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OPEN SURFACES WITH CONGRUENT GEODESICS

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The aim of this paper is to prove the Theorem: Let M be a complete non compact surface without boundary in the euclidean space $\mathbb{E}^3$. We suppose that all geodesics of M are congruent. Then M is an affine plane in $\mathbb{E}^3$.

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If M is a closed surface in the euclidean 3-space which has all its geodesics congruent, then M is a round sphere. Compactness of M, which implies that M is a diffeomorphic to a sphere, is crucial in establishing the above result (see [**3**]).

Similarly, in the study of manifolds with families of congruent curves, compactness is an essential hypothesis (see [**2**, **6**, **10**]).

In the present note following the principal ideas of [**3**] we are able, for the first time, to remove the compactness assumption. In fact we show:

Theorem. *Let M be a complete non-compact surface without boundary embedded in the euclidean space $\mathbb{E}^3$. We suppose that all geodesics of M are congruent. Then M is an affine plane in $\mathbb{E}^3$.*

In the course of the proof we will often refer to the compact case [**3**]. However, we will make this paper as self-contained as possible by introducing all necessary notation and definitions.

Proof of the theorem. We separate the proof in several lemmas.

Lemma 1. *The surface M is diffeomorphic to $\mathbb{R}^2$.*

Proof. At first we show that all congruent geodesics of M are simple curves diffeomorphic to $\mathbb{R}$.

Suppose that the geodesics of M have self-intersection points. We pick such a geodesic γ. In the following we suppose that all the parametrizations of the geodesics or of the geodesic arcs that we consider are by arc-length. Let f: $(-\infty, \infty) \to M$ be a parametrization of γ with $f(0) = p$ and let $\rho > 0$ such that $f/[0, \rho]$ has at least one self-intersection point. Since M is an open manifold it is well known that there exists a geodesic ray r: $[0, +\infty) \to M$ with $r(0) = p$. Therefore $r/[0, \rho]$ is an embedding in M and consequently $r'(0)\neq f'(0)$. We fix an orthonormal basis $\{e_1,e_2\}$ of T_pM with $e_1=r'(0)$, which induces an orientation on T_pM. We parametrize each unit vector v of T_pM by the oriented angle $\theta=\measuredangle(v,e_1)$, $0\leqq\theta<2\pi$; note that $\measuredangle(e_1,e_2)=\pi/2$.

Let now $\theta_0=\sup\{\theta\in[0,2\pi]$ such that every geodesic arc $g{:}[0,\rho]\to M$ with $g'(0)\in T_pM$ and $\measuredangle(g_0'(0),e_1)=\theta'<\theta$ is simple$\}$. Note that $\theta_0>0$ since the set of embeddings g: $[0.\rho]\to M$ is open in the space $C^\infty([0,\rho],M)$ [**5**]. Now we consider the geodesic g_0: $[0,\rho]\to M$ with $g_0'(0)\in T_pM$ and let $\measuredangle(g_0'(0),e_1)=\theta_0$. Claim: $g_0/[0,\rho]$ is a simple geodesic arc. From this we conclude that $\theta_0=2\pi$ which contradicts the hypothesis that $f/[0,\rho]$ has self intersection points. To prove the claim observe that if $g_0/[0,\rho]$ were not simple then every geodesic arc g: $[0,\rho]\to M$ ε-close to $g_0/[0,\rho]$, for ε small enough, would not be simple. But this contradicts the definition of θ_0.

Suppose now that $\pi_1(M)\neq 1$. It is well known (see for example [**4**, Ch. 10, Th. 13] that for every pair of points p,q (and hence for $p=q$) and for every arc $\alpha(p,q)$ joining p,q there is a geodesic arc $\gamma(p,q)$ in the homotopy class of $\alpha(p,q)$ with end points fixed. So if we take a noncontractible loop $\alpha(p,p)$ on M and if we consider a geodesic arc $\gamma(p,p)$ in the homotopy class of $\alpha(p,p)$ with p fixed, then the geodesic of M containing $\gamma(p,p)$ is either closed or it has self-intersection points. But this is impossible since we have proved that all geodesics of M are curves diffeomorphic to $\mathbb{R}$. Therefore $\pi_1(M)=1$ and M is diffeomorphic to $\mathbb{R}^2$. □

