. Here is what we imagine to be the simplest definition of the Agler class in infinitely many variables.

***Definition* 1.4** Given a function $f : Ball(c_{00}) \to \mathbb{C}$ we say f is in the *Agler class in infinite variables*, $\mathcal{A}_\infty$, if for every N, the restriction of f to $\mathbb{D}^N$ belongs to the Agler class in N variables, $\mathcal{A}_N$.

***Theorem* 1.5** *If $f \in \mathcal{A}_\infty$, then f has a transfer function realization: there exists a contractive operator V acting on $\mathbb{C} \oplus \bigoplus_{j=1}^\infty \mathcal{H}_j$ where $\mathcal{H}_1, \mathcal{H}_2, \dots$ are Hilbert spaces such that when we write V in block form $V = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ we have*

$$f(z) = A + B\Delta(z)(1 - D\Delta(z))^{-1}C \tag{1.3}$$

where $\Delta(z) = \sum_{j=1}^\infty z_j P_j$ and each P_j represents projection onto $\mathcal{H}_j$ within the direct sum $\bigoplus_{k=1}^\infty \mathcal{H}_k$. Letting $f_N(z) = f(z_1, \dots, z_N, 0, \dots)$, we have the following extension of von Neumann's inequality: for any tuple $T = (T_1, T_2, \dots)$ of commuting contractive operators (acting on a common Hilbert space) such that $\sup_j \|T_j\| < \infty$ we have that

$$\lim_{N\to\infty} f_N(T)$$

converges in the strong operator topology to a natural definition of $f(T)$ as

$$f(T) := (A \otimes I) + (B \otimes I)\Delta(T)(1 - (D \otimes I)\Delta(T))^{-1}(C \otimes I)$$

where $\Delta(T) := \sum_{j=1}^\infty P_j \otimes T_j$ (also convergent in the strong operator topology).

The transfer function formula (1.3) for f makes sense for $z \in Ball(\ell^\infty)$ and we have

$$\lim_{N\to\infty} f_N(z) = f(z)$$

for $z \in Ball(\ell^\infty)$. Thus, the transfer function formula readily shows that Agler class functions extend to $Ball(\ell^\infty)$. Our proof relies on a Montel theorem in infinite variables from [15] and does not use the intricate argument involving nets of points in $Ball(c_0)$ as in Davie-Gamelin [13] for the Schur class case.

The transfer function formula (1.3) is essentially equivalent (via a standard argument) to an Agler decomposition

$$1 - f(z)\overline{f(w)} = \sum_{j=1}^{\infty}(1 - \bar{w}_j z_j)K_j(z, w).$$

which we will show converges absolutely. Again, the K_j are positive semi-definite kernels on $Ball(\ell^\infty) \times Ball(\ell^\infty)$.

Remark 1.6 The paper Dritschel-McCullough [16] discusses a version of the Agler class in infinite variables via an approach to interpolation and realization formulas using *test functions*. Their definition of the Agler class in infinitely many variables is more expansive and allows for, for instance, linear functionals on ℓ^∞ that annihilate c_0 and do not have Agler decompositions in the sense of Theorem 1.5. This expanded Agler class has a more general type of Agler decomposition. See Proposition 5.4 of [16]. Here is a simplified version of an example they present.

Let $\alpha = (a_n)_{n=1}^{\infty}$ be an increasing sequence of positive real numbers that converge to some $a \in (0, 1)$; e.g. $a_n = \frac{1}{2} - \frac{1}{n+1}$. By the Hahn-Banach theorem, there exists a linear functional $L \in (\ell^\infty)^*$ such that $L(\alpha) = a$ and $\|L\| = 1$. We claim $L \notin \mathcal{A}_\infty$. If we had an Agler decomposition,

$$1 - L(z)\overline{L(w)} = \sum_{j=1}^{\infty}(1 - z_j\bar{w}_j)K_j(z, w)$$

with each K_j positive semi-definite on $Ball(\ell^\infty)$ and $\sum_{j=1}^{\infty} K_j(z, z) < \infty$, then inserting different combinations $z, w \in \{0, \alpha\}$ we have

$$1 = \sum_{j=1}^{\infty} K_j(0, 0), \qquad 1 = \sum_{j=1}^{\infty} K_j(\alpha, 0), \qquad 1 - a^2 = \sum_{j=1}^{\infty}(1 - a_j^2)K_j(\alpha, \alpha).$$

By Cauchy-Schwarz,

$$1 \leq \sum_{j=1}^{\infty} |K_j(\alpha, 0)| \leq \left(\sum_{j=1}^{\infty} K_j(0, 0)\right)^{1/2} \left(\sum_{j=1}^{\infty} K_j(\alpha, \alpha)\right)^{1/2}$$

so that $1 \leq \sum_{j=1}^{\infty} K_j(\alpha, \alpha)$. On the other hand, we have

$$(1 - a^2)(1 - \sum_{j=1}^{\infty} K_j(\alpha, \alpha)) = \sum_{j=1}^{\infty}(a^2 - a_j^2)K_j(\alpha, \alpha) \geq 0$$

and therefore this must equal zero which would imply $K_j(\alpha, \alpha) = 0$ for all j. This is a contradiction.

Theorem 1.5 is adapted to functions that have the added continuity that makes them completely determined by their values on $c_{00} \subset c_0$ and so our definition rules out functions that are zero on all of c_{00}. ⋄

Some aspects of interpolation in $\mathcal{A}_\infty$ are straightforward, however one important aspect has a subtlety.

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***Theorem* 1.7** *Let $X \subset Ball(\ell^\infty)$ be a finite subset and let $f : X \to \overline{\mathbb{D}}$ be a function. Consider the following conditions.*

(1) There exists $\tilde{f} \in \mathcal{A}_\infty$ with $\tilde{f}\big|_X = f$.
(2) There exist positive semi-definite functions $K_1, K_2, \dots$ on X such that for $z, w \in X$

$$1 - f(z)\overline{f(w)} = \sum_{j=1}^{\infty}(1 - z_j \bar{w}_j)K_j(z, w)$$

and for all $z \in X$

$$\sum_{j=1}^{\infty} K_j(z, z) < \infty.$$

(3) For every tuple $T = (T_1, T_2, \dots)$ of commuting, contractive, simultaneously diagonalizable matrices whose joint eigenspaces have dimension at most 1 and satisfy $\sigma(T) \subset X$, we have $\|f(T)\| \leq 1$.

