Proceedings of the Edinburgh Mathematical Society (2015) **58**, 485–497
DOI:10.1017/S0013091514000194

CLASSIFYING POLYGONAL ALGEBRAS BY THEIR K_0-GROUP

JOHAN KONTER[1] AND ALINA VDOVINA[2]

[1] *Department of Mathematics, Northwestern University, Evanston, IL 60208-2370, USA*
[2] *School of Mathematics and Statistics, University of Newcastle, Newcastle upon Tyne NE1 7RU, UK* (alina.vdovina@ncl.ac.uk)

(Received 10 September 2012)

Abstract We prove that every incidence graph of a finite projective plane allows a partitioning into incident point-line pairs. This is used to determine the order of the identity in the K_0-group of so-called polygonal algebras associated with cocompact group actions on $\tilde{A}_2$-buildings with three orbits. These C^*-algebras are classified by the K_0-group and the class $\mathbb{1}$ of the identity in K_0. To be more precise, we show that $2(q-1)\mathbb{1} = 0$, where q is the order of the links of the building. Furthermore, if $q = 2^{2l-1}$ with $l \in \mathbb{Z}$, then the order of $\mathbb{1}$ is $q-1$.

Keywords: Euclidean buildings; Kuntz–Krieger algebras; incidence graphs; finite projective planes

2010 *Mathematics subject classification:* Primary 46L80
Secondary 19K99

1. Introduction

The class of the identity in K_0 of different families of crossed product C^*-algebras has been of interest because it is a classifying invariant for these algebras (see, for instance, [**1**, **7**–**10**, **13**]). We will concentrate on a case associated with the universal covering building $\mathcal{B}$ of a polyhedron on three vertices such that the links are incidence graphs of finite projective planes. To these Euclidean buildings we associate a rank 2 Cuntz–Krieger algebra $\mathcal{O}_{\hat{M}}$ that we call a polygonal algebra (where the subscript $\hat{M}$ reflects the dependence of the algebra on two matrices $\hat{M}_1$, M_2 that are defined in terms of incidence relations in the building). In the first section we give a brief exposition of the classification of these polygonal algebras. The key point of the second section is the following theorem in incidence geometry, which might be considered interesting in its own right.

Theorem. *Every incidence graph of a finite projective plane allows a partitioning into incident point-line pairs.*

This theorem gives rise to a so-called *semi-basic subset* in the K_0-group of $\mathcal{O}_{\hat{M}}$. Using this semi-basic subset, we prove the following new results in the classification of polygonal algebras.

Theorem. *The identity* $\mathbb{1}$ *in* $K_0(\mathcal{O}_{\hat{M}})$ *satisfies* $2(q-1)\mathbb{1} = 0$, *where* q *is the order of the links of the building.*

Theorem. *If* $q = 2^{2l-1}$ *with* $l \in \mathbb{Z}$, *then the order of* $\mathbb{1}$ *is* $q-1$.

It is worth emphasizing that we allow the finite projective planes that form the links of $\mathcal{B}$ to be non-Desarguesian and that the results are independent of the structure of the group acting on the building; we only require that the action on $\mathcal{B}$ has three orbits.

2. Background

We start with a general polygonal presentation but we will restrict ourselves to $n = 3$ shortly afterwards.

Definition 2.1 (Cartwright *et al.* [4, 5]; Vdovina [15]). For $i = 1, \dots, n$, let G_i denote distinct connected bipartite graphs. For each i, we fix two sets B_i and W_i of 'black', respectively 'white', vertices in G_i that give a bipartition. The unions of these sets will be denoted by $B = \bigcup B_i$ and $W = \bigcup W_i$. A set $\mathfrak{K}$ of k-tuples $(x_1, x_2, \dots, x_k)$ with all $x_j \in B$ will be called a *polygonal presentation* over B compatible with a bijection $\lambda\colon B \to W$ if the following properties are satisfied.

(1) If $(x_1, x_2, \dots, x_k) \in \mathfrak{K}$, then all cyclic permutations of $(x_1, x_2, \dots, x_k)$ are elements of $\mathfrak{K}$.

(2) Given $x_1, x_2 \in B$, there are $x_3, \dots x_k \in B$ such that $(x_1, x_2, \dots, x_k) \in \mathfrak{K}$ if and only if $\lambda(x_1)$ and x_2 share an edge in some G_i.

(3) Given $x_1, x_2 \in B$, there is at most one set $\{x_3, \dots, x_k\} \subset B$ such that $(x_1, x_2, \dots, x_k) \in \mathfrak{K}$.

If a polygonal presentation exists with respect to λ, we will call λ a *basic bijection*.

Furthermore, we can associate a polyhedron K with every polygonal presentation $\mathfrak{K}$. For the set of cyclic permutations of the k-tuple $(x_1, \dots, x_k)$, we build a k-gon with labels $x_1, \dots, x_k$ on its directed edges. The polyhedron K is obtained by gluing all k-gons together by identifying the edges with the same label, preserving orientation. The links of the vertices of this polyhedron become exactly the graphs G_i. This allows one to construct polyhedra with specified properties, as is done in [**16**].

On the other hand, if one starts with a suitable polyhedron there is a natural way to construct a polygonal presentation. Write down the cycles of edges that correspond to faces and take the links as G_i. For the basic bijection we take the map sending a vertex in one link to the vertex in another link corresponding to the same edge in the polyhedron.

Definition 2.2. For the remainder of this paper, X is a polyhedron on three vertices, v_0, v_1 and v_2, with triangular faces such that all faces have three different vertices. We demand that the links, say G_0, G_1, G_2, respectively, are incidence graphs of finite projective planes, but they need not be isomorphic. The building $\mathcal{B}$ is defined as the universal covering of X. Let Γ denote the fundamental group, acting on $\mathcal{B}$ (from the left).

