

Methods of Hyperbolic Partial Differential Equations

The structures of a causal fermion system are all encoded in the family of physical wave functions (see Section 5.7). Consequently, the dynamics of the causal fermion is understood once we know how each physical wave function propagates in time. In many examples, the physical wave functions satisfy the *Dirac equation* (for the simplest example of this type, see Section 5.5). More abstractly, the dynamics of the physical wave functions is described by the *dynamical wave equation* (9.49). Moreover, we also encountered the *linearized field equations* (see Definition 8.1.2). We now turn attention to methods for solving these equations. In this chapter, we begin with linear partial differential equations like the Dirac equation. Causality is reflected in these equations in the fact that they are *hyperbolic*. As we shall see, the methods developed here will also be fruitful for the study of the linearized field equations, as will be explained in Chapter 14. With this in mind, the constructions here can be regarded as a technical preparation for Chapter 14. We remark that the adaptation of the methods to the dynamical wave equation will not be covered in this book; we refer the interested reader instead to [82].

13.1 The Cauchy Problem and Linear Symmetric Hyperbolic Systems

In this section, we shall prove that the Cauchy problem for the Dirac equation in the presence of an external potential has a unique global solution. Moreover, we will show that the finite speed of propagation as postulated by special relativity is indeed respected by the solutions of the Dirac equation. For later purposes, it is preferable to include an inhomogeneity. Thus, we consider the Cauchy problem in Minkowski space

$$(i\partial\!\!\!/ + \mathcal{B} - m)\psi = \phi \in C^\infty(\mathcal{M}, S\mathcal{M}), \quad \psi|_{t_0} = \psi_0 \in C^\infty(\mathbb{R}^3, S\mathcal{M}), \quad (13.1)$$

for a given inhomogeneity ϕ and initial data ψ_0 . In order to make the standard methods available, we multiply the equation by $-i\gamma^0$,

$$\mathbb{1}_{\mathbb{C}^4} \partial_t \psi + \gamma^0 \vec{\gamma} \vec{\nabla} \psi - i\gamma^0 (\mathcal{B} - m)\psi = -i\gamma^0 \phi. \quad (13.2)$$

Now the matrices in front of the derivatives are all Hermitian (with respect to the standard scalar product on $\mathbb{C}^4$). Moreover, the matrix in front of the time derivative is positive definite. Friedrichs [91] observed that these properties are precisely what is needed in order to get a well-posed Cauchy problem. He combined these properties in the notion of a *symmetric hyperbolic system*. We now give its

general definition. More specifically, we consider a system of N complex-valued equations with spatial coordinates $\vec{x} \in \mathbb{R}^m$ and time t in an interval $[0, T]$ with $T > 0$. The initial data will always be prescribed at time $t = 0$. For notational clarity, we denote partial derivatives in spatial directions by ∇ .

Definition 13.1.1 *A linear system of differential equations of the form*

$$A^0(t, \vec{x}) \partial_t u(t, \vec{x}) + \sum_{\alpha=1}^m A^\alpha(t, \vec{x}) \nabla_\alpha u(t, \vec{x}) + B(t, \vec{x}) u(t, \vec{x}) = w(t, \vec{x}), \quad (13.3)$$

with

$$A^0, A^\alpha, B \in C^\infty([0, T] \times \mathbb{R}^m, L(\mathbb{C}^N)), \quad w \in C^\infty([0, T] \times \mathbb{R}^m, \mathbb{C}^N), \quad (13.4)$$

is called **symmetric hyperbolic** if

- (i) *The matrices A^0 and A^α are Hermitian,*

$$(A^0)^\dagger = A^0 \quad \text{and} \quad (A^\alpha)^\dagger = A^\alpha, \quad (13.5)$$

where $\dagger$ denotes the adjoint with respect to the canonical scalar product on $\mathbb{C}^N$.

- (ii) *The matrix A^0 is uniformly positive definite, that is, there is a positive constant C such that*

$$A^0(t, \vec{x}) > C \mathbf{1}_{\mathbb{C}^N} \quad \text{for all } (t, \vec{x}) \in ([0, T] \times \mathbb{R}^m). \quad (13.6)$$

In the case $w \equiv 0$, the linear system is called **homogeneous**.

A good reference for linear symmetric hyperbolic systems is the book by John [107, Section 5.3] (who was Friedrichs' colleague at the Courant Institute). Our presentation was also influenced by [133, Chapter 8]. We remark that the concept of a symmetric hyperbolic system can be extended to nonlinear equations of the form

$$A^0(t, \vec{x}, u) \partial_t u(t, \vec{x}) + \sum_{\alpha=1}^m A^\alpha(t, \vec{x}, u) \nabla_\alpha u(t, \vec{x}) + B(t, \vec{x}, u) u = 0, \quad (13.7)$$

where the matrices A^0 and A^α should again satisfy the abovementioned conditions (i) and (ii). For details, we refer to [144, Section 16] or [135, Section 7]. Having the Dirac equation in mind, we always restrict attention to linear systems. We also note that an alternative method for proving the existence of fundamental solutions is to work with the so-called Riesz distributions (for a good textbook, see [6]). Yet another method is to work with estimates in the interaction picture [25]. For completeness, we finally note that the concept of symmetric hyperbolic systems was extended by Friedrichs to so-called symmetric *positive* systems [92].

It is a remarkable fact that all partial differential equations in relativistic physics as well as most wave-type equations can be rewritten as a symmetric hyperbolic system. As an illustration, we now explain this reformulation in the example of a scalar hyperbolic equation.

Example 13.1.2 Consider a scalar hyperbolic equation of the form

$$\begin{aligned} \partial_{tt}\phi(t, \vec{x}) &= \sum_{\alpha, \beta=1}^m a_{\alpha\beta}(t, \vec{x}) \nabla_{\alpha\beta}\phi \\ &+ \sum_{\alpha=1}^m b_{\alpha}(t, \vec{x}) \nabla_{\alpha}\phi + c(t, \vec{x}) \partial_t\phi + d(t, \vec{x}) \phi, \end{aligned} \quad (13.8)$$

with $(a_{\alpha\beta})$ a symmetric, uniformly positive matrix (in the case $a_{\alpha\beta} = \delta_{\alpha\beta}$ and $b, c, d = 0$, one gets the scalar wave equation). Now the initial data prescribes ϕ and its first time derivatives,

$$\phi|_{t=0} = \phi_0 \in C^\infty(M), \quad \partial_t\phi|_{t=0} = \phi_1 \in C^\infty(M). \quad (13.9)$$

In order to rewrite the equation as a symmetric hyperbolic system, we introduce the vector u with $k := m + 2$ components by

$$u_1 = \nabla_1\phi, \dots, u_m = \nabla_m\phi, \quad u_{m+1} = \partial_t\phi, \quad u_{m+2} = \phi. \quad (13.10)$$

Then, the system

$$\left\{ \begin{array}{l} \sum_{\beta=1}^m a_{\alpha\beta} \partial_t u_\beta - \sum_{\beta=1}^m a_{\alpha\beta} \nabla_\beta u_{m+1} = 0 \\ - \sum_{\alpha, \beta=1}^m a_{\alpha\beta} \nabla_\beta u_\alpha - \sum_{\alpha=1}^m b_\alpha u_\alpha + \partial_t u_{m+1} - c u_{m+1} - d u_{m+2} = 0 \\ 0 \qquad \qquad \qquad -u_{m+1} \qquad \qquad + \partial_t u_{m+2} = 0, \end{array} \right. \quad (13.11)$$

is symmetric hyperbolic (as one verifies by direct inspection). Also, a short calculation shows that if ϕ is a smooth solution of the scalar equation (13.8), then the corresponding vector u is a solution of the system (13.11). Conversely, assume that u is a smooth solution of (13.11) that satisfies the initial condition $u|_{t=0} = u_0$, where u_0 is determined by ϕ_0 and ϕ_1 via (13.10). Setting $\phi = u_{m+2}$, the last line in (13.11) shows that $u_{m+1} = \partial_t\phi$. Moreover, the first line in (13.11) implies that $\partial_t u_\beta = \nabla_\beta u_{m+1} = \partial_t \nabla_\beta \phi$. Integrating over t and using that the relation $u_\beta = \nabla_\beta \phi$ holds initially, we conclude that this relation holds for all times. Finally, the second line in (13.11) yields that ϕ satisfies the scalar hyperbolic equation (13.8). In this sense, the Cauchy problem for the system (13.11) is equivalent to that for the scalar equation (13.9). $\diamond$

This procedure works similarly for other physical equations like the Klein-Gordon or Maxwell equations. Exercise 13.1 is concerned with the example of the homogeneous Maxwell equations.