Then,

- *(1) and (2) are equivalent and imply item (3).*
- *Item (3) implies (1) and (2) when $X \subset \mathbb{D}_2^\infty$.*

In particular, (1),(2), and (3) are equivalent when $X \subset \mathbb{D}_2^\infty$.

The subtlety alluded to above is that we only obtain a full generalization of an Agler-Pick interpolation theorem when our interpolation points lie in $\mathbb{D}_2^\infty$ and we do not know to what extent this condition can be removed.

Remark 1.8 Going back to Remark 1.6 and the example discussed there, the interpolation problem $0 \in \ell^\infty \mapsto 0 \in \mathbb{C}, \alpha \mapsto a$ cannot be solved within $\mathcal{A}_\infty$, however, the associated function $f : \{0, \alpha\} \to \{0, a\}$, $f(0) = 0$, $f(\alpha) = a$, satisfies the condition of item (3) above. Indeed, if we have commuting simultaneously diagonalizable contractive matrices T_j with $\sigma(T_j) \subset \{0, a_j\}$, then $f(T_1, T_2, \dots)$ will be contractive by continuity. Specifically, if $\vec{b}_0$ is the eigenvector for 0 and $\vec{b}_1$ the eigenvector for α, then contractivity of T_j means

$$|T_j(\sum_{k=0,1} c_k\vec{b}_k)| \leq |\sum_{k=0,1} c_k\vec{b}_k|$$

for arbitrary $c_0, c_1 \in \mathbb{C}$. But $T_j(\sum_{k=0,1} c_k\vec{b}_k) = c_1 a_j \vec{b}_1$ and sending $j \to \infty$ we get

$$|f(T)(\sum_{k=0,1} c_k\vec{b}_k)| \leq |\sum_{k=0,1} c_k\vec{b}_k|.$$

It seems that a complete relaxation of the condition $X \subset \mathbb{D}_2^\infty$ to $X \subset Ball(\ell^\infty)$ would lead to the broader notion of Agler class constructed with the test function approach of [16]. It would be interesting if the condition $X \subset \mathbb{D}_2^\infty$ could be relaxed to $X \subset Ball(c_0)$ with a valid interpolation theorem in $\mathcal{A}_\infty$. ⋄

Referring to Remark 1.2, it is of interest to understand the image of the map

$$F \in \mathcal{A}_\infty \mapsto f(s) = F(p_1^{-s}, p_2^{-s}, \dots) \in \mathscr{H}^\infty$$

into the space $\mathscr{H}^\infty$ of convergent and bounded Dirichlet series in the right half plane in $\mathbb{C}$. We shall let $\mathscr{A}^\infty$ denote the image of the above map. (We caution that as we have defined things the functions in $\mathscr{A}^\infty$ are bounded by 1 whereas $\mathscr{H}^\infty$ is a Banach space of functions normed by supremum norm.) The following is basically a formality but worth pointing out. Let $\mathbb{C}_+ = \{z \in \mathbb{C} : \Re z > 0\}$ denote the right half plane.

***Theorem* 1.9** *Let $f \in \mathscr{H}^\infty$. The following are equivalent:*

(1) $f \in \mathscr{A}^\infty$
(2) There exist positive semi-definite kernels $K_1, K_2, \dots$ on $\mathbb{C}_+$ such that

$$1 - f(s)\overline{f(w)} = \sum_{j=1}^{\infty} (1 - p_j^{-(s+\bar{w})}) K_j(s,w)$$

and $\sum_{j=1}^\infty K_j(s,s) < \infty$ for each $s \in \mathbb{C}_+$.
(3) For every diagonalizable matrix M with 1 dimensional eigenspaces, with $\sigma(M) \subset \mathbb{C}_+$, and with the property $\|n^{-M}\| \leq 1$ for all $n \in \mathbb{N}$, we have

$$\|f(M)\| \leq 1.$$

Again, $p_1, p_2, \dots$ are the prime numbers. It would be interesting if the matrices M in item (3) had a simpler description. Functions in $\mathscr{A}^\infty$ satisfy a special von Neumann inequality.

***Theorem* 1.10** *Let $f \in \mathscr{A}^\infty$. Suppose M is a bounded operator on a Hilbert space such that $\sigma(M) \subset \mathbb{C}_+$ and $\|n^{-M}\| \leq 1$ for every $n \in \mathbb{N}$. Then,*

$$\|f(M)\| \leq 1$$

where n^{-M} and $f(M)$ are defined using the Riesz holomorphic functional calculus.

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2 Notations and background

Several spaces of sequences will be of interest.

- $\mathbb{N} = \{1, 2, \ldots\}$.
- $\ell^\infty = \ell^\infty(\mathbb{N}) = \{z = (z_j)_{j\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} : \|z\|_\infty := \sup_j |z_j| < \infty\}$
- $\ell^2 = \ell^2(\mathbb{N}) = \{z = (z_j)_{j\in\mathbb{N}} : \sum_{j=1}^\infty |z_j|^2 < \infty\}$
- $c_0 = c_0(\mathbb{N}) = \{z \in \ell^\infty : \lim_{j\to\infty} z_j = 0\}$.
- $c_{00} = c_{00}(\mathbb{N}) = \{z \in \ell^\infty : \exists N \in \mathbb{N}, z_j = 0 \text{ for } j > N\}$.
- $Ball(\ell^\infty) = \{z \in \ell^\infty : \|z\|_\infty < 1\}$ denotes the open unit ball of ℓ^∞.
- $\mathbb{D}^\infty = \mathbb{D}^{\mathbb{N}} = \{z \in \ell^\infty : \forall j, |z_j| < 1\}$.
- $Ball(c_0) = \{z \in c_0 : \|z\|_\infty < 1\}$.
- $Ball(c_{00}) = \{z \in c_{00} : \|z\|_\infty < 1\}$.
- We identify $\mathbb{D}^N$ with $\mathbb{D}^N \times \{(0, 0, \ldots)\} \subset Ball(c_{00})$.
- The Hilbert multidisk is the set $\mathbb{D}_2^\infty := \ell^2 \cap Ball(\ell^\infty)$; namely, the set of sequences $(z_j)_{j\in\mathbb{N}}$ such that $\sup_j |z_j| < 1$ and $\sum_j |z_j|^2 < \infty$. Note that for $z, w \in \mathbb{D}_2^\infty$ the infinite product
$$\prod_{j=1}^\infty \frac{1}{1 - \bar{w}_j z_j}$$
converges absolutely.
- We generally use standard modulus bars $|\cdot|$ for the modulus of complex numbers or vectors (in $\mathbb{C}^N$ or Hilbert space) while double bars $\|\cdot\|$ are reserved for operator norms or other norms as listed above.