Because $\Gamma\backslash\mathcal{B} \simeq X$, we get a labelling of the vertices of $\mathcal{B}$ as types 0, 1 and 2 which we let correspond to the labels of the vertices of X. So vertices of type i have G_i as link. The edges are directed and labelled with small roman letters with subscripts in such a manner that x_i denotes an edge with an origin of type $i+1$ and a terminal vertex of type $i-1$ modulo 3. The subsets B_i and W_i of G_i correspond to the outgoing edges and the incoming edges, respectively. Two vertices $b \in B_i$ and $w \in W_i$ are connected in G_i if there is a face of X that has b and w as edges. Suppose the order of the projective plane G_i is q. Then the properties of finite projective planes dictate that $|B_i| = |W_i| = q^2+q+1$. Furthermore, we can count that the number of triangles equals $(q+1)(q^2+q+1)$. Hence the order of every link is q. Another noteworthy remark is the fact that the incidence graph of a finite projective plane is a generalized 3-gon.

The universal covering $\mathcal{B}$ is a building of type $\tilde{A}_2$. The fundamental group Γ acts simply transitively on the three separate sets of vertices of type 0, 1 or 2.

Definition 2.3. We pick an origin O in $\mathcal{B}$. Let $\mathcal{T}$ denote the set of ordered triples (a_0, a_1, a_2), (a_1, a_2, a_0) and (a_2, a_0, a_1) in $\mathcal{B}$ such that a_0, a_1 and a_2 are the labels of the sides of a triangle that has O as one of its vertices. The set of triples starting with an edge of type 0, the set of triples starting with an edge of type 1 and the set of triples starting with an edge of type 2 will be denoted by $\mathcal{T}_0$, $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively.

Theorem 2.4. *The set $\mathcal{T}$ forms a polygonal presentation whose associated polygon is isomorphic to X.*

Proof. We start with the polyhedron X and apply the proposed construction: write down the cycles of edges that correspond to faces of X and we define λ as the map sending a vertex in one link to the vertex in another link corresponding to the same edge in X. Conditions 1 and 2 are immediately satisfied and condition 3 follows from the fact that the links G_i are incidence graphs of finite projective planes. This polygonal presentation is indeed $\mathcal{T}$, since $\mathcal{T}$ has exactly three triples representing every triangle in X. $\square$

We proceed with the set-up for polygonal algebras.

Definition 2.5. We define two matrices, $\hat{M}_1$ and $\hat{M}_2$, acting on the free $\mathbb{Z}$-module spanned by $\mathcal{T}$. Pick two elements $a = (a_{i-1}, a_i, a_{i+1}) \in \mathcal{T}_{i-1}$ and $b = (b_i, b_{i+1}, b_{i-1}) \in \mathcal{T}_i$. Then $\hat{M}_1(b, a) = 1$ if and only if there exists a $c_{i-1} \neq a_{i-1}$ such that (a_{i+1}, c_{i-1}, b_i) is an element of $\mathcal{T}$, and $\hat{M}_2(b, a) = 1$ if and only if there exists a $c_{i-1} \neq a_{i-1}$ such that (b_{i+1}, c_{i-1}, a_i) is an element of $\mathcal{T}$ (see figure 1). For all other combinations the corresponding entries of $\hat{M}_1$ and $\hat{M}_2$ are zero.

The size of these matrices is $|\mathcal{T}| = 3(q+1)(q^2+q+1)$, three times the number of triangles of X.

Theorem 2.6. *Each row and column of the $\{0,1\}$-matrices, $\hat{M}_1$ and $\hat{M}_2$, has exactly q^2 entries that are* 1.

Proof. We fix $a = (a_{i-1}, a_i, a_{i+1}) \in \mathcal{T}_{i-1}$ and count how many $b = (b_i, b_{i+1}, b_{i-1}) \in \mathcal{T}_i$ satisfy $\hat{M}_1(b, a) = 1$. By definition this is the case if and only if there exists a $c_{i-1} \neq a_{i-1}$

https://doi.org/10.1017/S0013091514000194 Published online by Cambridge University Press

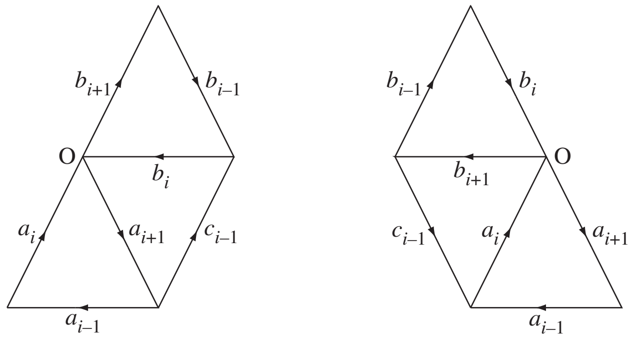

Figure 1. Showing $\hat{M}_1(b,a)=1$ and $\hat{M}_2(b,a)=1$.

such that (a_{i+1},c_{i-1},b_i) is an element of $\mathcal{T}$. Because the link of O is the incidence graph of a finite projective plane, we have $(q+1)-1=q$ choices for $b_i \neq a_i$ and subsequently q choices for $b_{i+1}\neq a_{i+1}$. The proof for $\hat{M}_2$ is similar. $\square$

Definition 2.7. For $m=(m_1,m_2)\in\mathbb{N}^2$ we let $[0,m]$ denote $\{0,\dots,m_1\}\times\{0,\dots,m_2\}$ and let e_j denote the jth standard unit basis vector. For every $m\in\mathbb{N}^2$ we define the *set of words of length* m as

$$W_m=\{w\colon [0,m]\to\mathcal{T};\ \hat{M}_j(w(l+e_j),w(l))=1 \text{ whenever } l,l+e_j\in[0,m]\}.$$

Let $W=\bigcup_{m\in\mathbb{N}^2}W_m$ denote the total set of words. We define the *shape* $\sigma(w)$ of a word $w\in W_m$ as $\sigma(w)=m$. We also define the *origin* and the *terminus* of a word via the maps $o\colon W_m\to\mathcal{T}$ and $t\colon W_m\to\mathcal{T}$ given by $o(w)=w(0)$ and $t(w)=w(m)$.