13.2 Finite Propagation Speed and Uniqueness of Solutions

For what follows, it is convenient to combine the time and spatial coordinates to a spacetime vector $x = (t, \vec{x}) \in [0, T] \times \mathbb{R}^m$. We denote the spacetime dimension by $n = m + 1$. Moreover, setting $\partial_0 \equiv \partial_t$, we use Latin spacetime indices $i \in \{0, \dots, m\}$ and employ the Einstein summation convention. Then, our linear system (13.3) can be written in the compact form

$$A^j(x) \partial_j u(x) + B(x) u(x) = w(x). \quad (13.12)$$

Next, a direction in spacetime can be described by a vector $\xi = (\xi_i)_{i=0,\dots,m} \in \mathbb{R}^{m+1}$. Contracting with the matrices $A^j(x)$, we obtain the Hermitian $N \times N$ -matrix

$$A(x, \xi) := A^j(x) \xi_j, \quad (13.13)$$

referred to as the *characteristic matrix*. Note that in the example of the Dirac equation (13.2), the index i is a vector index in Minkowski space, and ξ should be regarded as a co-vector (i.e., a vector in the cotangent bundle). One should keep in mind that, despite the suggestive notation, the equation (13.12) should not be considered as being manifestly covariant because it corresponds to the Hamiltonian formulation (13.2), where a time direction is distinguished.

The determinant of the characteristic matrix is referred to as the *characteristic polynomial*, being a polynomial in the components ξ_i . For our purposes, it is most helpful to consider whether the characteristic matrix is positive or negative definite. If the vector $\xi = (\tau, \vec{0})$ points in the time direction, then $A(x, \xi) = \tau A^0$, which in view of Definition 13.1.1 is a definite matrix. By continuity, $A(x, \xi)$ is definite if the spatial components of ξ are sufficiently small. In the example of the Dirac equation (13.2), the fact that

$$A(x, \xi) = \mathbb{1}\xi_0 + \gamma^0 \vec{\gamma} \vec{\xi} \quad \text{has eigenvalues} \quad \xi^0 \pm |\vec{\xi}|, \quad (13.14)$$

shows that $A(x, \xi)$ is definite if and only if ξ is a timelike vector. Moreover, it is positive definite if and only if ξ is future-directed and timelike. This suggests that the causal properties of the equation are encoded in the positivity of the characteristic matrix. We simply use this connection to *define* the causal structure for a general symmetric hyperbolic system.

Definition 13.2.1 *The vector $\xi \in \mathbb{R}^{m+1}$ is called **timelike** at the spacetime point x if the characteristic matrix $A(x, \xi)$ is definite. A timelike vector is called **future-directed** if $A(x, \xi)$ is positive definite. If the characteristic polynomial vanishes, then the vector ξ is called **lightlike**. A hypersurface $\mathcal{H} \subset [0, T] \times \mathbb{R}^m$ with normal ν is called **spacelike** if the matrix $A(x, \nu)$ is positive definite for all $x \in \mathcal{H}$.*

The notion of a *normal* used here requires an explanation. The simplest method is to represent the hypersurface locally as the zero set of a function $\phi(x)$. Then, the normal can be defined as the gradient of ϕ . In this way, the gradient is a co-vector, so that the contraction $A^j \nu_j = A^j \partial_j \phi$ is well defined without referring to a scalar product. In particular, the last definition is independent of the choice of a scalar product on spacetime vectors in $\mathbb{R}^n$. We always choose the normal to be future-directed, and we normalize it with respect to the Euclidean scalar product on $\mathbb{R}^{m+1}$, but these are merely conventions.

We shall now explain why and in which sense the solutions of symmetric hyperbolic systems comply with this notion of causality.

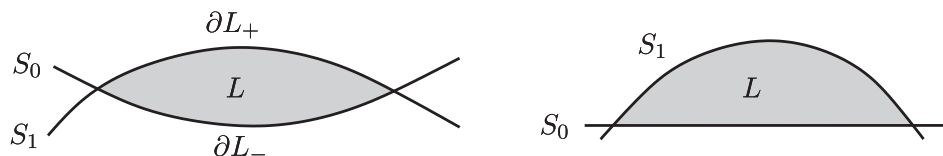


Figure 13.1 Lens-shaped regions.

Definition 13.2.2 Let u be a smooth solution of the linear symmetric hyperbolic system (13.12). A subset K of the initial value surface $\{t = 0\}$ determines the solution at a spacetime point $x \in [0, T] \times \mathbb{R}^m$ if every smooth solution of the system that coincides on K with u also coincides with u at x . The **domain of determination** of K is the set of all spacetime points at which the solution is determined by the initial data on K .

Definition 13.2.3 An open subset $L \subset (0, T) \times \mathbb{R}^m$ is called a **lens-shaped region** if L is relatively compact in $\mathbb{R}^n$ and if its boundary ∂L is contained in the union of two smooth hypersurfaces S_0 and S_1 whose intersection with $\bar{L}$ is spacelike. We set $(\partial L)_+ = \partial L \cap S_1$ and $(\partial L)_- = \partial L \cap S_0$, where we adopt the convention that $(\partial L)_+$ lies to the future of $(\partial L)_-$.

Figure 13.1 shows typical examples of lens-shaped regions. Often, one chooses the initial data surface as $S_0 = \{t = 0\}$. Moreover, it is often convenient to write the hypersurface S_1 as a graph $S_1 = \{(t, \vec{x}) \mid t = f(\vec{x})\}$. In this case, S_1 is the zero set of the function $\phi(t, \vec{x}) = t - f(\vec{x})$, and the normal ν is the gradient of this function, that is,

$$(\nu_j)_{j=0, \dots, m} = (1, \nabla_1 f, \dots, \nabla_m f). \quad (13.15)$$

We first consider the homogeneous equation

$$(A^j \partial_j + B)u = 0. \quad (13.16)$$

The idea for analyzing the domain of determination is to multiply this equation by a suitable test function and to integrate over a lens-shaped region. More precisely, we consider the equation

$$0 = \int_L e^{-Kt} 2\operatorname{Re}\langle u, (A^j \partial_j + B)u \rangle d^n x, \quad (13.17)$$

where $\langle \cdot, \cdot \rangle$ denotes the canonical scalar product on $\mathbb{C}^N$, and $K > 0$ a positive parameter to be determined later. Since the A^j are Hermitian, we have

$$\partial_j \langle u, A^j u \rangle = 2\operatorname{Re}\langle u, A^j \partial_j u \rangle + \langle u, (\partial_j A^j)u \rangle, \quad (13.18)$$

and using this identity in (13.17) gives

$$0 = \int_L e^{-Kt} \left(\partial_j \langle u, A^j u \rangle + \langle u, (B + B^* - (\partial_j A^j))u \rangle \right) d^n x. \quad (13.19)$$

In the first term, we integrate by parts with the Gauss divergence theorem,

$$\begin{aligned} \int_L e^{-Kt} \partial_j \langle u, A^j u \rangle \, d^n x &= K \int_L e^{-Kt} \langle u, A^0 u \rangle \, d^n x \\ &+ \int_{(\partial L)_+} e^{-Kt} \langle u, \nu_j A^j u \rangle \, d\mu_{\partial L_+} - \int_{(\partial L)_-} e^{-Kt} \langle u, \nu_j A^j u \rangle \, d\mu_{\partial L_-} . \end{aligned} \quad (13.20)$$

We now use (13.20) in (13.19) and solve for the surface integral over $(\partial L)_+$,

$$\begin{aligned} \int_{(\partial L)_+} e^{-Kt} \langle u, \nu_j A^j u \rangle \, d\mu_{\partial L_+} &= \int_{(\partial L)_-} e^{-Kt} \langle u, \nu_j A^j u \rangle \, d\mu_{\partial L_-} \\ &+ \int_L e^{-Kt} \langle u, (-K - B - B^* + (\partial_j A^j)) u \rangle \, d^n x . \end{aligned} \quad (13.21)$$

This identity is the basis for the following uniqueness results.

Theorem 13.2.4 *Let u_1 and u_2 be two smooth solutions of the linear symmetric hyperbolic system (13.3) that coincide on the past boundary of a lens-shaped region L ,*

$$u_1|_{(\partial L)_-} = u_2|_{(\partial L)_-} . \quad (13.22)$$

Then, u_1 and u_2 coincide in the whole set L .

Proof The function $u := u_1 - u_2$ is a solution of the homogeneous system (13.16) with $u|_{(\partial L)_-} = 0$. Hence, (13.21) simplifies to

$$\int_{(\partial L)_+} e^{-Kt} \langle u, \nu_j A^j u \rangle \, d\mu_{\partial L_+} = \int_L e^{-Kt} \langle u, (-K - B - B^* + \partial_j A^j) u \rangle \, d^n x . \quad (13.23)$$

Assume that u does not vanish identically in L . By choosing K sufficiently large, we can then arrange that the right-hand side becomes negative. However, since ∂L_+ is a spacelike hypersurface, the left-hand side is nonnegative. This is a contradiction. $\square$

As an immediate consequence, we obtain the following uniqueness result for solutions of the Cauchy problem.

Corollary 13.2.5 *Let u_1 and u_2 be two smooth solutions of the linear symmetric hyperbolic system (13.3) with the same initial at time $t = 0$. Then, $u_1 \equiv u_2$ in a neighborhood of the initial data surface.*

If the matrices A^j are uniformly bounded and A^0 is uniformly positive, then $u_1 \equiv u_2$ in $[0, T] \times \mathbb{R}^m$.

Proof The local uniqueness result follows immediately by covering the initial data surface by lens-shaped regions (see the left of Figure 13.2).

