Can. J. Math., Vol. XXVI, No. 3, 1974, pp. 593–599

# ADDITIVE DIVISIBILITY IN COMPACT TOPOLOGICAL SEMIRINGS

P. H. KARVELLAS

**1. Introduction.** A topological semiring $(S, +, \cdot)$ is a nonempty Hausdorff space $S$ on which are defined continuous and associative operations, termed addition $(+)$ and multiplication $(\cdot)$, such that the multiplication distributes over addition from left and right. The additive semigroup $(S, +)$ need not be commutative.

We prove that the set $A$ of additively divisible elements of a compact semiring $S$ is a two-sided multiplicative ideal, containing the set $E[+]$ of additive idempotents, with the property that $(A.S) \cup (S.A) \subset E[+]$. Several well-known corollaries are immediate consequences. Section one also extends material from Wallace [**11**]. The second section is devoted to the determination of the semiring multiplication when an $I$-semigroup addition has been specified on an interval of the real line.

Semigroup nomenclature from [**3**] will be used throughout. *Complex products* are given by

$$X.Y = \{xy : x \in X, y \in Y\} \quad \text{and} \quad X + Y = \{x + y : x \in X, y \in Y\}.$$

The nonempty subset $M$ of a semiring $S$ is a *multiplicative ideal* if $(S.M) \cup (M.S) \subset M$ and is an additive ideal if $(M + S) \cup (S + M) \subset M$. If the semiring is compact, then *minimal ideals* (*kernels*) exist for both the additive and multiplicative semigroups [**10**]. The *idempotent sets* are $E[+] = \{x : x = x + x\}$ and $E[\cdot] = \{x : x = x^2\}$. The *union of all additive subgroups* will be denoted by $H[+]$. Both idempotents and subgroups exist for the compact case [**10**]. Both $H[+]$ and $E[+]$ are two-sided multiplicative ideals although in general neither set need be closed under addition. For an element $x$ and positive integer $n$, interpret $nx$ as the $n$-fold sum of $x$.

**2. The set of additively divisible elements.** An element $x$ of a semiring $S$ is said to be *additively divisible* if for each positive integer $n$ there exists an element $y$ of $S$ such that $x = ny$. The set of additively divisible elements of a semiring will be denoted by $A$ and $N$ shall represent the positive integers. Nets will be written as $\{x_a\} (a \in D)$, $D$ being the directed set.

THEOREM 1. *Let $S$ be a compact topological semiring. The set $A$ of additively*

Received November 16, 1972 and in revised form, January 2, 1974. This research constitutes a part of the author's Ph.D. thesis written at the University of Houston under the direction of Professor M. Friedberg.



https://doi.org/10.4153/CJM-1974-056-7 Published online by Cambridge University Press

*divisible elements of* $S$ *is nonempty and topologically closed. Moreover,* $(A.S) \cup (S.A) \subset E[+] \subset A$ *and, if* $S$ *has a multiplicative identity, then* $E[+] = A$.

*Proof.* Because $(S, +)$ is a compact topological semigroup, $E[+]$ is nonvoid [**10**]. If $e = e + e$, then $e = ne$ for all $n$ in $N$, implying $E[+] \subset A$. Trivially $A$ is a closed set.

Let $a \in A$ and $s \in S$. For each integer $n$ in $N$ there exists $b_n \in S$ such that $a = nb_n$. Thus $as = (nb_n)s = b_n(ns)$ for each $n \in N$. From the compactness of $S$ the net $\{ns\}(n \in N)$ clusters to an additive idempotent $e$ [**6**, Theorem 1.1.10]. Denoting the convergent subnet by $N'$, there is a corresponding subnet of $\{b_n\}(n \in N')$ which must cluster to some element $b$ of $S$. Writing this convergent subnet as $N''$, $\{b_n\} \to b$ $(n \in N'')$ and $\{ns\} \to e$ $(n \in N'')$ are convergent nets. From the continuity of multiplication $\{b_n(ns)\} \to be$ $(n \in N'')$ is convergent. But $as = b_n(ns)$ for each $n \in N''$ and therefore $as = be = b(e + e) = be + be \in E[+]$. Thus $A.S \subset E[+]$ and similarly $S.A \subset E[+]$ also. Lastly, if the element 1 is an identity for multiplication, $A = A \cdot \{1\} \subset A.S \subset E[+] \subset A$, hence $A = E[+]$.