Theorem 2.8. *The matrices $\hat{M}_1$ and $\hat{M}_2$ satisfy the following conditions.*

(H1) $\hat{M}_1\hat{M}_2=\hat{M}_2\hat{M}_1$ *and this product is a $\{0,1\}$-matrix.*

(H2) *The directed graph with vertices $a\in\mathcal{T}$ and directed edges (a,b) whenever $M_i(b,a)=1$ for some i is irreducible.*

(H3) *For any $p\in\mathbb{N}^2$ there exists an $m\in\mathbb{N}^2$ and a $w\in W_m$ that is not p-periodic, i.e. there exists an l such that $w(l)$ and $w(l+p)$ are both defined, but not the same.*

Proof. Condition (H1) follows from the fact that the links are generalized 3-gons. If we take $a,d\in\mathcal{T}_i$ and $b\in\mathcal{T}_{i+1}$ with $\hat{M}_1(b,a)=1$ and $\hat{M}_2(d,b)=1$, then there is precisely one $c\in\mathcal{T}_{i-1}$ such that $\hat{M}_2(c,a)=1$, because the representatives of a, b, c in the link G_i are alternating sides of a 6-cycle that is fixed by two out of the three. Looking at the link G_{i-1} we find that $M_1(d,c)=1$ for this c (see figure 2). We conclude that $\hat{M}_1\hat{M}_2=\hat{M}_2\hat{M}_1$. And the fact that there is precisely one such $c\in\mathcal{T}_{i-1}$ implies that the product is a $\{0,1\}$-matrix.

For (H2) we prove that for every $a,b\in\mathcal{T}$ there exists an $r\in\mathbb{N}$ such that $\hat{M}_1^r(b,a)>0$. Suppose that $a\in\mathcal{T}_i$. By Theorem 2.6 it is obvious that either $b\in\mathcal{T}_i$ or there exists an $a'\in\mathcal{T}$ such that $\hat{M}_1^s(b,a')>0$, where s is 1 or 2. Because the proof of Theorem 1.3 in [**14**]

https://doi.org/10.1017/S0013091514000194 Published online by Cambridge University Press

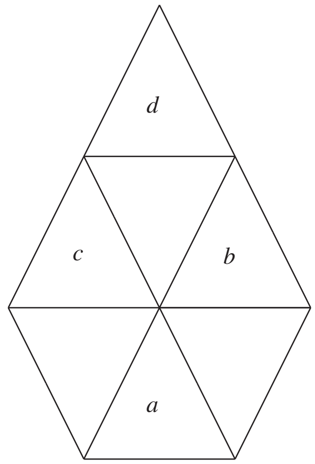

Figure 2. Property (H1) holds.

only uses that the link of every vertex of the building is an incidence graph of a finite projective plane, this theorem tells us that $\hat{M}_1^t(a', a) > 0$ for some natural number t.

For (H3) the proof of Proposition 7.9 in [**12**] carries over. □

Conditions (H1)–(H3) are the ones necessary to define a rank two Cuntz–Krieger algebra in accordance with the definition of such an algebra given in [**12**]. Condition (H1) implies that we can define a product in the following way. Let $u \in W_m$ and $v \in W_n$ with $t(u) = o(v)$. There is then a unique $w \in W_{m+n}$ such that

$$w|_{[0,m]} = u \quad \text{and} \quad w|_{[m,m+n]} = v,$$

where $w|_{[0,m]}$ is the restriction of the map w to $[0, m]$. We define this w to be the product of u and v. Condition (H2) says that we can always find a word with a given origin and terminus. The more technical condition (H3) is used to show that the algebra that we will construct now is simple (see Theorem 5.9 of [**12**]).

Definition 2.9. We define the *polygonal algebra* $\mathcal{O}_{\hat{M}}$ as the universal C^*-algebra generated by a family of partial isometries $\{s_{u,v}; u, v \in W \text{ and } t(u) = t(v)\}$ with relations

$$s_{u,v}^* = s_{v,u}, \tag{2.1}$$

$$s_{u,v} = \sum_{\substack{w \in W; \sigma(w) = e_j, \\ o(w) = t(u) = t(v)}} s_{uw,vw} \quad \text{for } 1 \leqslant j \leqslant r, \tag{2.2}$$

$$s_{u,v} s_{v,w} = s_{u,w}, \tag{2.3}$$

$$s_{u,u} s_{v,v} = 0 \qquad \text{for } u, v \in W_0,\ u \neq v. \tag{2.4}$$

Theorem 2.10. *Morita equivalence and stable isomorphism provide the same notion of equivalence for polygonal algebras. Furthermore, $\mathcal{O}_{\hat{M}}$ is classified up to isomorphism by its K_0-group, its K_1-group and the class of the identity in $K_0(\mathcal{O}_{\hat{M}})$.*

Proof. By Remark 6.15 of [**12**] the C^*-algebra $\mathcal{O}_{\hat{M}}$ is a separable nuclear unital purely infinite simple algebra that satisfies the universal coefficient theorem. It is well known that a separable C^*-algebra contains countable approximate identities, so by Theorem 1.2 of [**3**], Morita equivalence and stable isomorphism provide the same notion of equivalence. Because it is separable, nuclear, purely infinite, simple and satisfies the universal coefficient theorem, $\mathcal{O}_{\hat{M}}$ is classified up to isomorphism by its K-groups and the class of the identity in $K_0(\mathcal{O}_{\hat{M}})$, according to Theorem 4.2.4 of [**11**]. □

Let us recall two theorems which give us some handle on the K_0- and K_1-groups.

Theorem 2.11. *Let r be the rank of* $\operatorname{coker}(I-\hat{M}_1, I-\hat{M}_2)$ *and let T be the torsion part of* $\operatorname{coker}(I-\hat{M}_1, I-\hat{M}_2)$. *Then*

$$K_0(\mathcal{O}_{\hat{M}}) \simeq K_1(\mathcal{O}_{\hat{M}}) \simeq \mathbb{Z}^{2r} \oplus T.$$

Proof. Because Γ acts freely and with finitely many orbits on the vertex set, this is a direct consequence of Proposition 4.13, Lemma 5.1 and Lemma 6.1 of [**13**]. □

The calculation of the order of the identity in $K_0(\mathcal{O}_{\hat{M}})$ turns out to be quite difficult, but we are able to give some estimates. To find these bounds, we need one more tool for which we look at the boundary of $\mathcal{B}$. For any $x \in \partial\mathcal{B}$ there is a unique sector in the class x with base point O, which we will denote $[O, x)$ (Lemma VI.9.2 in [**2**]). The collection of sets of the form

$$\partial\mathcal{B}_y = \{x \in \partial\mathcal{B} \colon y \in [O, x)\}$$

with y running through $\mathcal{B}$ gives a base for a topology for $\partial\mathcal{B}$, with respect to which it is a totally disconnected compact Hausdorff space (§ 7 of [**12**]). This topology is independent of the base point O (see Lemma 2.5 of [**6**]). The action of Γ on $\mathcal{B}$ induces an action on $\partial\mathcal{B}$.