For the global uniqueness, for any $x_0 = (t_0, \vec{x}_0) \in [0, T] \times \mathbb{R}^m$ our task is to choose a lens-shaped region that contains x_0 and whose past boundary S_0 is contained in the surface $\{t = 0\}$. We need to rule out the situation shown on the right of Figure 13.2 that the hypersurface S_1 does not intersect S_0 , in which case we would not get a relatively compact lens-shaped region. To this end,

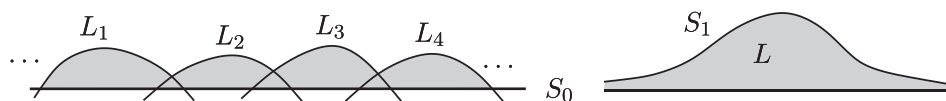


Figure 13.2 Coverings by lens-shaped regions.

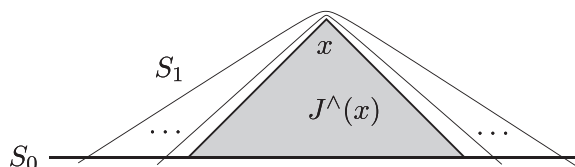


Figure 13.3 Approximating the light cone by lens-shaped regions.

we must use that the matrices A^j are uniformly bounded by assumption. As a consequence, there is $\varepsilon > 0$ such that the inequality $\|\nabla f\| \leq \varepsilon$ implies that the hypersurface $S_1 = \{(t = f(\vec{x}), \vec{x})\}$ is spacelike. Possibly after decreasing ε , we may choose

$$f(\vec{x}) = t_0 + \varepsilon \left(1 - \sqrt{1 + \|\vec{x} - \vec{x}_0\|^2}\right). \quad (13.24)$$

This concludes the proof. $\square$

By a suitable choice of lens-shaped region, one can get an upper bound for the maximal propagation speed. For the Dirac equation, where the causal structure of Definition 13.2.1 coincides in view of (13.14) with that of Minkowski space, one can choose for S_1 a family of spacelike hypersurfaces, which converge to the boundary of a light cone (see Figure 13.3). This shows that the maximal propagation speed for Dirac waves is indeed the speed of light (which, according to our conventions, is equal to one).

13.3 Global Existence of Smooth Solutions

In this section, we will show that, by refining the abovementioned uniqueness argument, we even obtain an existence proof. The close connection between existence and uniqueness for linear equations is a familiar theme in mathematics. The simplest setting where it occurs is in the study of the linear equation $Au = v$ with a given vector $v \in \mathbb{R}^n$ and a quadratic matrix A . In this case, the uniqueness of the solution implies that the matrix A is invertible, which in turn ensures existence. A more interesting example is Fredholm's alternative for compact operators (see, e.g., [131, Section VI.5]). The procedure for globally hyperbolic systems follows somewhat similar ideas. Here, the general strategy is to construct a bounded linear functional on a Hilbert space in such a way that the Fréchet–Riesz theorem (see Theorem 2.2.4) gives the desired solution.

Before beginning, we point out that, in view of uniqueness and finite propagation speed, it suffices to consider the problem in a bounded spatial region. Indeed, once we have constructed “local solutions” in small lens-shaped regions as shown on the left of Figure 13.2, uniqueness implies that these solutions agree in the overlap of

the lens-shaped regions, making it possible to “glue them together” to obtain the desired solution which is global in space. We will come back to this construction in more detail in the context of the Dirac equation in Sections 13.4 and 13.6 (see also Figure 13.4). Having this construction in mind, we may start from a local problem and to extend the coefficients of the symmetric hyperbolic system in an arbitrary way outside. Therefore, it is no loss of generality to consider a problem in the whole space $\mathbb{R}^m$. Choosing a bounded time interval $t \in [0, T]$ (where $t = 0$ is the initial time), we are led to considering the time strip

$$R_T := [0, T] \times \mathbb{R}^m. \quad (13.25)$$

We now write the linear system (13.12) as

$$Lu = w \quad \text{with} \quad L := A^j \partial_j + B, \quad (13.26)$$

where we again sum over $j = 0, \dots, m$. Again using that the system can be extended arbitrarily outside a bounded spatial region, we may assume that the functions A^j , B and w are uniformly bounded in R_T and that w has spatially compact support (meaning that $w(t, \cdot) \in C_0^\infty(\mathbb{R}^m)$ for all $t \in [0, T]$). Moreover, for convenience, we again assume smoothness of A^j , B and w . In the *Cauchy problem*, one seeks a solution to the equation (13.26) with prescribed initial data $u_0 \in C^\infty(\mathbb{R}^m)$ at time $t = 0$,

$$Lu = w, \quad u|_{t=0} = u_0 \in C_0^\infty(\mathbb{R}^m), \quad (13.27)$$

in $C^s(R_T)$. First of all, we may restrict attention to the case $u_0 \equiv 0$,

$$Lu = w, \quad u|_{t=0} \equiv 0. \quad (13.28)$$

In order to see this, let u be a solution to the abovementioned Cauchy problem. Choosing a function $v \in C^\infty(R_T)$ which at $t = 0$ coincides with u_0 . Then the function $\tilde{u} := (u - v)$ satisfies the equation $L\tilde{u} = \tilde{w}$ with $\tilde{w} = w + A^j \partial_j v + Bv$ and vanishes at $t = 0$. If conversely $\tilde{u}$ is a solution to the corresponding Cauchy problem with zero initial data, then $u := \tilde{u} + v$ is a solution of the original problem (13.27).

In preparation of the existence proof, we need to introduce the notion of a *weak solution*. In order to get into the weak formulation, we multiply the equation (13.26) by a test function $v(t, \vec{x})$ and integrate over R_T , giving rise to the equation

$$\langle v, Lu \rangle_{L^2(R_T)} = \langle v, w \rangle_{L^2(R_T)}, \quad (13.29)$$

with the L^2 -scalar product defined by

$$\langle v, w \rangle_{L^2(R_T)} := \int_0^T dt \int_{\mathbb{R}^m} \langle v(t, \vec{x}), w(t, \vec{x}) \rangle \, d^m x. \quad (13.30)$$

The next step is to integrate by parts, so that the derivatives act on the test function v . Before doing so, we need to specify the regularity of the test functions. To this end, for $\lambda \in [0, T]$ we consider the time strip

$$R_\lambda := [0, \lambda] \times \mathbb{R}^m. \quad (13.31)$$

We denote the s -times continuously differential functions on R_λ with spatially compact support by $C^s(R_\lambda)$. The function spaces

$$\underline{C^s(R_\lambda)} \quad \text{and} \quad \overline{C^s(R_\lambda)}, \quad (13.32)$$

are defined as the functions which in addition vanish at $t = 0$ and $t = \lambda$, respectively. As the space of test functions, we choose $\overline{C^1(R_T)}$; this guarantees that integrating by parts does not yield boundary terms at $t = T$. For a classical solution $u \in \underline{C^1(R_T)}$ (i.e., a solution with zero Cauchy data (13.28)), the boundary term at $t = 0$ also vanishes. We thus obtain

$$\langle v, w \rangle_{L^2(R_T)} = \langle \tilde{L}v, u \rangle_{L^2(R_T)} \quad \text{for all } v \in \overline{C^1(R_T)}, \quad (13.33)$$

where $\tilde{L}$ is the formal adjoint of L with respect to the scalar product (13.30), that is,

$$\tilde{L} := \tilde{A}^j \partial_j + \tilde{B} \quad \text{with} \quad \tilde{A}^j = -A^j \quad \text{and} \quad \tilde{B} = B^\dagger - (\partial_j A^j). \quad (13.34)$$

Now suppose that a function $u \in C^1(R_T)$ satisfies (13.33). Testing with functions $v \in \overline{C^1(R_T)} \cap C^1(R_T)$ which vanish both at times $t = 0$ and $t = T$, we can integrate by parts without boundary terms. Using a standard denseness argument, one finds that u solves the symmetric hyperbolic system (13.26). Next, testing with a function $v \in \overline{C^1(R_T)}$ which does *not* vanish at $t = 0$, only the boundary term remains, giving the equation

$$\int_{\mathbb{R}^m} \langle v(0, \vec{x}), u(0, \vec{x}) \rangle \, d^m x = 0 \quad \text{for all } v \in \overline{C^1(R_T)}, \quad (13.35)$$

which in turn implies that u vanishes initially. Thus u is a solution of the Cauchy problem (13.28). To summarize, for functions $u \in \overline{C^1(R_T)}$, the weak formulation (13.33) is equivalent to our Cauchy problem (13.26) and (13.28). Therefore, it is sensible to take (13.33) as the definition of a weak solution of the Cauchy problem. The main advantage of the weak formulation (13.33) is that it is well defined even for functions that are not differentiable.