The following result was obtained by Selden [**9**].

COROLLARY 2. *Let* $S$ *be a compact topological semiring, with* $S = (S.E[\cdot]) \cup (E[\cdot].S)$. *Then each additive subgroup of* $S$ *is totally disconnected.*

*Proof.* For each $a \in A$ there exists an element $t \in E[\cdot]$ such that either $a = at$ or $a = ta$. In either case $a \in E[+]$ and thus $A = E[+]$. Let $G$ be an additive subgroup of $S$ with additive identity $e$ and let $C$ be the identity component of $e$ in $G$. Then the topological closure $G^*$ of $G$ is compact and is a topological group. The identity component $C'$ of $e$ in $G^*$ contains $C$ and $C'$ is a continuum topological group. From a result of Mycielski [**5**], $C'$ is additively divisible and thus $C = C' = \{e\}$. Since translation is a homeomorphism, $G$ is totally disconnected.

The corollaries which follow can also be obtained from the results of Wallace [**11**]. We omit the proofs. A topological semiring $(S, +, \cdot)$ is a (topological) *distributive nearring* if $(S, +)$ is an algebraic group.

COROLLARY 3 [**2**]. *The multiplication on a compact and connected topological distributive nearring* $(R, +, \cdot)$ *is given by* $xy = 0$, *where* 0 *is the additive identity.*

COROLLARY 4 [**1**]. *Let* $R$ *be a compact, connected topological ring. Then* $R^2 = \{0\}$.

COROLLARY 5. *A compact topological ring with multiplicative identity is totally disconnected.*

The next result finds particular application in the characterization problem treated in section two.

COROLLARY 6. *Let $S$ be a compact semiring which is additively divisible. Then $S^2 \subset E[+]$. If also $S$ is connected and $E[+]$ is totally disconnected, then $S^2 = \{e\}$ for some element $e$ in $E[+]$.*

The first example will be used in our later work. The additions correspond to $I$-semigroups of types $J_1$ and $J_2$ [**4**].

*Example* 1. Let $P$ be the interval $[0, 1]$ of real numbers with addition $x + y = x * y$, where $*$ represents ordinary real number product, and let $A$ be the interval $[1/2, 1]$ with the addition $x + y = \max(1/2, x * y)$. Both additions are divisible. If both intervals are to be topological semirings, then $P^2 = \{0\}$ or $\{1\}$, while $Q^2 = \{1/2\}$ or $\{1\}$.

**3. Additively divisible semirings on intervals.** In this section the continuum $S$ shall be the interval $[z, u]$ of real numbers, with $z$ minimal and $u$ maximal in the left to right order on the line. Subcontinua will be written $[x, y]$, where $x \leqq y$. That is, $x = x \wedge y$ and $y = x \vee y$.

An *$I$-semigroup* is a topological semigroup which is both isomorphic and homeomorphic (*iseomorphic*) to a semigroup on $[0, 1]$, such that 0 and 1 act respectively as a zero and an identity for the semigroup operation. Pearson has given characterizations of the semiring addition when an $I$-semigroup multiplication has been specified on an interval [**7**; **8**]. In this section we shall consider the problem of determining the multiplication when an $I$-semigroup addition has been defined on the interval $S = [z, u]$.

There exist four possible types of $I$-semigroup additions [**4**, Theorem B]. These are listed below, with real number product written as $x * y$.

$J_1$: The interval $[0, 1]$ with addition $x + y = x * y$.

$J_2$: The interval $[1/2, 1]$ with addition $x + y = \max(1/2, x * y)$.

$J_3$: The interval $[z, u]$ with addition $x + y = x \wedge y$.