Definition 2.12. The *boundary operator algebra* is the full crossed product algebra $C(\partial\mathcal{B}) \rtimes \Gamma$, the universal C^*-algebra generated by the algebra of continuous functions $C(\partial\mathcal{B})$ and a fixed unitary representation π of Γ satisfying the covariant defining relation

$$f(\gamma^{-1}\omega) = \pi(\gamma) f \pi(\gamma)^{-1}(\omega)$$

for all $f \in C(\partial\mathcal{B})$, $\gamma \in \Gamma$ and $\omega \in \partial\mathcal{B}$.

It was shown in § 7 of [**12**] that the polygonal algebra $\mathcal{O}_{\hat{M}}$ and the boundary algebra $C(\partial\mathcal{B}) \rtimes \Gamma$ are isomorphic. The interplay of the two algebras becomes visible in the next series of very useful lemmas.

Definition 2.13. We pick a chamber $\mathfrak{t} \in \mathcal{B}$, called the *model triangle*, that has O as one of its vertices. Let A denote the set of non-degenerate simplicial maps $\mathfrak{t} \to \mathcal{B}$ and $\hat{A} = \Gamma\backslash A$. For such a map $\iota \colon \mathfrak{t} \to \mathcal{B}$ we define $\partial\mathcal{B}(\iota)$ as the subset of $\partial\mathcal{B}$ consisting of those boundary points that may be represented by sectors that originate at $\iota(O)$ and contain $\iota(\mathfrak{t})$. Let $\mathbf{1}_\iota$ denote the characteristic function of $\partial\mathcal{B}(\iota)$.

https://doi.org/10.1017/S0013091514000194 Published online by Cambridge University Press

Lemma 2.14. *There is a bijection between $\hat{A}$ and $\mathcal{T}$. This bijection sends an element of $\hat{A}$ to a triple in $\mathcal{T}_i$ if it has a representative that sends O to a vertex of type i.*

Proof. Every element of $\hat{A}$ contains exactly one representative $\iota\colon \mathfrak{t} \to \mathcal{B}$ such that it has O as one of its vertices. This yields that it may be represented by an ordered triple, (a_0, a_1, a_2), (a_1, a_2, a_0) or (a_2, a_0, a_1), depending on the type of $\iota(O)$. It is straightforward to check that this map is a bijection. □

Lemma 2.15. *Let $\iota_1, \iota_2\colon \mathfrak{t} \to \mathcal{B}$ be two representatives of some $t \in \mathcal{T} \simeq \hat{A}$. Then $[\mathbf{1}_{\iota_1}] = [\mathbf{1}_{\iota_2}]$ in $K_0(C(\partial\mathcal{B}) \rtimes \Gamma)$, which makes the assignment of $[\mathbf{1}_t] \in K_0(C(\partial\mathcal{B}) \rtimes \Gamma)$ to every element $t \in \mathcal{T}$ a well defined notion.*

Proof. This is an analogue of Lemma 8.1 of [**13**]. Because $\Gamma\iota_1 = \Gamma\iota_2$, we can find a $\gamma \in \Gamma$ such that $\iota_2 = \gamma\iota_1$. We compute that $\gamma\mathbf{1}_{\iota_1}\gamma^{-1}(\partial\mathcal{B}) = \gamma\mathbf{1}_{\iota_1}(\partial\mathcal{B}) = \partial\mathcal{B}(\gamma\iota_1)$, so $\gamma\mathbf{1}_{\iota_1}\gamma^{-1} = \mathbf{1}_{\gamma\iota_1} = \mathbf{1}_{\iota_2}$. Equivalent idempotents belong to the same class in K_0, hence we get $[\mathbf{1}_{\iota_1}] = [\mathbf{1}_{\iota_2}]$. Because all elements of $\mathcal{T}$ correspond to an equivalence class of injections of $\mathfrak{t}$ in $\mathcal{B}$, this proves the lemma. □

Lemma 2.16. *For the function $\mathbf{1}$ in $C(\partial\mathcal{B}) \rtimes \Gamma$, we have $[\mathbf{1}] = \sum_{a\in\mathcal{T}_0}[\mathbf{1}_a] = \sum_{b\in\mathcal{T}_1}[\mathbf{1}_b] = \sum_{c\in\mathcal{T}_2}[\mathbf{1}_c]$. For $b \in \mathcal{T}_i$, we have $[\mathbf{1}_b] = \sum_{a\in\mathcal{T}_{i-1}}\hat{M}_1(b,a)[\mathbf{1}_a]$ and $[\mathbf{1}_b] = \sum_{c\in\mathcal{T}_{i+1}}\hat{M}_2(b,c)[\mathbf{1}_c]$.*

Proof. Fix a point $O_0 \in \mathcal{B}$ of type 0. Every element $a \in \mathcal{T}_0$ contains a representative $\iota_a \in A$ that sends $O \in \mathfrak{t}$ to O_0. The statement $[\mathbf{1}] = \sum_{a\in\mathcal{T}_0}[\mathbf{1}_a]$ is a result from the previous lemma and the fact that the set $\{\iota_a(\mathfrak{t})\colon a \in \mathcal{T}_0\}$ contains all the triangles with vertex O_0 exactly once. Of course, the same can be done for points O_1 and O_2 with the sets $\{\iota_b(\mathfrak{t})\colon b \in \mathcal{T}_1\}$ and $\{\iota_c(\mathfrak{t})\colon c \in \mathcal{T}_2\}$.