Remark 2.1 We use the basics of positive semi-definite functions. Given a set X, a function $A : X \times X \to \mathbb{C}$ is positive semi-definite on X if for every finite subset $Y \subset X$ and every function $a : Y \to \mathbb{C}$ we have
$$\sum_{z,w\in Y} a(z)\overline{a(w)}A(z, w) \geq 0.$$
We write $A(z, w) \geq 0$ in this case. More generally, we write $A(z, w) \geq B(z, w)$ if $A - B \geq 0$. We frequently use the Schur product theorem which say that if $A(z, w), B(z, w) \geq 0$ then $A(z, w)B(z, w) \geq 0$. For a function $f : X \to \mathbb{C}$, we let $f \otimes \bar{f}$ denote the function $(z, w) \mapsto f(z)\overline{f(w)}$. Also, if $X \subset \mathbb{C}^{\mathbb{N}}$, we let $Z_j \otimes \bar{Z}_j$ denote the function $(z, w) \mapsto z_j \bar{w}_j$.

3 Proof of Theorem 1.5

Theorem 1.5 will be proven with three lemmas. The first lemma is our main advance while the other two are standard.

For the first lemma, let $\rho = (\rho_n)_{n\in\mathbb{N}} \in \mathbb{D}_2^\infty$ be a fixed square summable sequence of positive numbers. Let $\overline{\mathbb{D}}_\rho = \{z = (z_n)_{n\in\mathbb{N}} : |z_n| \leq \rho_n\}$ and let $\mathbb{D}_\rho = \{z \odot \rho : z \in Ball(c_0)\}$ where $\rho \odot z = (\rho_j z_j)_{j\in\mathbb{N}}$. (The notation is not entirely consistent but it is temporary.)

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***Lemma* 3.1** *Assume $f : Ball(c_{00}) \to \mathbb{C}$ is in the Agler class, $\mathcal{A}_\infty$.*

Then, for $j = 0, 1, 2, \dots$, there exist positive semi-definite kernels K_j on $\mathbb{D}_\rho$ such that

$$1 - f(z)\overline{f(w)} = K_0(z,w) + \sum_{j=1}^{\infty}(1 - z_j\bar{w}_j)K_j(z,w)$$

where the sum converges absolutely.

Note that we have introduced the term K_0 which is for convenience in the proof. This term can be absorbed into any of the other terms for instance as

$$(1 - z_1\bar{w}_1)\left(\frac{K_0(z,w)}{1 - z_1\bar{w}_1} + K_1(z,w)\right).$$

Since K_0 will be positive semi-definite and since $\frac{1}{1-z_1\bar{w}_1}$ is positive semi-definite, the product is positive semi-definite.

Remark 3.2 In the proof, we will use the Montel theorem given in [15] (Theorem 2.17). It states that for a separable normed vector space X, if we are given a sequence $(D_n)_{n\in\mathbb{N}}$ of $D_n \in H^\infty(Ball(X))$ with uniformly bounded supremum norms, say $\|D_n\|_\infty \leq 1$, then there exists a subsequence $(D_{n_j})_{j\in\mathbb{N}}$ that converges uniformly on compact subsets of $Ball(X)$ to some $D \in H^\infty(Ball(X))$ necessarily with $\|D\|_\infty \leq 1$. We will apply this to $X = c_0$ using compact sets of the form $\overline{\mathbb{D}}_\rho$ defined above. ⋄

Proof For each N, let $f_N(z_1, z_2, \dots) = f(z_1, \dots, z_N, 0, \dots)$. Since f restricted to $\mathbb{D}^N$ has an Agler decomposition, we can write

$$1 - f_N(z)\overline{f_N(w)} = \sum_{j=1}^{N}(1 - \bar{w}_j z_j)K_j^N(z,w)$$

for positive semi-definite kernels K_j^N that only depend on the first N variables $z_1, \dots, z_N, w_1, \dots, w_N$, are analytic in z, and are anti-analytic in w.

Note that for $z, w \in \mathbb{D}_2^\infty$, the product

$$S(z,w) = \prod_{j=1}^{\infty}\frac{1}{1 - \bar{w}_j z_j}$$

converges and is positive semi-definite. Note that

$$S_n(z,w) = (1 - \bar{w}_n z_n)S(z,w) = \prod_{j\neq n}\frac{1}{1 - \bar{w}_j z_j}$$

is also positive semi-definite and $S, S_n \geq 1$ using the partial order from Remark 2.1. Here '1' is the identically 1 function.

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Let $f_N \otimes \overline{f_N}$ denote the function $(z,w) \mapsto f_N(z)\overline{f_N(w)}$. We have the positive semi-definite function inequality

$$S \geq (1 - f_N \otimes \overline{f_N})S = \sum_{j=1}^{N} K_j^N S_j \geq \sum_{j=1}^{N} K_j^N \geq \sum_{j=1}^{M} K_j^N \tag{3.1}$$

for $M < N$ and by Cauchy-Schwarz we have for $z, w \in \mathbb{D}_2^\infty$

$$S(z,z)^{1/2} S(w,w)^{1/2} \geq |K_j^N(z,w)|.$$

Recall $\rho \odot z = (\rho_j z_j)_{j\in\mathbb{N}}$. For $z, w \in Ball(c_0)$, $K_j^N(\rho \odot z, \rho \odot \bar{w})$ is bounded and analytic in $Ball(c_0) \times Ball(c_0)$ with supremum norm bounded by $S(\rho, \rho)$. By the Montel theorem (see Remark 3.2), for each $j = 1, 2, \ldots$ in succession there is a subsequence of $N \in \mathbb{N}$ such that

$$K_j^N(\rho \odot z, \rho \odot \bar{w}) \overset{N\to\infty}{\to} K_j(\rho \odot z, \rho \odot \bar{w}) \text{ uniformly on compact subsets of } Ball(c_0). \tag{3.2}$$