$J_4$: The interval $[z, u]$ with the properties:

(1) $z$ is an additive zero, $u$ an additive identity;

(2) if $T$ is the closure of an interval in $S \backslash E[+]$, $T$ is iseomorphic to $J_1$ or $J_2$;

(3) if $x$ and $y$ are not in the closure of the same subinterval of $S \backslash E[+]$, $x + y = x \wedge y$.

All $I$-semigroup operations are divisible. In order to refer to an arbitrary $I$-semigroup operation on an interval $[x, y]$, either $x$ or $y$ is allowed to assume the role of the identity element. Henceforth we shall consider $(S, +, \cdot)$ to be a topological semiring on the interval $[z, u]$, where $(S, +)$ is one of the $J$-additions and $u$ is an additive identity.

If $(S, +)$ is either $J_1$ or $J_2$, the results of Example 1 are the only multiplications compatible with the addition. That is: $S^2 = \{z\}$ or $S^2 = \{u\}$. We require additional examples descriptive of the type of semiring obtainable when addition is $J_3$ or $J_4$.

*Example* 2. Let $T = [a, b]$ be an interval with min addition. If the multiplicative semigroup $(T, \cdot)$ is an $I$-semigroup, with either $a$ or $b$ as identity, the resulting structure is a semiring. Similarly if $x + y = x \vee y$ in $[a, b]$ and multiplication is any $I$-semigroup, $(T, +, \cdot)$ is again a topological semiring.

The next example exhibits many of the properties derived in the lemma which follows.

*Example* 3. Let $T = [0, 1/2]$ with ordinary multiplication and addition $x + y = x \wedge y$. If addition is given by $x + y = x \vee y$, $(T, +, \cdot)$ is another topological semiring on the same set.

LEMMA 7. *Let $T = [a, b]$ be an interval, with $J_4$ addition, endowed with a multiplication such that $E[\cdot] = \{a\}$ and $(T, +, \cdot)$ is a topological semiring. Then:*

(1) *$T^2$ is continued in the same subinterval $L$ of $E[+]$ which contains the element $a$.*

(2) *If $x$, $y$ and $w$ are in $T$, with $x \leqq y$, then $xw \leqq yw$ and $wx \leqq wy$; if $x \neq a$, then $xw, wx < x$.*

(3) *If $x \in T$, $xT = [a, xb]$, $Tx = [a, bx]$ and $T^2 = [a, b^2]$.*

*Proof*. The $J_4$ addition is divisible and thus $T^2 \subset E[+]$. Since $T^2$ is also connected and contains $a = a^2$, $T^2$ is wholly contained in $L$.

Addition in $E[+]$ is min. Let $x, y \in T$, with $x \leqq y$. If either $x$ or $y$ is in $E[+]$, then $x = x + y$. For any $w \in T$, $xw = xw + yw$ and $wx = wx + wy$. All elements are in $E[+]$, hence $xw \leqq yw$ and $wx \leqq wy$. The same computations are also valid if $x$ and $y$ are in different subintervals of $S \backslash E[+]$. If $x$ and $y$ are in the same subinterval $L$ of $S \backslash E[+]$, there exists $h \in L$ such that $y + h = x$. Then $x + y = y + y + h$ and, because $yw \in E[+]$, we obtain the result $xw + yw = xw \leqq yw$.

Let $x, w \in T$, with $x \neq a$. Then $x \neq x^2$ and $x < x^2$ implies that $x = x + x^2$, hence $x^2 = x^2 + x^3$. Adding $x$ to both sides, $x = x + x^3$ and, by induction, $x = x + x^n$ for all $n \geqq 2$. But $T$ is compact and the net of powers of $x$ must then cluster to $a$, implying that $x = x + a$, which is a contradiction. From $a = a + x$, $a = a^2 = a + xa \leqq xa$. Similarly $xa = xa + x^2 \leqq x^2 < x$. Now, if $x = xw$, then $x = xw^n$ for every integer $n \geqq 2$ and thus $x = xa$, a contradiction. Now, if $x < xw$, then $x = x + xw$, from which $xw = xw + xw^2$ and, using the same procedure as above, $x = x + xa$, which is another contradiction. Consequently $xw \leqq x$ and similarly $wx \leqq x$.