Take $b \in \mathcal{T}_i$ and pick a triangle in $\mathcal{B}$ with vertex O_{i-1} that represents b. Then $[\mathbf{1}_b]$ is given by $\sum[\mathbf{1}_a]$, where we sum over all the a having a representative in the sector spanned by the fixed triangle such that $\iota_a(O) = O_{i-1}$ (see figure 3). This is equivalent to saying $a \in \mathcal{T}_{i-1}$ and $\hat{M}_1(b,a) = 1$. Hence, $[\mathbf{1}_b] = \sum_{a\in\mathcal{T}_{i-1}}\hat{M}_1(b,a)[\mathbf{1}_a]$. Similar considerations apply to $\hat{M}_2$. □

Definition 2.17. We let $\mathbb{1}$ denote the equivalence class of the identity in both $K_0(C(\partial\mathcal{B}) \rtimes \Gamma)$ and $K_0(\mathcal{O}_{\hat{M}})$.

Since the algebras $\mathcal{O}_{\hat{M}}$ and $C(\partial\mathcal{B}) \rtimes \Gamma$ are isomorphic (by § 7 of [**12**] and Theorem 2.11), this abuse of notation will not cause any problems; the order of $\mathbb{1}$ is the same regardless of the group. The two counting Lemmas 2.15 and 2.16 may be used to find an upper and a lower bound for the order of $\mathbb{1}$ by employing similar methodology to that in Theorems 8.2 and 8.3 of [**13**].

Theorem 2.18. *The order of $\mathbb{1}$ in the K_0-groups divides $q^2 - 1$.*

https://doi.org/10.1017/S0013091514000194 Published online by Cambridge University Press

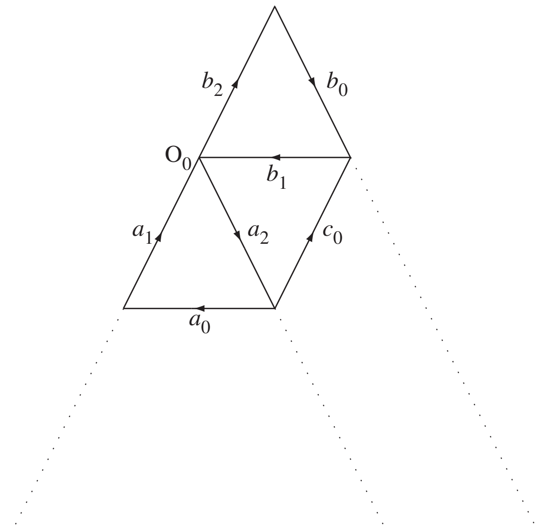

Figure 3. Showing $[\mathbf{1}_b] = \sum_{a\in\mathcal{T}_0} \hat{M}_1(b,a)[\mathbf{1}_a]$ for the example $i = 1$.

Proof. Lemma 2.16 states that for $b \in \mathcal{T}_i$ we have

$$[\mathbf{1}_b] = \sum_{a\in\mathcal{T}_{i-1}} \hat{M}_1(b,a)[\mathbf{1}_a],$$

and therefore the lemma implies that

$$\mathbb{1} = [\mathbf{1}] = \sum_{b\in\mathcal{T}_i}[\mathbf{1}_b] = \sum_{b\in\mathcal{T}_i}\sum_{a\in\mathcal{T}_{i-1}} \hat{M}_1(b,a)[\mathbf{1}_a].$$

So, using Theorem 2.6, this yields

$$\mathbb{1} = \sum_{b\in\mathcal{T}_i}\sum_{a\in\mathcal{T}_{i-1}} \hat{M}_1(b,a)[\mathbf{1}_a] = \sum_{a\in\mathcal{T}_{i-1}}\sum_{b\in\mathcal{T}_i} \hat{M}_1(b,a)[\mathbf{1}_a] = \sum_{a\in\mathcal{T}_{i-1}} q^2[\mathbf{1}_a] = q^2\mathbb{1}.$$

We conclude that $(q^2-1)\mathbb{1} = 0$, so $\operatorname{ord}\mathbb{1}\,|\,(q^2-1)$. □

Theorem 2.19. *The order of the identity in the K_0-groups is divisible by $(q-1)$ for $q \not\equiv 1 \bmod 3$ and by $\frac{1}{3}(q-1)$ for $q \equiv 1 \bmod 3$.*

Proof. Because of the isomorphism between $C(\partial\mathcal{B}) \rtimes \Gamma$ and $\mathcal{O}_{\hat{M}}$, the order of $\mathbb{1} = \sum_{b\in\mathcal{T}_i}[\mathbf{1}_b]$ in $K_0(C(\partial\mathcal{B}) \rtimes \Gamma)$ is equal to the order of $\sum_{b\in\mathcal{T}_i}[\mathbf{1}_b]$ when interpreted as an element of $\operatorname{coker}(I-\hat{M}_1, I-\hat{M}_2)$. By Theorem 2.6, the only relations between generators of $\operatorname{coker}(I-\hat{M}_1, I-\hat{M}_2)$ are the ones we also found in Lemma 2.16, which expressed a generator $[\mathbf{1}_b]$ in precisely q^2 generators. This implies that we can define a homomorphism

https://doi.org/10.1017/S0013091514000194 Published online by Cambridge University Press

$\varphi\colon \operatorname{coker}(I-\hat{M}_1, I-\hat{M}_2) \to \mathbb{Z}/(q^2-1)$ by sending every generator to 1. We also know that the sum $\sum_{b\in\mathcal{T}_i}[\mathbf{1}_b]$ has exactly $(q+1)(q^2+q+1)$ terms. Thus

$$\varphi\biggl(\sum_{b\in\mathcal{T}_i}[\mathbf{1}_b]\biggr) \equiv (q+1)(q^2+q+1) \equiv 3(q+1) \mod (q^2-1)$$

and $\operatorname{ord}(3(q+1)) = \operatorname{ord}(\varphi(\mathbb{1}))|\operatorname{ord}\mathbb{1}$. For the order of $3(q+1)$ in $\mathbb{Z}/(q^2-1)$, we find

$$\begin{aligned}\frac{q^2-1}{\gcd(q^2-1,3(q+1))} &= \frac{q^2-1}{(q+1)\gcd(q-1,3)}\\ &= \frac{q-1}{\gcd(q-1,3)}\\ &= \begin{cases} q-1 & \text{if } q \not\equiv 1 \bmod 3\\ \frac{1}{3}(q-1) & \text{if } q\equiv 1 \bmod 3.\end{cases}\end{aligned}$$

We conclude that $\operatorname{ord}\mathbb{1}$ is a multiple of $(q-1)$ for $q\not\equiv 1 \bmod 3$, and a multiple of $\frac{1}{3}(q-1)$ for $q\equiv 1 \bmod 3$. □

3. The semi-basic subset

Theorem 2.18 gives an upper bound on the order of $\mathbb{1}$ that is quadratic in q. The goal of this section is to give a bound linear in q. At the end we will also combine both upper bounds in a special case.