Now we consider a fixed curve Γ_0 in $\mathbb{E}^3$ such that every geodesic of M is congruent to Γ_0. If Γ_0 is a plane curve or if the curvature of Γ_0 is constant then in each case we can easily deduce that all points of M are umbilical and consequently M is an affine plane in $\mathbb{E}^3$. We next assume that Γ_0 is not a plane curve as well as that the curvature of Γ_0 is not constant and we will prove that this assumption is incompatible with the hypothesis that all geodesics of M are congruent. Let $\alpha(s)$, $s\in(-\infty,\infty)$ be a parametrization by arc length of Γ_0 and let $k(s)$, $\tau(s)$ be the curvature and torsion functions of $\alpha(s)$ respectively.

We denote by $<,>$ the usual inner product in $\mathbb{E}^3$ and by A the shape operator of M. Let v_p be a vector in the unit tangent bundle $S(M)$ of M. There exists a unique geodesic γ: $(-\infty,\infty)\to M$ such that $\gamma(0)=p$, $\gamma'(0)=v_p$. We denote by $\kappa(v_p)$, $\tau(v_p)$ the normal curvature and torsion of γ at p, and we have that:

$$\kappa(v_p)=<Av_p,v_p>,\tau(v_p)=<Av_p,Jv_p>,$$

where by Jv_p we denote the vector that we obtain if we rotate v_p counterclockwise in T_pM by $\pi/2$.

In what follows we will refer to them as the curvature and torsion of vectors of $S(M)$.

Lemma 2. (a) *Let r: $S(M)\to\mathbb{R}^+$ be the differentiable function defined by $r(v_p)=|\kappa(v_p)|$ and let k_0 be a non-critical value of $k(s)$. Then the set $r^{-1}(k_0)$ is a closed surface in $S(M)$.*

(b) *We can choose the non-critical value k_0 such that there exists a component C of $r^{-1}(k_0)$ which contains only non-principal vectors. Moreover, for each v_p in C, $\tau(v_p)=$* constant $\neq 0$.

Proof. For the proof of (a) we remark that if k_0 is a non-critical value of $k(s)$ then r is of rank 1 on $r^{-1}(k_0)$ (for more details see the proof of Proposition 2 in [3]).

For the proof of (b) we consider a non-umbilical point q in M; remark that such a point exists since the function $k(s)$ is not constant. Now we can choose a non-principal vector w_q in T_qM such that $r(w_q)=k_0$ and k_0 is a non-critical value of $k(s)$. Among the components of the surface $r^{-1}(k_0)$ consider that one which contains the vector w_q and denote it by C. We can prove that $\tau(v_p)=\text{constant}\neq 0$ for each v_p in C which implies that all the vectors of C are non-principal (for more details see the proof of the lemma in [3]).

Lemma 3. *Let π: $C\to M$ be the projection in M with $\pi(v_p)=p$. Then the pair (C,π) is a covering space of M.*

Proof. As in Proposition 3 of [3] we prove that π has rank 2 at every v_p in C, so π is a local diffeomorphism. We next show that π is onto by proving that $\pi(C)$ is an open and closed subset in M. Since π is a local diffeomorphism we get that $\pi(C)$ is an open subset of M and next we will prove that $\pi(C)=\overline{\pi(C)}$ which implies that $\pi(C)$ is also closed in M. Let $p\in\overline{\pi(C)}$, then there is a sequence p_n in $\pi(C)$ which converges to p. Let $v_n\in C$ with $\pi(v_n)=p_n$. Since M is diffeomorphic to $\mathbb{R}^2$ we have that $S(M)$ is diffeomorphic to $M\times S^1$ under a diffeomorphism F. Let $(p_n,\theta_n)=F(v_n)$. The space S^1 is compact so there exists a subsequence θ_{n_κ} of θ_n converging to a $\theta\in S^1$. Consequently $(p_{n_\kappa},\theta_{n_\kappa})$ converges to (p,θ); hence the subsequence v_{n_κ} of v_n converges to a v in C since C is a closed subset in $S(M)$. It follows that $p=\pi(v)=\lim_\kappa \pi(v_{n_\kappa})$ belongs to $\pi(C)$ which implies that $\pi(C)=\overline{\pi(C)}$. □

Observe that M is simply connected and therefore has no non-trivial covering spaces. So the projection π: $C\to M$ is a diffeomorphism. This permits the construction of a non-vanishing vector field X on M such that $r(X_p)=\text{constant}$ for each p in M.