The limiting functions, K_j, are necessarily positive semi-definite because this property is preserved under limits. By a standard diagonal argument, we can find a common subsequence of N such that for all j, (3.2) holds. Recall $\mathbb{D}_\rho = \{z \odot \rho : z \in Ball(c_0)\}$. By (3.1), for $z, w \in \mathbb{D}_\rho$, $S \geq \sum_{j=1}^{M} K_j^N$ for $M < N$ and sending $N \to \infty$ we have $S \geq \sum_{j=1}^{M} K_j$. Finally, we can send $M \to \infty$ to obtain $S \geq \sum_{j=1}^{\infty} K_j$ with absolute convergence. Absolute convergence can be proven by looking at the diagonal $z = w$ first and then applying Cauchy-Schwarz. To finish the proof, we show

$$K_0(z,w) := 1 - f(z)\overline{f(w)} - \sum_{j=1}^{\infty}(1 - z_j\bar{w}_j)K_j(z,w)$$

is positive semi-definite. Here we emphasize that f on $\mathbb{D}_\rho$ is defined via f's holomorphic extension from $Ball(c_{00})$ to $Ball(c_0)$ as in the traditional Schur class of infinitely many variables.

Now, for $N < M$

$$1 - f_M(z)\overline{f_M(w)} - \sum_{j=1}^{N}(1 - z_j\bar{w}_j)K_j^M(z,w) = \sum_{j=N+1}^{M}(1 - z_j\bar{w}_j)K_j^M(z,w)$$

Let $Z_j \otimes \overline{Z_j}$ denote the function $(z,w) \mapsto z_j\bar{w}_j$. Since $S \geq K_j^M$ we see that

$$1 - f_M \otimes \overline{f_M} - \sum_{j=1}^{N}(1 - Z_j \otimes \overline{Z_j})K_j^M \geq -\sum_{j=N+1}^{M}(Z_j \otimes \overline{Z_j})S.$$

This inequality means that for any finite subset $Y \subset \mathbb{D}_\rho$ and any function $a : Y \to \mathbb{C}$

$$\sum_{z,w\in Y}(1 - f_M(z)\overline{f_M(w)} - \sum_{j=1}^{N}(1 - z_j\bar{w}_j)K_j^M(z,w))a(z)\overline{a(w)} \geq -\sum_{z,w\in Y}\sum_{j=N+1}^{M} z_j\bar{w}_j S(z,w)a(z)\overline{a(w)}.$$

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(See Remark 2.1.) Now,

$$\sum_{z,w\in Y}\sum_{j=N+1}^{M} z_j\bar{w}_j S(z,w)a(z)\overline{a(w)} \leq S(\rho,\rho)\sum_{j=N+1}^{M}\rho_j^2\left(\sum_{z\in Y}|a(z)|\right)^2 \leq S(\rho,\rho)\sum_{j=N+1}^{\infty}\rho_j^2\left(\sum_{z\in Y}|a(z)|\right.$$

Sending $M \to \infty$

$$\sum_{z,w\in Y}(1-f(z)\overline{f(w)}-\sum_{j=1}^{N}(1-z_j\bar{w}_j)K_j(z,w))a(z)\overline{a(w)} \geq -S(\rho,\rho)\sum_{j=N+1}^{\infty}\rho_j^2\left(\sum_{z\in Y}|a(z)|\right)^2$$

and then sending $N \to \infty$

$$\sum_{z,w\in Y}(1-f(z)\overline{f(w)}-\sum_{j=1}^{\infty}(1-z_j\bar{w}_j)K_j(z,w))a(z)\overline{a(w)} \geq 0$$

which proves $K_0(z,w)$ is positive semi-definite. ■

***Lemma* 3.3** *Let $X \subset Ball(\ell^\infty)$. Assume $f : X \to \mathbb{C}$ is a function such that for $j = 0, 1, 2, \ldots$, there exist positive semi-definite kernels $K_j : X \times X \to \mathbb{C}$ on X such that*

$$1 - f(z)\overline{f(w)} = \sum_{j=1}^{\infty}(1 - z_j\bar{w}_j)K_j(z,w)$$

where the sum converges absolutely. Then, there exists a contractive operator V acting on $\mathbb{C} \oplus \bigoplus_{j=1}^{\infty}\mathcal{H}_j$ where $\mathcal{H}_1, \mathcal{H}_2, \ldots$ are Hilbert spaces such that when we write V in block form $V = \begin{pmatrix} A & B \\ C & D\end{pmatrix}$ we have

$$f(z) = A + B\Delta(z)(1 - D\Delta(z))^{-1}C$$

$$K_j(z,w) = C^*(I - \Delta(w)^*D^*)^{-1}P_j(I - D\Delta(z))^{-1}C$$

where $\Delta(z) = \sum_{j=1}^{\infty} z_jP_j$ and each P_j represents projection onto $\mathcal{H}_j$ within the direct sum $\bigoplus_{k=1}^{\infty}\mathcal{H}_k$. With these formulas, f and K_j extend to $Ball(\ell^\infty)$ and there exists a positive semi-definite kernel $K_0(z,w)$ such that

$$1 - f(z)\overline{f(w)} = K_0(z,w) + \sum_{j=1}^{\infty}(1 - z_j\bar{w}_j)K_j(z,w)$$

holds on $Ball(\ell^\infty) \times Ball(\ell^\infty)$.

The proof is a standard lurking isometry argument (for those who know what that is) that we include for completeness (for those who do not).

Proof By the Moore-Aronszajn theorem on reproducing kernel Hilbert space, we can factor $K_j(z,w) = K_{j,z}^*K_{j,w}$ for $K_{j,z}$ an element of some Hilbert space $\mathcal{H}_j$. We write $K_{j,z}^*K_{j,w}$ instead of $\langle K_{j,w}, K_{j,z}\rangle$ and view $K_{j,z}^*$ as an element of the dual $\mathcal{H}_j^*$ of $\mathcal{H}_j$.