Definition 3.1. A subset $\mathcal{R}_i \subset \mathcal{T}_i$, for some fixed $i\in\{0,1,2\}$, is called a semi-basic subset if $\mathcal{R}_i$ consists of q^2+q+1 triples (b_i, b_{i+1}, b_{i-1}) such that all b_i and all b_{i+1} occur exactly once.

For our purpose, the existence of such a semi-basic subset will be vital.

Lemma 3.2. *Giving a semi-basic subset $\mathcal{R}_i$ is equivalent to giving a partitioning of G_{i+1} into incident point-line pairs.*

Proof. Because $\mathcal{T}$ is a polygonal presentation, there is a basic bijection λ that sends b_i to $w_i := \lambda(b_i)$ in the link G_{i+1}. By definition of a polygonal presentation, we know that w_i and b_{i+1} are incident in G_{i+1}. Because $|W_{i+1}| = |B_{i+1}| = q^2+q+1$, the q^2+q+1 pairs (w_i, b_{i+1}) form a partitioning of G_{i+1} into incident point-line pairs if and only if $\{w_i\} = W_{i+1}$ and $\{b_{i+1}\} = B_{i+1}$. □

Theorem 3.3. *Every incidence graph of a finite projective plane allows a partitioning into incident point-line pairs.*

Proof. We fix an incidence graph G of a finite projective plane of order q. Let P denote the set of points, let L denote the set of lines and let $V = P\cup L$ denote the set of vertices of G.

Let m denote the maximal possible number of disjoint incident point-line pairs. We choose such a maximal collection $C = \{(p_i, l_i)\colon 1\leqslant i\leqslant m\}$ of disjoint incident point-line

https://doi.org/10.1017/S0013091514000194 Published online by Cambridge University Press

pairs. We call a point or a line *married* if it is an element of a pair in C, and specifically it is married to the other element of that pair. We call a vertex of G *lonely* if it is not married. Furthermore, we define a pairing $\langle\cdot,\cdot\rangle\colon P\times L\to\{0,1,2\}$ by

$$\langle p,l\rangle=\begin{cases}2 & \text{if } p \text{ and } l \text{ are married,}\\ 1 & \text{if } p \text{ and } l \text{ are only incident,}\\ 0 & \text{otherwise.}\end{cases}\tag{3.1}$$

If $m=q^2+q+1$ we are done, so suppose that $m<q^2+q+1$ and choose a lonely point p_0 and a lonely line l_0. The set

$$X=\{p\colon \langle p,l_0\rangle=1\}$$

of points incident to l_0 is of cardinality $|X|=q+1$ by the properties of a finite projective plane. If there were a lonely point p in X, then we could add the incident point-line pair (p,l_0) to C thus contradicting the maximality of C. Hence, all points in X are married and the cardinality of the set $X'=\{l\colon \exists p_1 \text{ such that } \langle p_1,l\rangle=2,\ \langle p_1,l_0\rangle=1\}$ of lines married to a point in X is given by $|X'|=|X|=q+1$. We take it one step further with the set

$$Y=\{p\colon \exists l_1,p_1 \text{ such that } \langle p,l_1\rangle=1,\ \langle p_1,l_1\rangle=2,\ \langle p_1,l_0\rangle=1\}$$

of points incident to lines of X', with the exception of those that are married to lines of X'. Suppose that Y contains a lonely point p. We could then replace the pair $(p_1,l_1)\in C$ as in the definition of Y by the pairs (p,l_1) and (p_1,l_0), hence contradicting the maximality of C. We conclude that all points in Y are married. As a multiset, the cardinality of Y is given by $q\cdot|X'|=q(q+1)$: for every line in X' we get $(q+1)-1=q$ points that are incident to it but not married to it. However, because G is a generalized 3-gon, the cardinality of Y as a set is less. To estimate $|Y|$, we randomly order the lines of X' and we count sequentially how many points each line adds to Y. When we count the points on the jth line, we already counted all the points on $j-1$ lines, so maybe $j-1$ points on the jth line are already counted. But at least $q-(j-1)$ of the points are not yet counted. Because X' consists of $q+1$ lines, summing over all lines in X' gives the estimate

$$|Y|\geqslant q+(q-1)+\cdots+1+0=\tfrac{1}{2}q(q+1).$$

Now we approach from p_0 by defining

$$Y'=\{l\colon \exists p_2,l_2 \text{ such that } \langle p_0,l_2\rangle=1,\ \langle p_2,l_2\rangle=2,\ \langle p_2,l\rangle=1\}.$$

By analogous arguments to those we used for Y, all lines in Y' are married and $|Y'|\geqslant\frac{1}{2}q(q+1)$. Finally, we also consider the set

$$Z=\{p\colon \exists l_3,p_2,l_2 \text{ such that } \langle p_0,l_2\rangle=1,\ \langle p_2,l_2\rangle=2,\ \langle p_2,l_3\rangle=1,\ \langle p,l_3\rangle=2\}$$

of points married to lines in Y' and we remark that $|Z|=|Y'|$.