Our next step is to derive so-called *energy estimates* for a given solution $u \in C^1(R_T)$. To this end, we return to the formula for the divergence (13.18) and using the equation (13.26), we obtain

$$\partial_j \langle u, A^j u \rangle + \langle u, Cu \rangle = 2 \operatorname{Re} \langle u, w \rangle, \quad (13.36)$$

$$C := B + B^* - (\partial_j A^j). \quad (13.37)$$

Next, we integrate (13.36) over R_λ , integrate by parts and use that the initial values at $t = 0$ vanish. We thus obtain

$$\begin{aligned} E(\lambda) &:= \int_{t=\lambda} \langle u, A^0 u \rangle \, d^m x \\ &= \int_0^\lambda dt \int_{\mathbb{R}^m} \left(2 \operatorname{Re} \langle u, w \rangle - \langle u, Cu \rangle \right) \, d^m x. \end{aligned} \quad (13.38)$$

Since the matrix C is uniformly bounded and A_0 is uniformly positive, there is a constant $K > 1$ such that

$$|\langle u, Cu \rangle| \leq K \langle u, A^0 u \rangle. \quad (13.39)$$

Moreover, the linear term in u can be estimated with the Schwarz inequality by

$$2 \operatorname{Re} \langle u, w \rangle \leq \mu \langle u, u \rangle + \frac{1}{\mu} \langle w, w \rangle \leq \langle u, A^0 u \rangle + \frac{1}{\mu^2} \langle w, A^0 w \rangle, \quad (13.40)$$

with a suitable constant $\mu > 0$. Applying these estimates in (13.38) gives

$$E(\lambda) \leq (K+1) \int_0^\lambda E(t) \, dt + \frac{1}{\mu^2} \int_{R_\lambda} \langle w, A^0 w \rangle \, d^n x. \quad (13.41)$$

Writing this inequality as

$$\frac{d}{d\lambda} e^{-(K+1)\lambda} \int_0^\lambda E(t) \, dt \leq e^{-(K+1)\lambda} \frac{1}{\mu^2} \int_{R_\lambda} \langle w, A^0 w \rangle \, d^n x, \quad (13.42)$$

we can integrate over λ to obtain

$$\int_0^T E(\lambda) \, d\lambda \leq \frac{e^{(K+1)T} - 1}{K+1} \frac{1}{\mu^2} \int_{R_T} \langle w, A^0 w \rangle \, d^n x. \quad (13.43)$$

Finally, we apply the mean value theorem and use that the exponential function is monotone to conclude that

$$\int_0^T E(\lambda) \, d\lambda \leq \frac{T}{\mu^2} e^{(K+1)T} \int_{R_T} \langle w, A^0 w \rangle \, d^n x. \quad (13.44)$$

This is the desired energy estimate.

Before going on, we point out that the notion of “energy” used for the quantity $E(\lambda)$ does in general *not* coincide with the physical energy. In fact, for the Dirac equation (13.2), $E(\lambda)$ has the interpretation as the electric charge. Following Example 13.1.2, for the scalar wave equation $\square\phi = 0$, we find

$$E(\lambda) = \int_{\mathbb{R}^m} \left(|\partial_t \phi|^2 + |\nabla \phi|^2 + |\phi|^2 \right) d^m x. \quad (13.45)$$

This differs from the physical energy by the last summand $|\phi|^2$ (and an overall factor of two). The name “energy” for $E(\lambda)$ was motivated by the fact, considering only the highest derivative terms, the expression (13.45) is indeed the physical energy. We point out that, in contrast to the physical energy, the quantity $E(\lambda)$ does in general depend on time. The point is that (13.44) gives an a priori control of the energy in terms of the inhomogeneity. The exponential factor in (13.44) can be understood in analogy to a Grönwall estimate (for the classical Grönwall estimate see, e.g., [1, Lemma 1.15 in Section VII.1]).

For the following construction, it is convenient to introduce on $C^1(R_T)$ the scalar product

$$(u, v) = \int_{R_T} \langle u, A^0 v \rangle \, d^n x. \quad (13.46)$$

We denote the corresponding norm by $\|\cdot\|$. Setting furthermore

$$\Gamma^2 = \frac{T}{\mu^2} e^{(K+1)T}, \quad (13.47)$$

the energy estimate can be written in the compact form

$$(u, u) \leq \Gamma^2 (w, w). \quad (13.48)$$

This inequality holds for every solution u of the differential equation $Lu = w$ which vanishes at $t = 0$. Noting that every function $u \in \overline{C^1(R_T)}$ is a solution of this differential equation with inhomogeneity $w := Lu$, we obtain

$$\|u\| \leq \Gamma \|Lu\| \quad \text{for all } u \in \overline{C^1(R_T)}. \quad (13.49)$$

This is the form of the energy estimates suitable for an abstract existence proof.

Note that the operator $\tilde{L}$ in (13.34) is also symmetric hyperbolic and has the same boundedness and positivity properties as L . Hence, repeating the abovementioned arguments, we obtain similar to (13.49) the “dual estimate”

$$\|v\| \leq \tilde{\Gamma} \|\tilde{L}v\| \quad \text{for all } v \in \overline{C^1(R_T)}. \quad (13.50)$$

We now want to show the existence of weak solutions with the help of the Fréchet–Riesz theorem (see Theorem 2.2.4 in the preliminaries or, e.g., [131, 116]). To this end, we first introduce on $\overline{C^1(R_T)}$ yet another scalar product denoted by

$$\langle v, v' \rangle = (\tilde{L}v, \tilde{L}v'). \quad (13.51)$$

This scalar product is indeed positive definite, because for any $v \neq 0$,

$$\langle v, v \rangle = (\tilde{L}v, \tilde{L}v) \geq \tilde{\Gamma}^{-2} (v, v) \neq 0, \quad (13.52)$$

where in the last step we applied (13.50). Forming the completion, we obtain the Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$. We denote the corresponding norm by $\|\cdot\|$. In view of (13.50) and (13.51), we know that every vector $v \in \mathcal{H}$ is a function in $L^2(R_T, d^n x)$. Moreover, we know from (13.51) that $\tilde{L}v$ is also in $L^2(R_T, d^n x)$. We remark that, in the language of functional analysis, the space $\mathcal{H}$ can be identified with the Sobolev space $W^{1,2}(R_T)$, but we do not need this here.

We now consider for $w \in C^0(R_T)$ and $v \in \overline{C^1(R_T)}$ the linear functional $\langle v, w \rangle_{L^2(R_T)}$. In view of the estimate

$$|\langle v, w \rangle_{L^2(R_T)}| \leq \|v\|_{L^2(R_T)} \|w\|_{L^2(R_T)} \leq \frac{\tilde{\Gamma}}{C} \|w\|_{L^2(R_T)} \|v\|, \quad (13.53)$$

this functional is continuous in $v \in \mathcal{H}$. The Fréchet–Riesz theorem shows that there is $U \in \mathcal{H}$ with

$$\langle v, w \rangle_{L^2(R_T)} = \langle v, U \rangle = (\tilde{L}v, \tilde{L}U) \quad \text{for all } v \in \mathcal{H}. \quad (13.54)$$

Rewriting the last scalar product as

$$(\tilde{L}v, \tilde{L}U) = \langle Lv, A^0 \tilde{L}U \rangle_{L^2(R_T)}, \quad (13.55)$$

one sees that the function $u := A^0 \tilde{L}U \in L^2(R_T, \mathrm{d}^n x)$ satisfies the equation (13.33) and is thus the desired weak solution. Note that all our methods apply for arbitrarily large T . We have thus proved the global existence of weak solutions.

We next want to show that the solutions are smooth. Thus our task is to show that our constructed weak solution u is of the class $C^s(R_\lambda)$, where $s \geq 1$ can be chosen arbitrarily large. We first show that a linear symmetric hyperbolic system can be “enlarged” to include the partial derivatives of ϕ .

Lemma 13.3.1 *Suppose that the system $A^j \partial_j u + Bu = w$ is symmetric hyperbolic. Then there is a symmetric hyperbolic system of the form*

$$\tilde{A}^j \partial_j \Psi + \tilde{B} \Psi = \tilde{w}, \quad (13.56)$$

for the vector $\Psi := (\partial_t u, \nabla_1 u, \dots, \nabla_m u, u) \in \mathbb{C}^{(n+1)N}$.

Proof Let i be a fixed spacetime index. We differentiate the equation $Lu = w$,

$$\partial_i w = \partial_i Lu = L \partial_i u + (\partial_i A^j) \partial_j u + (\partial_i B) u. \quad (13.57)$$

This equation can be written as

$$A^j \partial_j \Psi_i + \sum_{j=1}^n \tilde{B}_i^j \Psi_j + (\partial_i B) u = \tilde{w}_i, \quad (13.58)$$

where we set

$$\tilde{B}_i^j = B \delta_i^j + (\partial_i A^j) \quad \text{and} \quad \tilde{w}_i = \partial_i w. \quad (13.59)$$

Combining these equations with the equation $Lu = w$, we obtain a system of the form (13.56), where the matrices $\tilde{A}^j$ are block diagonal in the sense that

$$\tilde{A}^j = ((\tilde{A}^j)_\beta^\alpha)_{\alpha, \beta=0, \dots, m+1} \quad \text{with} \quad (\tilde{A}^j)_\beta^\alpha = A^j \delta_\beta^\alpha. \quad (13.60)$$

Obviously, this system is again symmetric hyperbolic. $\square$

Iterating this lemma, we obtain (at least in principle) a symmetric hyperbolic system for u and all its partial derivatives up to any given order s . Since the corresponding weak solution is in $L^2(R_T)$, we conclude that u and all its weak partial derivatives are square integrable. The next lemma, which is a special case of the general Sobolev embedding theorems (see, e.g., [32, Section II.5.] or [143, Section 4]), gives smoothness of the solution.