For $x, y \in T$, $a = a + y$ and $y = b + y$, hence $xa \leqq xy \leqq xb$ and thus $xT \subset [xa, xb]$. But $xT$ is connected and contains both $xa$ and $xb$, so $xT = [xa, xb]$. If $a < xa$, there exists a positive integer $n$ such that $x^n \in [a, xa)$. Because $n \neq 1$, we have the result

$$x^n = x^n + xa = x(x^{n-1} + a) = xa$$

which is a contradiction. Analogously one shows that $Tx = [a, bx]$ and $T^2 = [a, b^2]$.

https://doi.org/10.4153/CJM-1974-056-7 Published online by Cambridge University Press

*Example* 4. Let $T = [a, b]$ with $J_4$ addition and let $(\cdot)$ be a continuous multiplication defined on $T$ such that: (1) $E[\cdot] = \{a\}$; (2) if $x \leqq y$, and $w \in T$, then $xw \leqq yw$ and $wx \leqq wy$; (3) if $x \neq a$, then $xw, wx < x$ for all $w \in T$; (4) multiplication distributes over addition. Then $(T, +, \cdot)$ is a topological semiring with $J_4$ addition.

The existence of such a multiplication is obvious, since $T^2 = \{a\}$ satisfies the first three postulates and distributes over addition. It would seem that any solution yielding a complete characterization of the multiplication in Lemma 7 would require a knowledge of the topological semigroups which can exist on the interval $[0, 1/2]$, in which 0 is the only multiplicative idempotent.

Because $J_3$ is a special case of $J_4$, it is only necessary to consider the latter. The last example is representative of a topological semiring with $J_4$ addition.

*Example* 5. Let $S = [z, u]$ be a real number interval, with $J_4$ addition, in which $u$ is the additive identity. Choose any four points $s, e, f$ and $t$ in the same subinterval $L$ of $E[+]$, where $z \leqq s \leqq e \leqq f \leqq t \leqq u$. Label the resulting intervals as $A = [z, e]$, $K = [e, f]$ and $B = [f, u]$, where $A$ is the union of $C = [z, s]$ and $D = [s, e]$, while $B$ is the union of the subintervals $I = [f, t]$ and $R = [t, u]$. The multiplication on $S$ will be defined so that the set $E[\cdot]$ of multiplicative idempotents lies entirely in $[s, t]$, $S^2 \subset L$ and $K$ is the multiplicative kernel with left-trivial multiplication. Addition in $E[+]$ is min and the subintervals $D$, $K$ and $I$ will be contained in $L$. The multiplication is as follows.

In $K = [e, f]$: $xy = x$ and $ks = k$ for $k \in K$, $s \in S$.
In $I = [f, t]$: $x + y = x \wedge y$ and multiplication is an $I$-semigroup with identity $t$ and kernel $\{f\}$.
In $D = [s, e]$: $x + y = x \vee y$ and multiplication is an $I$-semigroup with identity $s$ and kernel $\{e\}$.
In $R = [t, u]$: $E[\cdot] \cap R = \{t\}$ and multiplication satisfies the four properties of Example 4.
In $C = [z, s]$: $E[\cdot] \cap C = \{s\}$ and multiplication is the analogue of Example 4 with $\{s\}$ acting as the multiplicative kernel.
In $F = [f, u]$: $xy = yx = x$ for $x \in I$, $y \in R$.
In $A = [z, e]$: $xy = yx = y$ for $x \in C$, $y \in D$.
Complex Products: $B.A = B.K = \{f\}$ and $A.B = A.K = \{e\}$.

The resulting structure $(S, +, \cdot)$ is a topological semiring, with $J_4$ addition and multiplicative kernel $K$: the subintervals $C, D, I, R, A, B$ and $K$ are subsemirings. Since products of elements from different subintervals are either trivial or left-trivial in $K$, the multiplication is easily verified to be continuous and distributive over the addition.