We note that $X \cap Y = \emptyset$, for the existence of an element in this intersection would imply a 4-cycle in G, while G is a generalized 3-gon. By the inclusion–exclusion principle, we get

$$|X \cup Y| = |X| + |Y| - |X \cap Y| = |X| + |Y| \geqslant (q+1) + \tfrac{1}{2}q(q+1).$$

We compute that

$$|X \cup Y| + |Z| \geqslant (q+1) + \tfrac{1}{2}q(q+1) + \tfrac{1}{2}q(q+1) = q^2 + 2q + 1 > q^2 + q + 1 = |P| \geqslant |X \cup Y \cup Z|.$$

Again by the inclusion–exclusion principle, we find that the intersection of $X \cup Y$ and Z is non-empty. Suppose that there is an element p_3 in the intersection $X \cap Z$. By the definitions of X and Z, there exist l_3, p_2 and l_2 such that $\langle p_0, l_2 \rangle = 1$, $\langle p_2, l_2 \rangle = 2$, $\langle p_2, l_3 \rangle = 1$, $\langle p_3, l_3 \rangle = 2$ and $\langle p_3, l_0 \rangle = 1$. Now we may replace the married pairs (p_2, l_2) and (p_3, l_3) in C by the newly-weds (p_0, l_2), (p_2, l_3) and (p_3, l_0). This contradicts the maximality of C. Suppose on the other hand that there is an element p_3 in the intersection $Y \cap Z$. Then there exist l_3, p_2, l_2, p_1, and l_1 such that $\langle p_0, l_1 \rangle = 1$, $\langle p_1, l_1 \rangle = 2$, $\langle p_2, l_1 \rangle = 1$, $\langle p_2, l_2 \rangle = 2$, $\langle p_2, l_3 \rangle = 1$, $\langle p_3, l_3 \rangle = 2$ and $\langle p_3, l_0 \rangle = 1$. Now we replace the married pairs (p_1, l_1), (p_2, l_2) and (p_3, l_3) in C by the newly-weds (p_0, l_1), (p_1, l_2), (p_2, l_3) and (p_3, l_0), again contradicting the maximality of C.

We conclude that the condition that C is a maximal collection of distinct incident point-line pairs is not unifiable with $|C| < q^2 + q + 1$. Hence, the maximal possible number of distinct incident point-line pairs is $q^2 + q + 1$ and a collection of $q^2 + q + 1$ of these pairs gives the desired partitioning. $\square$

Combining Lemma 3.2 and Theorem 3.3, we have proven that every polygonal presentation admits a semi-basic subset. Fix a semi-basic subset $\mathcal{R}_i$ of $\mathcal{T}$. By cyclically reordering $\mathcal{T}_0$, $\mathcal{T}_1$ and $\mathcal{T}_2$, we may assume without loss of generality that $i = 1$. We also define $\mathcal{R}_0 = \{(b_0, b_1, b_2) \colon (b_1, b_2, b_0) \in \mathcal{R}_1\}$.

Theorem 3.4. *The multiset* $\hat{M}_1(\mathcal{R}_1) := \{a \colon \hat{M}_1(b, a) = 1,\ b \in \mathcal{R}_1\}$ *consists of* $q - 1$ *copies of every element of* $\mathcal{T}_0$ *and one extra copy of* $\mathcal{R}_0$. *The multiset* $\hat{M}_2(\mathcal{R}_0) := \{a \colon \hat{M}_2(c, a) = 1, c \in \mathcal{R}_0\}$ *consists of* $q - 1$ *copies of every element of* $\mathcal{T}_1$ *and one extra copy of* $\mathcal{R}_1$.

Proof. For every $a \in \mathcal{T}_0$, we will count how often a occurs in the image of $\hat{M}_1$. Notice that a choice of a fixes a_0, a_1 and a_2. (See figure 4 for the structure of $\hat{M}_1$.)

We have $(q+1) - 1 = q$ options for $b_1 \neq a_1$ such that b_1 and a_2 are incident in the link of O. Because $\mathcal{R}_1$ is a semi-basic subset, b_1 fixes $b \in \mathcal{R}_1$. So we have q options for b such that $\hat{M}_1(b, a) = 1$. Now we consider two cases.

- If $a = (a_0, a_1, a_2) \in \mathcal{R}_0$, then $(b_1, a_2, c_0) \notin \mathcal{R}_1$ for every couple of b_1 and c_0, since $b_1 \neq a_1$ and (a_1, a_2, a_0) is the only triple of $\mathcal{R}_1$ having a_2 as an element. So a is in the image of all the b such that $b_1 \neq a_1$ and such that b_1 and a_2 are incident in the link of O. Hence a occurs precisely q times in $\hat{M}_1(\mathcal{R}_1)$.

- If $a \in \mathcal{T}_0 - \mathcal{R}_0$, then there exists a couple b_1 and c_0 such that $(b_1, a_2, c_0) \in \mathcal{R}_1$, because the semi-basic subset $\mathcal{R}_1$ must contain a triple that contains a_2. Hence, a occurs strictly less than q times in $\hat{M}_1(\mathcal{R}_1)$.

https://doi.org/10.1017/S0013091514000194 Published online by Cambridge University Press

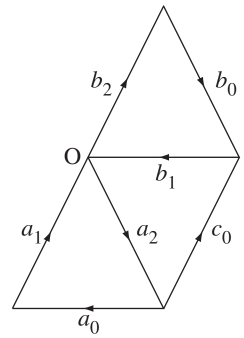

Figure 4. Showing $\hat{M}_1((b_1, b_2, b_0), (a_0, a_1, a_2)) = 1$.

By definition, we find that $|\mathcal{R}_i| = q^2 + q + 1$. We also know that $\hat{M}_1$ has q^2 entries that are equal to 1 per column and per row by Theorem 2.6. Hence the number of elements of $\hat{M}_1(\mathcal{R}_i)$ as a multiset is $|\hat{M}_1(\mathcal{R}_i)| = q^2(q^2 + q + 1)$. We also easily count that $|\mathcal{T}_0| = (q + 1)(q^2 + q + 1)$. Combining these observations, we find the equality $|\hat{M}_1(\mathcal{R}_1)| = q|\mathcal{R}_0| + (q - 1)(|\mathcal{T}_0| - |\mathcal{R}_0|)$. This implies that $a \in \mathcal{T}_0 - \mathcal{R}_0$ has to occur precisely $q - 1$ times.