Lemma 13.3.2 *Let $s > \frac{m}{2}$ be an integer. If a function g on $\mathbb{R}^m$ is s times weakly differentiable and*

$$\int_{\mathbb{R}^m} |\nabla^\alpha g|^2 \, \mathrm{d}^m x < C \quad (13.61)$$

for all multi-indices α with $|\alpha| \leq s$, then g is bounded, $g \in L^\infty(\mathbb{R}^m)$. Likewise, if g is $s + l + 1$ times weakly differentiable with $l \geq 1$ and (13.61) holds for all α with $|\alpha| \leq s + l + 1$, then $g \in C^l(\mathbb{R}^m)$.

Proof We apply the Schwarz inequality to the Fourier transform,

$$\begin{aligned} |g(x)|^2 &= \left| \int_{\mathbb{R}^m} \frac{d^m k}{(2\pi)^m} \hat{g}(k) e^{-ikx} \right|^2 \\ &= \left| \int_{\mathbb{R}^m} \frac{d^m k}{(2\pi)^m} (1 + |k|^2)^{-\frac{s}{2}} (1 + |k|^2)^{\frac{s}{2}} \hat{g}(k) e^{-ikx} \right|^2 \\ &\leq c_m \int_{\mathbb{R}^m} \frac{d^m k}{(2\pi)^m} (1 + |k|^2)^s |\hat{g}(k)|^2, \end{aligned} \quad (13.62)$$

where the constant c_m is finite due to our choice of s ,

$$c_m = \int_{\mathbb{R}^m} \frac{d^m k}{(2\pi)^m} (1 + |k|^2)^{-s} < \infty. \quad (13.63)$$

Using the Plancherel formula together with the fact that a factor k^2 corresponds to a Laplacian in position space, we obtain

$$\int_{\mathbb{R}^m} \frac{d^m k}{(2\pi)^m} (1 + |k|^2)^s |\hat{g}(k)|^2 = \sum_{\ell=0}^s \binom{n}{\ell} \|\nabla^\ell g\|_{L^2(\mathbb{R}^m)}^2 < c. \quad (13.64)$$

Hence $\sqrt{c_m c}$ is an L^∞ -bound for g .

Next, if g is $s + 1$ times weakly differentiable, then $\|Dg\|_{L^\infty(\mathbb{R}^m)} < c$. As a consequence, the mean value theorem yields $|g(x) - g(y)| \leq c|x - y|$, so that g is Lipschitz continuous. Finally, if g is $s + l + 1$ times weakly differentiable, then all partial derivatives $\nabla^\alpha g$ of order $|\alpha| \leq l$ are Lipschitz continuous, so that $g \in C^l(\mathbb{R}^m)$. $\square$

More precisely, in order to apply this lemma, we fix a time t and consider the solution $u(\lambda, \cdot)$. The identity (13.38) implies that $E(\lambda)$ is controlled in terms of $\|w\|$ and $\|u\|$. After iteratively applying Lemma 13.3.1, we conclude that the weak derivatives of $u(\lambda, \cdot)$ exist to any order and are in $L^2(\mathbb{R}^m)$. It follows that $u(\lambda, \cdot)$ is smooth. Finally, one uses the equation to conclude that u is also smooth in the time variable.

The results of this section can be summarized as follows.

Theorem 13.3.3 *Consider the Cauchy problem*

$$\begin{aligned} (A^0 \partial_t + \sum_{\alpha=1}^m A^\alpha \nabla_\alpha + B)u &= w \in C_0^\infty([0, T] \times \mathbb{R}^m), \\ u|_{t=0} &= u_0 \in C_0^\infty(\mathbb{R}^m). \end{aligned} \quad (13.65)$$

Assume that the matrices A^0 , A^j and B as well as the functions w and u_0 are smooth. Moreover, assume that all these functions as well as all their partial derivatives are uniformly bounded (where the bound may depend on the order of the derivatives). Then the Cauchy problem has a smooth solution on $[0, T] \times \mathbb{R}^m$.

This theorem also applies in the case $T = \infty$, giving the global existence of a smooth solution.

We finally show that the solutions depend smoothly on parameters.

Corollary 13.3.4 *Suppose that the matrices A^j, B and the functions w, u_0 depend smoothly on a parameter λ . Then the family of solutions $u(\lambda)$ is also smooth in λ .*

Proof First, as explained after (13.28), we may restrict attention to the case $u_0 = 0$. Differentiating the equation $Lu = w$ with respect to λ , we obtain

$$Lu_\lambda = (\partial_\lambda L)u + \partial_\lambda w =: \tilde{w}, \quad (13.66)$$

where u_λ stands for the formal derivative $\partial_\lambda u$. This is a symmetric hyperbolic system for u_λ . According to Theorem 13.3.3, we know that u and therefore $\tilde{w}$ are smooth. Hence, applying this theorem again, we conclude that there exists a smooth solution u_λ . Considering the limit of the difference quotients, one verifies that u_λ really coincides with $\partial_\lambda u(\lambda)$ for our given family of solutions $u(\lambda)$. The higher λ -derivatives can be treated inductively. $\square$

13.4 The Causal Dirac Green's Operators in Minkowski Space

We now want to apply the previous general existence and uniqueness results to the Cauchy problem (13.1) for the Dirac equation in Minkowski space in the presence of an external potential $\mathcal{B}$.

Theorem 13.4.1 *Consider the Cauchy problem for the Dirac equation (13.1) for smooth initial data ψ_0 , a smooth inhomogeneity ϕ and a smooth matrix-valued potential $\mathcal{B} \in C^\infty(\mathcal{M}, \mathbb{C}^{4 \times 4})$. Then there is a unique global smooth solution $\psi \in C^\infty(\mathcal{M}, S\mathcal{M})$.*

Proof Writing the Dirac equation in the Hamiltonian form (13.2), we obtain a symmetric hyperbolic system. In view of the uniqueness result for smooth solutions of Corollary 13.2.5, it suffices to construct a smooth solution at any given time $T \in \mathbb{R}$. It suffices to consider the case $T > t_0$, because otherwise we reverse the time direction. Moreover, we can arrange by a time shift that $t_0 = 0$.

We cannot apply Theorem 13.3.3 directly because the coefficient functions in (13.2) do not need to be bounded, nor are our initial values compactly supported. For this reason, we need to construct local solutions and “glue them together” using linearity: We first extend the initial data ψ_0 smoothly to the time strip R_T and consider the Cauchy problem for $\tilde{\psi} := \psi - \psi_0$:

$$(i\partial\!\!\!/ + \mathcal{B} - m)\tilde{\psi} = \tilde{\phi} \in C^\infty(\mathcal{M}, S\mathcal{M}), \quad \tilde{\psi}|_{t_0} = 0. \quad (13.67)$$

We let $(\eta_k)_{k \in \mathbb{N}}$ be a smooth partition of unity of $\mathbb{R}^m$ with $\eta_k \in C_0^\infty(\mathbb{R}^m)$ (for details, see, e.g., [136, Theorem 2.13]). We extend these functions to static functions on R_T (i.e., $\eta_k(t, \vec{x}) := \eta_k(\vec{x})$). Given $k \in \mathbb{N}$, we first solve the Cauchy problem for the inhomogeneity $\eta_k \tilde{\phi}$. We choose a compact set $K \subset \mathbb{R}^m$ such that $[0, T] \times K$ contains the causal future of the support of $(\eta_k \tilde{\phi})$ (see Figure 13.4; more specifically, we could choose $K = B_{2T}(\text{supp } \eta_k)$). Next, we choose a smooth, compactly

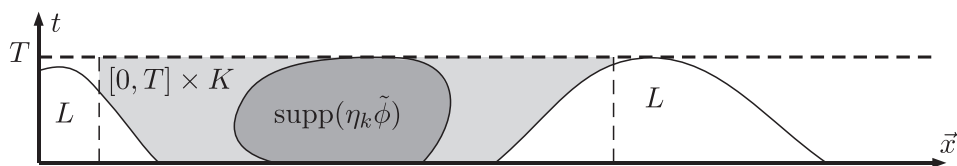


Figure 13.4 Construction of local solutions.

supported function $\theta \in C_0^\infty(\mathbb{R}^m)$ with $\theta|_K \equiv 1$. We again extend θ to a static function on R_T .