THEOREM 8. *Let $(S, +, \cdot)$ be a $J_4$ addition topological semiring on the interval $[z, u]$ of real numbers. Then:*

(1) *There exist elements $s, e, f$ and $t$, all in the same subinterval $L$ of $E[+]$, such that $K[\cdot] = [e, f]$, $E[\cdot] \subset [s, t]$, where $z \leqq s \leqq e \leqq f \leqq t \leqq u$. Moreover, $xy = x$ or $xy = y$ for all $x$ and $y$ in $K[\cdot]$.*

*Assuming that multiplication in* $K[\cdot]$ *is left-trivial* $(xy = x)$, *and labelling the resulting subintervals as* $A = [z, e]$, $B = [f, u]$, $C = [z, s]$, $D = [s, e]$, $I = [f, t]$ *and* $R = [t, u]$, *then:*

(2) $(A, +, \cdot)$ *and* $(B, +, \cdot)$ *are subsemirings of* $S$, *with respective multiplicative kernels* $\{e\}$ *and* $\{f\}$.

(3) $B.A = B.K[\cdot] = \{f\}$ *and* $A.B = A.K[\cdot] = \{e\}$.

(4) $(I, +, \cdot)$ *and* $(D, +, \cdot)$ *are subsemirings of* $S$, *contained in* $L$, *and have min addition and I-semigroup multiplications.*

(5) $(R, +, \cdot)$ *and* $(C, +, \cdot)$ *are subsemirings, with* $E[\cdot] \cap R = \{t\}$ *and* $E[\cdot] \cap C = \{s\}$; *the multiplication is given by Example* 4.

(6) *For* $x \in C$, $y \in D$, $xy = yx = y$; *for* $x \in I$, $y \in R$, $xy = yx = x$.

(7) $S^2 \subset L \subset E[+]$.

*Proof*. Because $(S, \cdot)$ is a compact and connected semiring, the multiplicative kernel $K[\cdot]$ must be a closed subinterval of $S$ contained in $E[+]$. Denote the kernel by $K[\cdot] = [e, f]$. Connectivity requires that the kernel be contained in a single component $L$ of $E[+]$. Similarly $E[\cdot]$ is closed, requiring that elements $s$ and $t$ exist such that $s = s \wedge x$ and $x = x \wedge t$ for all $x \in E[\cdot]$. Because $K[\cdot]$, unless trivial, has a cutpoint, multiplication in the kernel is either left- or right-trivial from [**6**, Corollary to Theorem 2.4.6]. We assume the former. Thus for $k \in K[\cdot]$, $s \in S$, $ks = k(ks) = k$ and $K[\cdot] \subset E[\cdot]$, requiring that $z \leqq s \leqq e \leqq f \leqq t \leqq u$.

Consider the subinterval $A = [z, e] = \{x : x = x + e\}$. Because $(A, +)$ is a subsemigroup we need only demonstrate closure under multiplication. For $x, y \in A$ we obtain

$$\begin{aligned} xy = (x + e)(y + e) &= xy + ey + xe + e^2 \\ &= xy + ey + xe + e \end{aligned}$$

implying that $xy = xy + e \in A$. Note that $ex, xe \in K[\cdot] \cap A = \{e\}$, and therefore $\{e\}$ is the multiplicative kernel. Similarly $(B, +, \cdot)$ is a subsemiring with multiplicative kernel $\{f\}$.

Recall that $Bf = \{f\}$ and $bk \in K[\cdot]$ for $b \in B, k \in K[\cdot]$. Since $f = b + f$, we obtain

$$bk = bk + f = bk + fk = (b + f)k = fk = f$$

and thus $B.K[\cdot] = \{f\}$. Analogously $A.K[\cdot] = \{e\}$.