We conclude that the multiset $\hat{M}_1(\mathcal{R}_1) := \{a\colon \hat{M}_1(b, a) = 1,\ b \in \mathcal{R}_1\}$ consists of q copies of $\mathcal{R}_0$ and $q-1$ copies of $\mathcal{T}_0 - \mathcal{R}_0$, which is equivalent to the first statement of the theorem. The multiset $\hat{M}_2(\mathcal{R}_0)$ may be handled in the same way. □

We improve the upper bound for the order of identity, which was quadratic in q, to one that is linear in q.

Theorem 3.5. *The identity* $\mathbb{1}$ *in the* K_0*-group satisfies* $2(q - 1)\mathbb{1} = 0$.

Proof. We recall that, for $b \in \mathcal{T}_i$, we have $[\mathbf{1}_b] = \sum_{a\in\mathcal{T}_{i-1}} \hat{M}_1(b, a)[\mathbf{1}_a]$. By the first statement of the above theorem, we find that

$$\sum_{b\in\mathcal{R}_1}[\mathbf{1}_b] = \sum_{b\in\mathcal{R}_1}\sum_{a\in\mathcal{T}_0}\hat{M}_1(b,a)[\mathbf{1}_a] = (q-1)\sum_{a\in\mathcal{T}_0}[\mathbf{1}_a] + \sum_{a\in\mathcal{R}_0}[\mathbf{1}_a] = (q-1)\mathbb{1} + \sum_{a\in\mathcal{R}_0}[\mathbf{1}_a]. \quad (3.2)$$

On the other hand, for $a \in \mathcal{T}_{i-1}$, we also have $[\mathbf{1}_a] = \sum_{b\in\mathcal{T}_i} \hat{M}_2(a, b)[\mathbf{1}_b]$, in particular for $i = 1$. The second statement of the above theorem gives a description of how $\hat{M}_2$ acts on $\mathcal{R}_0$. We use this on equation (3.2), and we find that

$$\sum_{b\in\mathcal{R}_1}[\mathbf{1}_b] = (q-1)\mathbb{1} + \sum_{a\in\mathcal{R}_0}\sum_{b\in\mathcal{T}_1}\hat{M}_2(a,b)[\mathbf{1}_b] = 2(q-1)\mathbb{1} + \sum_{b\in\mathcal{R}_1}[\mathbf{1}_b]. \quad (3.3)$$

We conclude that $2(q - 1)\mathbb{1} = 0$. □

In one particular family of values for q, the above theorems fit together very nicely. In this case, they precisely fix the order of the identity in the K_0-groups.

Theorem 3.6. *If* $q = 2^{2l-1}$, $l \in \mathbb{Z}$, *then the order of* $\mathbb{1}$ *is* $q - 1$.

https://doi.org/10.1017/S0013091514000194 Published online by Cambridge University Press

Proof. Because $q+1 \equiv 1 \mod 2$, Theorem 2.18 and Theorem 3.5 together imply that $(q-1)\mathbb{1} = 0$, and hence $\operatorname{ord} \mathbb{1} | q-1$. On the other hand, we know that $q \not\equiv 1 \mod 3$, so Theorem 2.19 implies that $q-1 | \operatorname{ord} \mathbb{1}$. $\square$

References

1. C. Anantharaman-Delaroche, C^*-algèbres de Cuntz–Krieger et groupes Fuchsiens, in *Proc. 16th Conf. Operator Theory: Operator Algebras and Related Topics, Timişoara, Romania, July 2–10, 1996*, pp. 17–35 (Theta Foundation, Bucharest, 1997).
2. K. Brown, *Buildings* (Springer, 1989).
3. L. Brown, P. Green and M. Rieffel, Stable isomorphism and strong Morita equivalence of C^*-algebras, *Pac. J. Math.* **71**(2) (1977), 349–363.
4. D. Cartwright, A. Mantero, T. Steger and A. Zappa, Groups acting simply transitively on the vertices of a building of type $\tilde{A}_2$, I, *Geom. Dedicata* **47**(2) (1993), 143–166.
5. D. Cartwright, A. Mantero, T. Steger and A. Zappa, Groups acting simply transitively on the vertices of a building of type $\tilde{A}_2$, II: the cases $q = 2$ and $q = 3$, *Geom. Dedicata* **47**(2) (1993), 167–223.
6. D. Cartwright, W. Młotkowski and T. Steger, Property (T) and $\tilde{A}_2$ groups, *Annales Inst. Fourier* **44**(1) (1994), 213–248.
7. A. Connes, Cyclic cohomology and the transverse fundamental class of a foliation, in *Geometric Methods in Operator Algebras, Kyoto, 1983*, Pitman Research Notes in Mathematics Series, Volume 123, pp. 52–144 (Longman, 1986).
8. G. Cornelissen, O. Lorscheid and M. Marcolli, On the K-theory of graph C^*-algebras, *Acta Appl. Math.* **102**(1) (2008), 57–69.
9. H. Moriyoshi and T. Natsume, The Godbillon–Vey cocycle and longitudinal Dirac operators, *Pac. J. Math.* **172** (1996), 483–539.
10. T. Natsume, Euler characteristic and the class of unit in K-theory, *Math. Z.* **194** (1987), 237–243.
11. N. C. Phillips, A classification theorem for nuclear purely infinite simple C^*-algebras, *Documenta Math.* **5** (2000), 49–114.
12. G. Robertson and T. Steger, Affine buildings, tiling systems and higher rank Cuntz–Krieger algebras, *J. Reine Angew. Math.* **513** (1999), 115–144.
13. G. Robertson and T. Steger, Asymptotic K-theory for groups acting on $\tilde{A}_2$ building, *Can. J. Math.* **53**(4) (2001), 809–833.
14. G. Robertson and T. Steger, Irreducible subshifts associated with $\tilde{A}_2$ buildings, *J. Combin. Theory* A **103** (2003), 91–104.
15. A. Vdovina, Combinatoral structure of some hyperbolic buildings, *Math. Z.* **241**(3) (2002) 471–478.
16. A. Vdovina, Polyhedra with specified links, in *Séminaire de Théorie spectrale et géométrie, Grenoble, 2002–2003*, Volume 21, pp. 37–42.