We now consider the modified Cauchy problem

$$\left(\mathbb{1}_{\mathbb{C}^4} \partial_t \psi + \gamma^0 \vec{\gamma} \vec{\nabla} + \theta(-i\gamma^0(\mathcal{B} - m)) \right) \tilde{\psi}_k = -i\gamma^0 \eta_k \tilde{\phi}, \quad \tilde{\psi}_k|_{t_0} = 0. \quad (13.68)$$

Now the coefficients in the PDE are uniformly bounded, and the inhomogeneity has compact support. Therefore, we can apply Theorem 13.3.3 to obtain a global smooth solution. Due to finite propagation speed (see Theorem 13.2.4, where we choose lens-shaped regions L as shown in Figure 13.4), this solution vanishes outside K . Therefore, it is also a solution of the unmodified Dirac equation, with initial data $\eta_k \tilde{\phi}$.

Finally, summing over k gives the desired solution of the original Cauchy problem,

$$\psi := \sum_{k=1}^{\infty} \tilde{\psi}_k. \quad (13.69)$$

Here the series converges because, again due to finite propagation speed, it is locally finite. $\square$

We next explain how the previous existence and uniqueness results give rise to the existence of causal Green's operators, being defined as integral operators with distributional kernels. These kernels are often referred to as *Green's functions*. Our main tool is the *Schwartz kernel theorem*. We do not give a proof of this more advanced result of distribution theory but refer instead to [105, Section 5.2] or [143, Section 4.6]. For better consistency with the notation in the perturbative treatment in Chapter 18, from now on we denote the objects in the presence of an external potential with an additional tilde. We begin with a representation formula for the solution of the Cauchy problem in terms of a distribution.

Theorem 13.4.2 *Assume that the external potential $\mathcal{B}$ is smooth and that $\mathcal{B}$ and all its partial derivatives are uniformly bounded in Minkowski space. Then for any t, t_0 there is a unique distribution $\tilde{k}_m(t, \cdot; t_0, \cdot) \in \mathcal{D}'(\mathbb{R}^3 \times \mathbb{R}^3)$ such that the solution of the Cauchy problem (17.1) has the representation*

$$\psi(t, \vec{x}) = 2\pi \int_N \tilde{k}_m(t, \vec{x}; t_0, \vec{y}) \gamma^0 \psi_0(\vec{y}) \, d^3y. \quad (13.70)$$

The integral kernel k_m is also a distribution in spacetime, $k_m \in \mathcal{D}'(M \times M)$ It is a distributional solution of the Dirac equation,

$$(i\partial_x + \mathcal{B} - m) \tilde{k}_m(x, y) = 0. \quad (13.71)$$

Proof Combining the energy estimates with the Sobolev embedding of Lemma 13.3.2 shows that there is $k \in \mathbb{N}$ and a constant $C = C(t, t_0, \vec{x}, \mathcal{B})$ such that the solution $\psi(t, \cdot)$ of the Cauchy problem is bounded in terms of the initial data by

$$|\psi(t, \vec{x})| \leq C |\psi_0|_{C^k}, \quad (13.72)$$

where $|\psi|^2 := \langle \psi | \gamma^0 \psi \rangle$, and the C^k -norm is defined by

$$|\psi_0|_{C^k} = \max_{|\beta| \leq k} \sup_{\vec{x} \in \mathbb{R}^3} |\nabla^\beta \psi_0(\vec{x})|. \quad (13.73)$$

Moreover, this estimate is locally uniform in $\vec{x}$, meaning that for any compact set $K \subset \mathbb{R}^3$, there is a constant C such that (13.72) holds for all $\vec{x} \in K$. This makes it possible to apply the Schwartz kernel theorem [105, Theorem 5.2.1], showing that $\tilde{k}_m(t, \cdot; t_0, \cdot) \in \mathcal{D}'(\mathbb{R}^3 \times \mathbb{R}^3)$.

Next, we note that the constant C in (13.72) can also be chosen locally uniformly in t and t_0 . Thus, after evaluating weakly in t and t_0 , we may again apply the Schwartz kernel theorem to obtain that $\tilde{k}_m \in \mathcal{D}'(M \times M)$. Finally, the distributional equation (13.71) follows immediately from the fact that (13.70) satisfies the Dirac equation for any choice of ψ_0 . $\square$

The distribution $\tilde{k}_m$ is referred to as the *causal fundamental solution*. Encoding the whole Dirac dynamics, it plays a fundamental role in the analysis of the Dirac equation. In the next step, we introduce the *causal Green's operators* by decomposing $\tilde{k}_m$ in time. Namely, for any t, t_0 we introduce the distribution $\tilde{s}_m^\vee(t, \cdot; t_0, \cdot), \tilde{s}_m^\wedge(t, \cdot; t_0, \cdot) \in \mathcal{D}'(\mathbb{R}^3 \times \mathbb{R}^3)$ by

$$\begin{cases} \tilde{s}_m^\vee(t, \cdot; t_0, \cdot) = 2\pi i \tilde{k}_m(t, \cdot; t_0, \cdot) \Theta(t_0 - t) \\ \tilde{s}_m^\wedge(t, \cdot; t_0, \cdot) = -2\pi i \tilde{k}_m(t, \cdot; t_0, \cdot) \Theta(t - t_0) \end{cases} \quad (13.74)$$

(where Θ denotes the Heaviside function). In this way, we introduce the causal fundamental solutions for any given t_0 and t as distributions on $\mathbb{R}^3 \times \mathbb{R}^3$. Alternatively, they can also be introduced as bi-distributions in spacetime, as is shown in the next lemma.

Theorem 13.4.3 *Assume that the external potential $\mathcal{B}$ is smooth and that $\mathcal{B}$ and all its partial derivatives are uniformly bounded in Minkowski space. Then there are unique distributions*

$$\tilde{s}_m^\vee, \tilde{s}_m^\wedge \in \mathcal{D}'(\mathcal{M} \times \mathcal{M}) \quad (13.75)$$

which satisfy the distributional equations

$$(i\partial_x + \mathcal{B} - m) \tilde{s}_m(x, y) = \delta^4(x - y) \quad (13.76)$$

and are supported in the upper respectively lower light cone,

$$\text{supp } \tilde{s}_m^\vee(x, \cdot) \subset J_x^\vee, \quad \text{supp } \tilde{s}_m^\wedge(x, \cdot) \subset J_x^\wedge. \quad (13.77)$$

Proof It is clear by construction and the fact that the constant C in (13.72) can be chosen locally uniformly in x and y that the causal Green's operators are well-defined distributions in $\mathcal{D}'(M \times M)$. The support property (13.77) follows

immediately from finite propagation speed as explained at the end of Section 13.2. The uniqueness of the causal Green's operators is clear from the uniqueness of solutions of the Cauchy problem. In order to derive the distributional equations (13.76), we only consider the retarded Green's operator (the argument for the advanced Green's operator is analogous). Then, according to (13.70) and (13.74),

$$\Theta(t - t_0) \psi(t, \vec{x}) = i \int_N \tilde{s}_m^\wedge(t, \vec{x}; t_0, \vec{y}) \gamma^0 \psi_0(\vec{y}) d^3 y, \quad (13.78)$$

where ψ is the solution of the corresponding Cauchy problem. Applying the Dirac operator in the distributional sense yields

$$i\gamma^0 \delta(t - t_0) \psi_0(t, \vec{x}) = i(\mathcal{D}_x - m) \int_N \tilde{s}_m^\wedge(t, \vec{x}; t_0, \vec{y}) \gamma^0 \psi_0(\vec{y}) d^3 y. \quad (13.79)$$

We now choose the initial values as the restriction of a test function in spacetime, $\psi_0 = \phi|_{t=t_0}$ with $\phi \in C_0^\infty(M, SM)$. Then we can integrate over t_0 to obtain

$$i\gamma^0 \phi(x) = (\mathcal{D}_x - m) \int_M \tilde{s}_m^\wedge(x, y) i\gamma^0 \phi(y) d^4 y. \quad (13.80)$$

This gives the result. $\square$

We remark that, turning the abovementioned argument around, we can also use the causal Green's operators in order to define the causal fundamental solution as a bi-distribution in spacetime,

$$\tilde{k}_m := \frac{1}{2\pi i} (\tilde{s}_m^\vee - \tilde{s}_m^\wedge) \in \mathcal{D}'(\mathcal{M} \times \mathcal{M}). \quad (13.81)$$

The causal fundamental solution has the remarkable property that it relates the scalar product with the inner product obtained by integrating the spin inner product over spacetime. We now explain this relation step by step. Given two wave functions ψ and ϕ (not necessarily solutions of the Dirac equation), we want to integrate their pointwise inner product $\prec \psi | \phi \succ_x$ over spacetime (as already done in the preliminaries in (1.42) and (4.58)). In order to ensure that this integral is well defined, it suffices to assume that one of the functions is compactly supported. We thus obtain the sesquilinear pairing

$$\begin{aligned} \langle . | . \rangle : C^\infty(\mathcal{M}, SM) \times C_0^\infty(\mathcal{M}, SM) &\rightarrow \mathbb{C}, \\ \langle \psi | \phi \rangle &= \int_{\mathcal{M}} \prec \psi | \phi \succ_x d\mu_{\mathcal{M}} \end{aligned} \quad (13.82)$$

(here $C^\infty(\mathcal{M}, SM)$ are again the smooth sections of the spinor bundle, and $C_0^\infty(\mathcal{M}, SM)$ denotes the smooth sections with compact support). Restricting the first argument to compactly supported wave functions, we obtain an inner product,

$$\langle . | . \rangle : C_0^\infty(\mathcal{M}, SM) \times C_0^\infty(\mathcal{M}, SM) \rightarrow \mathbb{C}, \quad (13.83)$$

referred to as the *spacetime inner product* (we remark that this inner product space can be extended to a Krein space; we refer the interested reader to [45, §1.1.5]). Alternatively, one can also restrict the first argument of $\langle . | . \rangle$ to smooth Dirac

solutions and extend by approximation to the whole Hilbert space $\mathcal{H}_m$, giving the sesquilinear pairing

$$\langle \cdot | \cdot \rangle : \mathcal{H}_m \times C_0^\infty(\mathcal{M}, S\mathcal{M}) \rightarrow \mathbb{C}. \quad (13.84)$$

The following proposition goes back to John Dimock (see [27, Proposition 2.2]).