Lemma 4. *The set of non-critical values k_0 of the curvature function $k(s)$, such that some component of $r^{-1}(k_0)$ contains non-principal vectors, is dense into the range R of $k(s)$.*

Proof. At first we know by Sard's theorem [5] that the set of non-critical values of $k(s)$ is dense in R. Let k_0 be a non-critical value of $k(s)$ such that $r^{-1}(k_0)$ contains only principal vectors. Let v_p be such a vector in $r^{-1}(k_0)$. We distinguish 2 cases:

(1) *The point p is non-umbilical.* Suppose without loss of generality that $k_0=r(v_p)$ is the minimum normal curvature at p. Then for each $\varepsilon>0$ there is a non-critical value k_1 of $k(s)$ in $[k_0,k_0+\varepsilon)$ such that $r^{-1}(k_1)$ contains a non-principal vector w_p and hence all the vectors in the connected component of $r^{-1}(k_1)$ which contains w_p are non-principal. To find such a non-critical value k_1 it is sufficient to note that if we consider an open neighbourhood U of v_p in $S_p(M)=\{v\in T_pM\colon |v|=1\}$, sufficiently small, then $r(U)$ is of the form $[k_0,k_0+\delta)$, $\delta>0$ and $\tau(v)\neq 0$ for each $v\in U-\{v_p\}$.

(2) *The point p is umbilical.* Let O be the set of umbilical points of M. Then there is not an open neighbourhood U of p in M with $U\subset O$. If such an open subset existed,

then U should be a piece of a plane or of a sphere (Th. 2-2 of [8]). So the value of $r(v_p)=k_0$ should be a critical value of $k(s)$ which is absurd. Therefore we can obviously find a sequence p_n of non-umbilical points in M converging to p. Now using case (1) above we can find a sequence of non-principal vectors (v_n), $v_n \in T_pM$ such that: the sequence (v_n) converges to v_p and the values $r(v_n)=k_n$ are non-critical values of $k(s)$ for each $n=1,2,\ldots$. This completes the proof of Lemma 4. □

Now we can finish the proof of the theorem:

The range $R(p)$ of the function $r/S_p(M)$ is obviously a closed subset of R and $R(p)\subset R$, for each p in M. By Lemmas 3 and 4 at every p in M there are unit tangent vectors v_i such that the values $r(v_i)$ form a dense subset in R. Therefore $R(p)=\overline{R(p)}=R$. This implies readily that the Gaussian curvature K of M is constant. If $K>0$ then M is compact (Th. 8–18 of [9]) which is impossible. On the other hand a complete surface M of constant negative curvature cannot be embedded in $\mathbb{E}^3$ (Th. 5–12 of [8]). Therefore the curvature K of M is equal to zero which implies that M is a generalized cylinder (Th. 5–9 of [8]), and since all geodesics of M are congruent, M will be necessarily an affine plane. But in this case, all geodesics of M are straight lines which contradicts the assumption that the curvature function $k(s)$ is not constant. Therefore $k(s)$ is a constant function and, as explained above, this implies that M is an affine plane. □

Remark. In a similar way we can prove the same result for open surfaces M embedded in the hyperbolic space $\mathbb{H}^3$. However, since we have not a complete idea for the surfaces of constant curvature in $\mathbb{H}^3$ (see [9, p. 163]) we proceed as follows: With exactly the same reasonings we conclude that if Γ_0 is not a plane curve and if the curvature of Γ_0 is not constant then the functions of principal curvatures remains constant on M. Therefore M is an isoparametric surface in $\mathbb{H}^3$. The classification of these surfaces which have at most two distinct principal curvatures [1], [7] gives that M is either a geometric sphere or a geometric cylinder or a geodesic plane in $\mathbb{H}^3$, and since $\pi_1(M)=1$ the result follows.

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