For elements $a \in A$ and $b \in B$, $e = eb = b + e = ae$ and $a = a + e$. Consequently

$$\begin{aligned} ab + e = ab + eb &= (a + e)b = ab \\ &= ab + ae = a(b + e) = ae = e \end{aligned}$$

and hence $A.B = \{e\}$. Similarly $B.A = \{f\}$ from the equations

$$\begin{aligned} ba + f = ba + bf &= b(a + f) = ba \\ &= ba + fa = (b + f)a = fa = f. \end{aligned}$$

https://doi.org/10.4153/CJM-1974-056-7 Published online by Cambridge University Press

Of the nine set products possible from $A$, $B$ and $K[\cdot]$, $K[\cdot]$, $B^2$ and $A^2$ are yet to be determined. Consider the subsemiring $B = [f, u]$, which is the union of $I = [f, t]$ and $R = [t, u]$. Since $I = \{x : f = f + x, x = x + t\}$ and $I.f = f.I = \{f\}$, $\{f, t\} \subset (tI) \cap (It)$ and therefore $I = tI = It$. The element $t$ is a two-sided multiplicative identity for $I$.

Noting that $([s, t])^2$ contains both $s$ and $t$, and that $S^2 \subset E[+]$, $[s, t] \subset E[+]$ and, indeed, $[s, t] \subset L$. Therefore for $x, y \in I$, $x + y = x \wedge y$. Now

$$xy = (x + t)(y + t) = xy + ty + xt + t = xy + y + x + t,$$

so $xy \leqq t$. But $xy \in B$ so $(I, +, .)$ is a subsemiring. Since $I$ is irreducibly connected between the multiplicative zero element $f$ and the multiplicative identity $t$, $(I, .)$ must be an $I$-semigroup from the analysis in [**4**]. In a similar fashion $(D, +, \cdot)$ is a subsemiring, where $(D, \cdot)$ is an $I$-semigroup with multiplicative identity element $s$.

Because $s, t \in E[+]$, both $R = [t, u]$ and $C = [z, s]$ are additive subsemigroups. Let $x, y \in R$. Then $t = t + x = t + y$ and

$$t = (t + x)(t + y) = t + xt + ty + xy = t + xy$$

which proves closure of $R$ under multiplication. Analogously one shows that $(C, \cdot)$ is a subsemigroup. Lemma 7 can now be applied.

For elements $x \in I$, $y \in R$, we have that $x = xt$ and $t = ty$ and therefore $xy = (xt)y = x(ty) = xt = x = yx$. A similar result holds for multiplication between $C$ and $D$.

Lastly, $S^2 \subset E[+]$ as remarked earlier and is a connected set. Consequently $S^2 \subset L$.

## References


**1.** H. Anzai, *On compact topological rings*, Proc. Japan Acad. *19* (1943), 613–615.

**2.** J. C. Beidleman and R. H. Cox, *Topological near-rings*, Arch. Math. *18* (1967), 485–492.

**3.** A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*, Volume I (American Mathematical Society, Providence, 1961).

**4.** P. S. Mostert and A. L. Shields, *On the structure of semigroups on a compact manifold with boundary*, Ann. of Math. *65* (1957), 117–143.

**5.** J. Mycielski, *Some properties of connected compact groups*, Colloq. Math. *5* (1958), 162–166.

**6.** A. B. Paalman de Miranda, *Topological semigroups* (Mathematisch Centrum, Amsterdam, 1964).

**7.** K. R. Pearson, *Interval semirings on $R_1$ with ordinary multiplication*, J. Austral. Math. Soc. *6* (1966), 273–288.

**8.** ——— *Certain topological semirings on $R_1$*, J. Austral. Math. Soc. *8* (1968), 171–182.

**9.** J. Shelden, *A note on compact semirings*, Proc. Amer. Math. Soc. *15* (1964), 882–886.

**10.** A. D. Wallace, *The structure of topological semigroups*, Bull. Amer. Math. Soc. *61* (1955), 95–112.

**11.** ——— *Dispersal in compact semimodules*, J. Austral. Math. Soc. *13* (1972), 338–342.



*Department of Electrical Engineering,*
*University of Alberta,*
*Edmonton, Alberta*