Proposition 13.4.4 *For any $\psi_m \in \mathcal{H}_m$ and $\phi \in C_0^\infty(\mathcal{M}, S\mathcal{M})$,*

$$(\psi_m | \tilde{k}_m \phi)_m = \langle \psi_m | \phi \rangle. \quad (13.85)$$

Proof We first give the proof under the additional assumption that the Dirac solution has spatially compact support, $\psi_m \in C_{\text{sc}}^\infty(\mathcal{M}, S\mathcal{M})$. We choose Cauchy surfaces $\mathcal{N}_+$ and $\mathcal{N}_-$ lying in the future and past of $\text{supp } \phi$, respectively. Let Ω be the spacetime region between these two Cauchy surfaces, that is, $\partial\Omega = \mathcal{N}_+ \cup \mathcal{N}_-$. Then, according to (13.109) and using again the notation (4.57),

$$\begin{aligned} (\psi_m | \tilde{k}_m \phi)_m &= (\psi_m | \tilde{k}_m \phi)_{\mathcal{N}_+} = \frac{i}{2\pi} (\psi_m | \tilde{s}_m^\wedge \phi)_{\mathcal{N}_+} \\ &= \frac{i}{2\pi} \left[(\psi_m | \tilde{s}_m^\wedge \phi)_{\mathcal{N}_+} - (\psi_m | \tilde{s}_m^\wedge \phi)_{\mathcal{N}_-} \right] \\ &= i \int_\Omega \nabla_j \prec \psi_m | \gamma^j \tilde{s}_m^\wedge \phi \succ_x \, d\mu(x), \end{aligned} \quad (13.86)$$

where in the last line we applied the Gauss divergence theorem and used (15.2). Using that ψ_m satisfies the Dirac equation, a calculation similar to (1.35) yields

$$\begin{aligned} (\psi_m | \tilde{k}_m \phi)_m &= \int_\Omega \prec \psi_m | (\mathcal{D} - m) \tilde{s}_m^\wedge \phi \succ_x \, d\mu(x) \\ &\stackrel{(16.8)}{=} \int_\Omega \prec \psi_m | \phi \succ_x \, d\mu(x). \end{aligned} \quad (13.87)$$

As ϕ is supported in Ω , we can extend the last integration to all of $\mathcal{M}$, giving the result.

In order to extend the result to general $\psi_m \in \mathcal{H}_m$, we use the following approximation argument. Let $\psi_m^{(n)} \in \mathcal{H}_m \cap C_{\text{sc}}^\infty(\mathcal{M}, S\mathcal{M})$ be a sequence which converges in $\mathcal{H}_m$ to ψ_m . Then obviously $(\psi_m^{(n)} | \tilde{k}_m \phi)_m \rightarrow (\psi_m | \tilde{k}_m \phi)_m$. In order to show that the right-hand side of (13.85) also converges, it suffices to prove that $\psi_m^{(n)}$ converges in $L_{\text{loc}}^2(\mathcal{M}, S\mathcal{M})$ to ψ_m . Thus let $K \subset \mathcal{M}$ be a compact set contained in the domain of a chart (x, U) . Using Fubini's theorem, we obtain for any $\psi \in \mathcal{H}_m \cap C_{\text{sc}}^\infty(\mathcal{M}, S\mathcal{M})$ the estimate

$$\int_K \prec \psi | \psi \succ \, d\mu_{\mathcal{M}} = \int dx^0 \int \prec \psi | \psi \succ \sqrt{|g|} \, d^3x \leq C(K) (\psi | \psi)_m. \quad (13.88)$$

Applying this estimate to the functions $\psi = \psi_m^{(n)} - \psi_m^{(n')}$, we see that $\psi_m^{(n)}$ converges in $L^2(K, S\mathcal{M})$ to a function $\tilde{\psi}$. This implies that $\psi_m^{(n)}$ converges to $\tilde{\psi}$ pointwise almost everywhere (with respect to the measure $d\mu_{\mathcal{M}}$). Moreover, the convergence of $\psi_m^{(n)}$ in $\mathcal{H}_m$ to ψ_m implies that the restriction of $\psi_m^{(n)}$ to any Cauchy surface $\mathcal{N}$ converges to $\psi_m|_{\mathcal{N}}$ pointwise almost everywhere (with respect to the measure $d\mu_{\mathcal{N}}$). It follows that $\tilde{\psi} = \psi_m|_K$, concluding the proof. $\square$

Corollary 13.4.5 *The operator $\tilde{k}_m$, (13.109), is symmetric with respect to the inner product (13.82).*

Proof Using Proposition 13.4.4, we obtain for all $\phi, \psi \in C_0^\infty(\mathcal{M}, S\mathcal{M})$,

$$\langle \tilde{k}_m \phi | \psi \rangle = (\tilde{k}_m \phi | \tilde{k}_m \psi)_m = \langle \phi | \tilde{k}_m \psi \rangle, \quad (13.89)$$

concluding the proof. $\square$

13.5 A Polynomial Estimate in Time

We now derive an estimate which shows that the solutions of the Dirac equation increase at most polynomially in time. This result will be needed in Section 17.2.1. For the proof, we adapt standard methods of the theory of partial differential equations to the Dirac equation. In generalization of (16.22), we denote the spatial Sobolev norms by

$$\|\phi\|_{W^{a,2}}^2 = \sum_{\alpha \text{ with } |\alpha| \leq a} \int_{\mathbb{R}^3} |\nabla^\alpha \phi(\vec{x})|^2 d^3x. \quad (13.90)$$

Lemma 13.5.1 *We are given two nonnegative integers a and b as well as a smooth time-dependent potential $\mathcal{B}$. In the case $a > 0$ and $b \geq 0$, we assume furthermore that the spatial derivatives of $\mathcal{B}$ decay faster than linearly for large times in the sense that*

$$|\nabla \mathcal{B}(t)|_{C^{a-1}} \leq \frac{c}{1 + |t|^{1+\varepsilon}} \quad (13.91)$$

for suitable constants $c, \varepsilon > 0$. Then there is a constant $C = C(c, \varepsilon, a, b)$ such that every family of solutions $\psi \in \mathcal{H}^\infty$ of the Dirac equation (1.39) for varying mass parameter can be estimated for all times in terms of the boundary values at $t = 0$ by

$$\|\partial_m^b \psi_m|_t\|_{W^{a,2}} \leq C (1 + |t|^b) \sum_{p=0}^b \|\partial_m^p \psi_m|_{t=0}\|_{W^{a,2}}. \quad (13.92)$$

Proof We choose a multi-index α of length $a := |\alpha|$ and a nonnegative integer b . Differentiating the Dirac equation (1.39) with respect to the mass parameter and to the spatial variables gives

$$(i\partial\!\!\!/ + \mathcal{B} - m) \nabla^\alpha \partial_m^b \psi_m = b \nabla^\alpha \partial_m^{b-1} \psi_m - \nabla^\alpha (\mathcal{B} \partial_m^b \psi_m) + \mathcal{B} \nabla^\alpha \partial_m^b \psi_m. \quad (13.93)$$

Introducing the abbreviations

$$\begin{aligned} \Xi &:= \nabla^\alpha \partial_m^b \psi_m \quad \text{and} \\ \phi &:= b \nabla^\alpha \partial_m^{b-1} \psi_m - \nabla^\alpha (\mathcal{B} \partial_m^b \psi_m) + \mathcal{B} \nabla^\alpha \partial_m^b \psi_m, \end{aligned} \quad (13.94)$$

we rewrite this equation as the inhomogeneous Dirac equation

$$(\mathcal{D} - m) \Xi = \phi. \quad (13.95)$$

A calculation similar to current conservation yields

$$\begin{aligned} &-i\partial_j \prec \Xi | \gamma^j \Xi \succ \\ &= \prec (\mathcal{D} - m) \Xi | \Xi \succ - \prec \Xi | (\mathcal{D} - m) \Xi \succ = \prec \phi | \Xi \succ - \prec \Xi | \phi \succ. \end{aligned} \quad (13.96)$$

Integrating over the equal time hypersurfaces and using the Schwarz inequality, we obtain

$$|\partial_t(\Xi|_t|\Xi|_t)| \leq 2 \|\Xi|_t\|_t \|\phi|_t\|_t \quad (13.97)$$

and thus

$$\left| \partial_t \|\Xi|_t\| \right| \leq \|\phi|_t\|_t. \quad (13.98)$$

Substituting the specific forms of Ξ and ϕ and using the Schwarz and triangle inequalities, we obtain the estimate

$$\begin{aligned} \left| \partial_t \|\nabla^\alpha \partial_m^b \psi_m|_t\| \right| &\leq b \|\nabla^\alpha \partial_m^{b-1} \psi_m|_t\|_t \\ &+ c a |\nabla \mathcal{B}(t)|_{C^{a-1}} \|\partial_m^b \psi_m|_t\|_{W^{a-1,2}}, \end{aligned} \quad (13.99)$$

where we used the notation (17.8).