The following map

$$\begin{pmatrix} 1 \\ z_1(K_{1,z})^* \\ z_2(K_{2,z})^* \\ \vdots \end{pmatrix} \mapsto \begin{pmatrix} f(z) \\ (K_{1,z})^* \\ (K_{2,z})^* \\ \vdots \end{pmatrix}$$

initially defined for vectors indexed by $z \in X$ extends linearly and in a well-defined way to a contractive operator V from $\mathbb{C} \oplus \bigoplus_{j=1}^{\infty} \mathcal{H}_j^*$ to $\mathbb{C} \oplus \bigoplus_{j=1}^{\infty} \mathcal{H}_j^*$. We write V in block form $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $A \in \mathbb{C} \cong \mathcal{B}(\mathbb{C}, \mathbb{C})$, $B \in \mathcal{B}(\bigoplus_{j=1}^{\infty} \mathcal{H}_j^*, \mathbb{C})$, $C \in \mathcal{B}(\mathbb{C}, \bigoplus_{j=1}^{\infty} \mathcal{H}_j^*)$, $D \in \mathcal{B}(\bigoplus_{j=1}^{\infty} \mathcal{H}_j^*)$, using the notation $\mathcal{B}(X, Y)$ to denote the bounded linear operators from X to Y (as well as $\mathcal{B}(X) = \mathcal{B}(X, X)$). Let $\Delta(z) \in \mathcal{B}(\bigoplus_{j=1}^{\infty} \mathcal{H}_j^*)$ be the diagonal operator sending

$$(h_j)_{j\in\mathbb{N}} \in \bigoplus_{j=1}^{\infty} \mathcal{H}_j^* \mapsto (z_j h_j)_{j\in\mathbb{N}} \in \bigoplus_{j=1}^{\infty} \mathcal{H}_j^*.$$

Let $F(z) := (K_{j,z}^*)_{j\in\mathbb{N}} \in \bigoplus_{j=1}^{\infty} \mathcal{H}_j^*$. Then,

$$V \begin{pmatrix} 1 \\ \Delta(z)F(z) \end{pmatrix} = \begin{pmatrix} f(z) \\ F(z) \end{pmatrix}$$

which implies

$$A + B\Delta(z)F(z) = f(z) \tag{3.3}$$

$$C + D\Delta(z)F(z) = F(z) \tag{3.4}$$

Solving for $F(z)$ and then $f(z)$ we obtain

$$F(z) = (I - D\Delta(z))^{-1}C$$
$$f(z) = A + B\Delta(z)(I - D\Delta(z))^{-1}C.$$

Note the expressions on the right are defined as written for $z \in Ball(\ell^\infty)$. Evidently, $F(w)^* P_j F(z)$ extends $K_j(z, w)$. Since V is contractive, we can factor $I - V^*V = W^*W$ for some operator W. Let

$$G(z) = W \begin{pmatrix} 1 \\ \Delta(z)F(z) \end{pmatrix}$$

so that

$$\begin{pmatrix} 1 \\ \Delta(w)F(w) \end{pmatrix}^* \begin{pmatrix} 1 \\ \Delta(z)F(z) \end{pmatrix} = \begin{pmatrix} f(w) \\ F(w) \end{pmatrix}^* \begin{pmatrix} f(z) \\ F(z) \end{pmatrix} + G(w)^*G(z).$$

This rearranges into

$$1 - f(z)\overline{f(w)} = F(w)^*(I - \Delta(w)^*\Delta(z))F(z) + G(w)^*G(z)$$

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and since

$$F(w)^*(I - \Delta(w)^*\Delta(z))F(z) = \sum_{j=1}^{\infty}(1 - \bar{w}_j z_j)F(w)^* P_j F(z)$$

we have the desired extension of the Agler decomposition using $K_0(z, w) = G(w)^* G(z)$. ■

Lemmas 3.1 and 3.3 establish most of Theorem 1.5. For the remaining part, we need a basic estimate on transfer function formulas in order to establish the full von Neumann inequality for the Agler class.

***Lemma* 3.4** *Suppose V is a contractive operator acting on $\mathbb{C} \oplus \bigoplus_{j=1}^{\infty} \mathcal{H}_j$, where $\mathcal{H}_1, \mathcal{H}_2, \ldots$ are Hilbert spaces. Writing V in block form $V = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ we define for $z \in Ball(\ell^\infty)$*

$$f(z) = A + B\Delta(z)(1 - D\Delta(z))^{-1}C$$

where $\Delta(z) = \sum_{j=1}^{\infty} z_j P_j$ and each P_j represents projection onto $\mathcal{H}_j$ within the direct sum $\bigoplus_{k=1}^{\infty} \mathcal{H}_k$. Then, for $z, w \in Ball(\ell^\infty)$

$$f(z) - f(w) = B(1 - \Delta(z)D)^{-1}\Delta(z - w)(1 - D\Delta(w))^{-1}C.$$

Proof

$$\begin{aligned} f(z) - f(w) &= B(\Delta(z)(1 - D\Delta(z))^{-1} - \Delta(w)(1 - D\Delta(w))^{-1})C \\ &= B(\Delta(z)((1 - D\Delta(z))^{-1} - (1 - D\Delta(w))^{-1}) + \Delta(z - w)(1 - D\Delta(w))^{-1})C \\ &= B(\Delta(z)(1 - D\Delta(z))^{-1})(D\Delta(z - w))(1 - D\Delta(w))^{-1} + \Delta(z - w)(1 - D\Delta(w))^{-1})C \\ &= B(\Delta(z)(1 - D\Delta(z))^{-1})D + 1)\Delta(z - w)(1 - D\Delta(w))^{-1})C \\ &= B(1 - \Delta(z)D)^{-1}\Delta(z - w)(1 - D\Delta(w))^{-1}C. \end{aligned}$$

■

Now we finish the proof of Theorem 1.5. As written in Theorem 1.5, for an infinite tuple $T = (T_1, T_2, \ldots)$ of operators on a common Hilbert space $\mathcal{H}$ satisfying

$$\|T\|_\infty := \sup_j \|T_j\| < 1$$

we define

$$f(T) := (A \otimes I) + (B \otimes I)\Delta(T)(1 - (D \otimes I)\Delta(T))^{-1}(C \otimes I)$$

where $\Delta(T) := \sum_{j=1}^{\infty} P_j \otimes T_j$. Note that this definition does not require the operators $(T_j)_j$ to pairwise commute but if they do then each T_j commutes with $f(T)$ since each $I \otimes T_j$ commutes with $\Delta(T)$.

Since $(D \otimes I)\Delta(T)$ is strictly contractive, $f(T)$ equals the absolutely convergent sum

$$(A \otimes I) + (B \otimes I)\Delta(T)\sum_{j=0}^{\infty}((D \otimes I)\Delta(T))^j(C \otimes I)$$

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and substituting $z \in Ball(\ell^\infty)$ we obtain an absolutely convergent homogeneous expansion for $f(z)$

$$f(z) = A + B\Delta(z)\sum_{j=0}^{\infty}((D\Delta(z))^j C. \tag{3.5}$$

Before we finish the proof of Theorem 1.5 we make some clarifications about our functional calculus.

Remark 3.5 In finitely many variables we stated that our convention/definition for $f(T)$ is via an absolutely convergent power series expansion. Therefore, it should be pointed out that this new formulation of "$f(T)$" using the transfer function formula matches the old one. All that really needs to be said is that when we insert $z = (z_1, \dots, z_N, 0, \dots)$ into (3.5) the homogeneous terms $B\Delta(z)(D\Delta(z))^j C$ are homogeneous polynomials and since f is analytic on $\mathbb{D}^N$, the monomial sum we obtain from expanding $B\Delta(z)(D\Delta(z))^j C$ is absolutely convergent in $\mathbb{D}^N$. Thus, evaluating $f(T)$ at a finite tuple $T = (T_1, T_2, \dots, T_N, 0, \dots)$ can either be evaluated using the transfer function formula or the absolutely convergent power series. ⋄

The proof of Lemma 3.4 extends directly to prove that for another such tuple S acting on the same Hilbert space $\mathcal{H}$ as T we have

$$f(T) - f(S) = (B \otimes I)(1 - \Delta(T)(D \otimes I))^{-1}\Delta(T - S)(1 - (D \otimes I)\Delta(S))^{-1}(C \otimes I).$$

This implies the estimate that for $x \in \mathcal{H}$

$$|(f(T)-f(S))x|^2 \leq \|(B\otimes I)(1-\Delta(T)(D\otimes I))^{-1}\|^2|\Delta(T-S)(1-(D\otimes I)\Delta(S))^{-1}(C\otimes x)|^2.$$

Letting $T^{(N)} = (T_1, \dots, T_N, 0, \dots)$ we have $f(T^{(N)}) = f_N(T)$ and

$$\begin{aligned}|(f_N(T) - f(T))x|^2 &\leq (1 - \|T\|_\infty)^{-2}|\Delta(T^{(N)} - T)(1 - (D \otimes I)\Delta(T))^{-1}(C \otimes x)|^2 \\ &= (1 - \|T\|_\infty)^{-2}\sum_{j=N+1}^{\infty}|(P_j \otimes T_j)(1 - (D \otimes I)\Delta(T))^{-1}(C \otimes x)|^2\end{aligned}$$

and this goes to 0 as $N \to \infty$. Thus, $f_N(T) \to f(T)$ in the strong operator topology. This concludes the proof of Theorem 1.5.

4 Proof of Theorem 1.7

That (1) implies (2) follows from Lemmas 3.1 and 3.3. Proving (2) implies (1) is a standard lurking isometry argument that is somewhat similar to our proof of Theorem 1.5. Proving (1) implies (3) consists mostly of technicalities that we discuss next. The proof of (3) implies (1) is the main contribution below.

Regarding (1) implies (3), by definition, we can make sense of $\tilde{f}(T)$ for $\tilde{f} \in \mathcal{A}_\infty$ when $(\|T_j\|)_{j\in\mathbb{N}} \in Ball(c_{00})$ and Theorem 1.5 lets us make sense of it when $(\|T_j\|)_{j\in\mathbb{N}} \in Ball(\ell^\infty)$. To prove $\|\tilde{f}(T)\| \leq 1$ when T consists of *matrices* that are commuting, contractive, simultaneously diagonalizable with joint spectrum $\sigma(T) \subset X$ and eigenspaces of dimension at most 1, we can give a continuity argument. First, $\|\tilde{f}(rT)\| \leq 1$ holds

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for $r < 1$ because we will have

$$\tilde{f}_N(rT) \to \tilde{f}(rT)$$

in the strong operator topology as $N \to \infty$—as in previous sections $\tilde{f}_N$ refers to the restriction of $\tilde{f}$ to $\mathbb{D}^N$. Note that $\tilde{f}_N(rT)$ is defined in terms of the absolutely convergent power series of $\tilde{f}_N$. Next, we use the diagonalizability properties of T; let $\vec{b}(z)$ be the eigenvector associated to joint eigenvalue $z \in \sigma(T)$. Then, for any function $a : \sigma(T) \to \mathbb{C}$ we have

$$|\tilde{f}(rT) \sum_{z\in\sigma(T)} a(z)\vec{b}(z)| = |\sum_{z\in\sigma(T)} \tilde{f}(rz)a(z)\vec{b}(z)| \leq |\sum_{z\in\sigma(T)} a(z)\vec{b}(z)|.$$

We can send $r \nearrow 1$ to conclude $\|\tilde{f}(T)\| \leq 1$. This proves (1) implies (3).

Our main contribution is the proof of (3) implies (2) assuming the finite set X belongs to $\mathbb{D}_2^\infty$. This is a modification of the finite variable cone separation argument; the main difference being Lemma 4.1 below. Consider the following cone of functions on $X \times X$

$$C = \{(z,w) \mapsto \sum_{j=1}^{\infty}(1 - z_j\bar{w}_j)A_j(z,w) : \\ A_1, A_2, \dots \text{ are positive semi-definite functions on } X; \\ \forall z \in X, \sum_{j=1}^{\infty} A_j(z,z) < \infty\Big\}.$$

***Lemma* 4.1** *C is closed.*

Proof Let

$$C_n(z,w) = \sum_{j=1}^{\infty}(1 - z_j\bar{w}_j)A_{n,j}(z,w)$$

define a sequence of functions in C that converges to the function $C : X \times X \to \mathbb{C}$ pointwise; in particular, each $A_{n,j}$ is positive semi-definite. We must show $C \in C$.

Let $\delta : X \times X \to \mathbb{C}$ denote the function $\delta(z,w) = 1$ if $z = w$ and $\delta(z,w) = 0$ if $z \neq w$. There necessarily exist constants c_1, c_2 such that for all n, $c_1\delta \geq C_n \geq c_2\delta$. This looks more familiar when we view our functions on $X \times X$ as matrices. Since $X \subset \mathbb{D}_2^\infty$ we can define

$$S(z,w) = \prod_{j=1}^{\infty} \frac{1}{1 - z_j\bar{w}_j} \text{ and } S_j(z,w) = (1 - z_j\bar{w}_j)S(z,w) = \prod_{i\neq j} \frac{1}{1 - z_i\bar{w}_i};$$

which are positive semi-definite and satisfy $S, S_j \geq 1$, with '1' representing the identically 1 function. Some parts of what follow are similar to the proof of Lemma 3.1. Now, for any $i \in \mathbb{N}$

$$c_1 S \geq C_n S = \sum_{j=1}^{\infty} S_j A_{n,j} \geq \sum_{j=1}^{\infty} A_{n,j} \geq A_{n,i} \geq 0$$

which shows the matrices $A_{n,i}$ are uniformly bounded. Let $c_3 > 0$ satisfy $c_3\delta \geq c_1 S$.

Using a diagonal argument we can select a subsequence such that each $A_{n,i}$ converges to a positive semi-definite matrix A_i as $n \to \infty$. Also, for each N we have $C_n S \geq \sum_{j=1}^N S_j A_{n,j} \geq \sum_{j=1}^N A_{n,j}$. Sending $n \to \infty$ we have $CS \geq \sum_{j=1}^N S_j A_j \geq \sum_{j=1}^N A_j$ and sending $N \to \infty$ we have $CS \geq \sum_{j=1}^\infty S_j A_j \geq \sum_{j=1}^\infty A_j$ where the sums converge absolutely. Next, recalling $Z_j \otimes \bar{Z}_j$ denotes the function $(z, w) \mapsto z_j \bar{w}_j$ we have

$$\begin{aligned} C_n - \sum_{j=1}^N (1 - Z_j \otimes \bar{Z}_j) A_{n,j} &= \sum_{j=N+1}^\infty (1 - Z_j \otimes \bar{Z}_j) A_{n,j} \\ &\geq -c_3 \sum_{j=N+1}^\infty (Z_j \otimes \bar{Z}_j)\delta \\ &\geq -c_3 \max\{\sum_{j=N+1}^\infty |z_j|^2 : z \in X\}\delta \end{aligned}$$

The last inequality amounts to the fact that for any function $a : X \to \mathbb{C}$ we have

$$\begin{aligned} &\sum_{z,w\in X} \sum_{j=N+1}^\infty (z_j \bar{w}_j)\delta(z, w) a(z)\overline{a(w)} \\ &= \sum_{z\in X} \sum_{j=N+1}^\infty |z_j|^2 |a(z)|^2 \\ &\leq \max\{\sum_{j=N+1}^\infty |z_j|^2 : z \in X\} \sum_{z\in X} |a(z)|^2. \end{aligned}$$

Setting $M_N = \max\{\sum_{j=N+1}^\infty |z_j|^2 : z \in X\}$ we have $M_N \to 0$ since $X \subset \ell^2$. Sending $n \to \infty$ we have

$$C - \sum_{j=1}^N (1 - Z_j \otimes \bar{Z}_j) A_j \geq -c_3 M_N \delta$$

and finally sending $N \to \infty$ we have

$$A_0 := C - \sum_{j=1}^\infty (1 - Z_j \otimes \bar{Z}_j) A_j \geq 0.$$

Finally, A_0 can be absorbed into any other term, say A_1 as $\tilde{A}_1 = A_1 + \frac{1}{1 - Z_1 \otimes \bar{Z}_1} A_0$ to see that

$$C = (1 - Z_1 \otimes \bar{Z}_1)\tilde{A}_1 + \sum_{j=2}^\infty (1 - Z_j \otimes \bar{Z}_j) A_j$$

is of the desired form. ∎

What follows is now standard. Suppose $f : X \to \overline{\mathbb{D}}$ is a function with the assumed property in item (3). Suppose the function on $X \times X$, $F = 1 - f \otimes \bar{f}$ is not in the

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cone $\mathcal{C}$. By the Hahn-Banach hyperplane separation theorem, there exists a function $B : X \times X \to \mathbb{C}$ with $B(z,w) = \overline{B(w,z)}$ such that

$$\sum_{z,w\in X} F(z,w)B(z,w) < 0 \text{ and for all } C \in \mathcal{C} \text{ we have } \sum_{z,w\in X} C(z,w)B(z,w) \geq 0.$$

The second condition implies $B \geq 0$ by setting $C = (1 - Z_1 \otimes \bar{Z}_1)\frac{a\otimes\bar{a}}{1-Z_1\otimes\bar{Z}_1}$ for an arbitrary function $a : X \to \mathbb{C}$ and observing

$$\sum_{z,w\in X} C(z,w)B(z,w) = \sum_{z,w} a(z)\overline{a(w)}B(z,w) \geq 0.$$

Next, we factor $B(w,z) = \overline{B(z,w)} = \vec{b}(z)^*\vec{b}(w)$ for vectors $\vec{b}(z) \in \mathbb{C}^r$ where r is the rank of the matrix$(B(z,w))_{z,w\in X}$.

Choosing now $C_j = (1 - Z_j \otimes \bar{Z}_j)(a \otimes \bar{a})$ for an arbitrary $a : X \to \mathbb{C}$

$$\begin{aligned} 0 &\leq \sum_{z,w\in X} (1 - z_j\bar{w}_j)a(z)\overline{a(w)}B(z,w) \\ &= \sum_{z,w\in X} (1 - z_j\bar{w}_j)a(z)\overline{a(w)}\vec{b}(z)^t\overline{\vec{b}(w)} \\ &= \left|\sum_{w\in X} a(w)\vec{b}(w)\right|^2 - \left|\sum_{w\in X} w_j a(w)\vec{b}(w)\right|^2 \end{aligned}$$

which proves that maps $T_j : \sum_{w\in X} a(w)\vec{b}(w) \mapsto \sum_{w\in X} w_j a(w)\vec{b}(w)$ are well-defined, contractive, diagonalizable. Setting $T = (T_1, T_2, \dots)$, we have assumed that $f(T)$ is contractive which means for all functions $a : X \to \mathbb{C}$

$$\left|\sum_{w\in X} a(w)\vec{b}(w)\right|^2 - \left|\sum_{w\in X} f(w)a(w)\vec{b}(w)\right|^2 \geq 0$$

and this means

$$\sum_{z,w\in X} (1 - f(z)\overline{f(w)})B(z,w)a(z)\overline{a(w)} \geq 0$$

and this means

$$FB = (1 - f \otimes \bar{f})B \geq 0$$

(recall $F = 1 - f \otimes f$) and in particular

$$\sum_{z,w\in X} F(z,w)B(z,w) \geq 0$$

which is a contradiction. This proves $F \in \mathcal{C}$ or more precisely, there exist positive semi-definite matrices $A_1, A_2, \dots$ with rows and columns indexed by S such that

$$1 - f(z)\overline{f(w)} = \sum_{j=1}^{\infty}(1 - z_j\bar{w}_j)A_j(z,w)$$

and $\sum_{j=1}^{\infty} A_j(z,z) < \infty$. This proves that (2) implies (3) as well as the full proof of Theorem 1.7.

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5 Proof of Theorem 1.9

The equivalence of (1) and (2) is straightforward based on previous results. The implication (1) implies (3) follows from Theorem 1.7.

To prove (3) implies (2), we note that our function $f : \mathbb{C}_+ \to \overline{\mathbb{D}}$ gives rise to a function $F : X \to \overline{\mathbb{D}}$ where

$$X = \{\pi(s) := (p_1^{-s}, p_2^{-s}, \dots) : s \in \mathbb{C}_+\} \subset Ball(c_0)$$

and $F(\pi(s)) = f(s)$. We can apply the interpolation result of Theorem 1.7 as follows. Let $Y \subset \{s \in \mathbb{C} : \Re s > 1/2\}$ be a countable set with a limit point in $\{s \in \mathbb{C} : \Re s > 1/2\}$. We want a limit point so that it is a determining set for holomorphic functions and we want $\Re s > 1/2$ so that $\pi(s) \in \mathbb{D}_2^\infty$. Let $\bigcup_{n=1}^\infty Y_n = Y$ be an increasing union of finite sets. Set $X_n = \pi(Y_n)$, $X_\infty = \pi(Y)$.

Fix n and let $T = (T_1, T_2, \dots)$ be an infinite tuple of contractive, commuting matrices with $\sigma(T) \subset X_n$ and 1 dimensional joint eigenspaces. The eigenvalues of T_1 are of the form $p_1^{-s} = 2^{-s}$ for $s \in Y_n$.

We can take $M = -\log T_1/\log 2$ using the principal log and then $T_j = p_j^{-M}$ for all j. By assumption, $\|T_j\| = \|p_j^{-M}\| \le 1$ and this extends to all natural numbers by factoring into primes: $\|n^{-M}\| \le 1$ for all $n \in \mathbb{N}$. By assumption (3), $\|f(M)\| = \|F(T)\| \le 1$. Therefore, by the implication (3) $\implies$ (1) in Theorem 1.7, there exists $G_n \in \mathcal{A}_\infty$ such that $G_n|_{X_n} = F|_{X_n}$. In particular, $g_n(s) := G_n(\pi(s)) \in \mathscr{A}^\infty$ and agrees with f_n on Y_n.

By the Montel Theorem of Remark 3.2, since $G_n \in \mathcal{A}_\infty \subset \mathcal{S}_\infty$, there is a subsequence of $n \in \mathbb{N}$ such that $G_n \to G \in \mathcal{S}_\infty$ uniformly on compact subsets of $Ball(c_0)$. The limit G necessarily agrees with F on X_∞. Since membership in $\mathcal{A}_\infty$ can be tested by checking whether G restricted to $\mathbb{D}^N$ belongs to $\mathcal{A}_N$ by Theorem 1.5, and since this in turn can be tested by examining G on finite subsets of $\mathbb{D}^N$, we see that the limit G belongs to $\mathcal{A}_\infty$ since each G_n belongs to $\mathcal{A}_\infty$. Finally, $g = G \circ \pi \in \mathscr{A}^\infty$ and agrees with f on the set of uniqueness Y, so we see that $g = f \in \mathscr{A}^\infty$.

6 Proof of Theorem 1.10

Given $f \in \mathscr{A}^\infty$, let $F \in \mathcal{A}_\infty$ be the Bohr lift of f; namely, $f(s) = F(p_1^{-s}, p_2^{-s}, \dots)$. Again, forming $F_N(z_1, z_2, \dots) = F(z_1, \dots, z_N, 0, \dots)$, we see that the functions $f_N(s) := F_N(p_1^{-s}, p_2^{-s}, \dots)$ converge uniformly to f on the half planes $\{s \in \mathbb{C} : \Re s \ge \epsilon\}$ for each $\epsilon > 0$. The power series for F_N converges absolutely on $\mathbb{D}^N$ and therefore the Dirichlet series for f_N converges to f_N absolutely on $\mathbb{C}_+$. Because of this the holomorphic functional calculus evaluation $f_N(M)$ agrees with $F_N(p_1^{-M}, p_2^{-M}, \dots)$. Since $F_N \in \mathcal{A}_\infty$,

$$\|f_N(M)\| = \|F_N(p_1^{-M}, p_2^{-M}, \dots)\| \le 1.$$

Since f_N converges uniformly to f on a half-plane containing $\sigma(M)$, $f_N(M) \to f(M)$ in operator norm and therefore $\|f(M)\| \le 1$ as desired. This completes the proof.

Acknowledgments

Thank you to the referee for a thoughtful report and for pointing out several typos. Thank you to Professor Daniel Alpay for bringing the reference [8] to our attention. This research was partially supported by NSF grant DMS-2247702.

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https://doi.org/10.4153/S0008439525100878 Published online by Cambridge University Press

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