We now proceed inductively in the maximal total order $a+b$ of the derivatives. In the case $a=b=0$, the claim follows immediately from the unitarity of the time evolution. In order to prove the induction step, we note that in (13.99), the order of differentiation of the wave function on the right-hand side is smaller than that on the left-hand side at least by one. In the case $a=0$ and $b \geq 0$, the induction hypothesis yields the inequality

$$|\partial_t \|\partial_m^b \psi_m|_t\| \leq b \|\partial_m^{b-1} \psi_m|_t\| \leq b C (1 + |t|^{b-1}) \sum_{p=0}^{b-1} \|\partial_m^p \psi_m|_{t=0}\|, \quad (13.100)$$

and integrating this inequality from 0 to t gives the result. In the case $a > 0$ and $b \geq 0$, we apply (13.91) together with the induction hypothesis to obtain

$$\left| \partial_t \|\partial_m^b \psi_m|_t\|_{W^{a,2}} \right| \leq b C (1 + |t|^{b-1}) \sum_{p=0}^{b-1} \|\partial_m^p \psi_m|_{t=0}\|_{W^{a,2}} \quad (13.101)$$

$$+ c C \frac{1 + |t|^b}{1 + |t|^{1+\varepsilon}} \sum_{p=0}^b \|\partial_m^p \psi_m|_{t=0}\|_{W^{a-1,2}}. \quad (13.102)$$

Again integrating over t gives the result. $\square$

13.6 The Cauchy Problem in Globally Hyperbolic Spacetimes

We conclude this chapter by extending the global existence and uniqueness result for the Dirac equation to curved spacetime. These results were already stated in Section 4.5. We are now in the position for giving the proof. The reader not interested in or not familiar with curved spacetime may skip this section. We note that more details on the geometric properties of globally hyperbolic spacetimes can be found in [6, Section 3.2].

Proof of Theorem 4.5.1. As explained in the proof of Theorem 13.4.1, by considering the Cauchy problem for $\psi - \psi_0$ one may reduce the problem to that of zero initial data zero. Moreover, choosing a partition of unity (η_k) subordinate to the

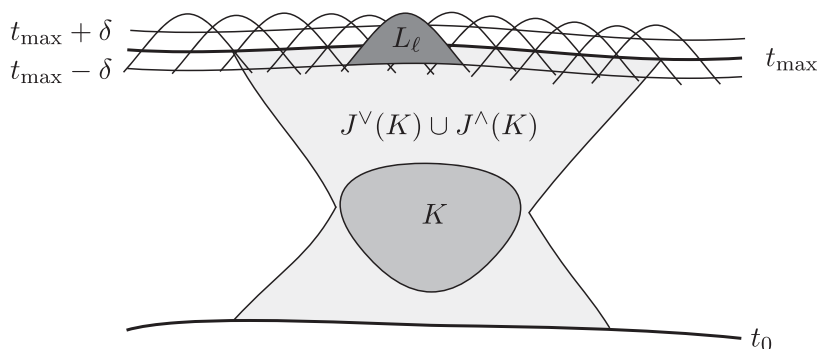


Figure 13.5 Global solutions in globally hyperbolic spacetimes.

charts of a given atlas, it suffices to consider the compactly supported inhomogeneity $\eta_k \phi$ (the sum over k is again locally finite, as explained in the proof of Theorem 13.4.1). In view of these constructions, it remains to consider the Cauchy problem

$$(\mathcal{D} - m)\psi = \phi \in C_0^\infty(\mathcal{M}, S\mathcal{M}), \quad \psi|_{\mathcal{N}_{t_0}} = 0. \quad (13.103)$$

We denote the support of ϕ by K .

Clearly, in local charts, the Dirac equation can be written as a symmetric hyperbolic system. Therefore, the results in Sections 13.2 and 13.3 yield the existence and uniqueness of solutions of the Cauchy problem in local charts. This also yields global uniqueness: Let ψ and $\tilde{\psi}$ be two smooth solutions to the Cauchy problem (13.103). Then their difference $\Xi := \tilde{\psi} - \psi$ is a homogeneous solution that vanishes at time t_0 . In view of a possible time reversal, it suffices to consider the solution in the future of t_0 . Thus let $x \in \mathcal{M}$ be in the future of t_0 . Then the past light cone $J^\wedge(x)$ intersects the future of t_0 in a compact set,

$$J^\wedge(x) \cap \left(\bigcup_{t \geq t_0} \mathcal{N}_t \right) \quad \text{is compact.} \quad (13.104)$$

Therefore, we can choose $\delta > 0$ such that for every $\hat{t}$, there is a finite number of lens-shaped regions which cover the time strip

$$J^\wedge(x) \cap \left(\bigcup_{t=\hat{t}}^{\hat{t}+\delta} \mathcal{N}_t \right). \quad (13.105)$$

On each lens-shaped regions, the solution for the Cauchy problem with zero initial data vanishes identically. Therefore, we can proceed inductively to conclude that $\Xi(x) = 0$. Since x is arbitrary, the solution Ξ vanishes identically on $\mathcal{M}$.

In order to prove global existence, we proceed indirectly. In view of a possible time reversal, it suffices to consider the Cauchy problem to the future. Thus suppose that the solution exists only up to finite time $t_{\max}$ (see Figure 13.5). Due to finite propagation speed, the solution is supported in the domain of causal dependence of K ,

$$\text{supp } \psi \subset J^\vee(K) \cup J^\wedge(K). \quad (13.106)$$

By properties of globally hyperbolic spacetimes, the intersection D of this set with the Cauchy surface $\mathcal{N}_{t_{\max}}$ is compact. Covering D by a finite number of charts, we choose δ such that the sets $J^\vee(D) \cup J^\wedge(D) \cap \mathcal{N}_t$ lie in the domain of these charts for all $t \in [t_{\max} - \delta, t_{\max} + \delta]$. Next, we choose a finite number of lens-shaped regions L_ℓ which also cover all these sets (see again Figure 13.5). In each of these lens-shaped regions, we can solve the Cauchy problem with initial data at time $t_{\max} - \delta$. In this way, we get a solution up to time $t_{\max} + \delta$. This is a contradiction, thereby proving that the solution must exist at all times. $\square$

Proof of Theorem 4.5.2. By extending the initial data ψ_0 to a smooth and compactly supported function in spacetime and considering the Cauchy problem for $\psi - \psi_0$, it again suffices to consider the case of zero initial data (13.103). The solution constructed subsequently the proof of Theorem 4.5.1 was supported in $J^\vee(K) \subset J^\wedge(K)$. By general properties of globally hyperbolic manifolds, the intersection of this set with every Cauchy surface is compact. This concludes the proof. $\square$

As also explained in Section 13.4 in Minkowski space, in curved spacetime the solution of the Cauchy problem can be expressed in terms of the causal fundamental solution k_m . The *retarded* and *advanced Green's operators* $s_m^\wedge$ and $s_m^\vee$ are linear mappings (for details, see, e.g., [27, 6])

$$s_m^\wedge, s_m^\vee : C_0^\infty(\mathcal{M}, S\mathcal{M}) \rightarrow C_{\text{sc}}^\infty(\mathcal{M}, S\mathcal{M}). \quad (13.107)$$

They satisfy the defining equation of the Green's operator

$$(\mathcal{D} - m)(s_m^{\wedge, \vee} \phi) = \phi. \quad (13.108)$$

Moreover, they are uniquely determined by the condition that the support of $s_m^\wedge \phi$ (or $s_m^\vee \phi$) lies in the future (respectively the past) of $\text{supp } \phi$. The *causal fundamental solution* k_m is introduced by

$$k_m := \frac{1}{2\pi i} (s_m^\vee - s_m^\wedge) : C_0^\infty(\mathcal{M}, S\mathcal{M}) \rightarrow C_{\text{sc}}^\infty(\mathcal{M}, S\mathcal{M}) \cap \mathcal{H}_m. \quad (13.109)$$

Note that it maps to solutions of the Dirac equation.

Proposition 13.6.1 *The solution of the Cauchy problem (4.5.1) has the representation*

$$\psi(x) = 2\pi \int_{\mathcal{N}} k_m(x, y) \psi \psi_{\mathcal{N}}(y) \, d\mu_{\mathcal{N}}(y), \quad (13.110)$$

where $k_m(x, y)$ is the causal fundamental solution (13.109).

Proof Let us consider a point x in the future of $\mathcal{N}$ (the case for the past is analogous). In this case, due to (13.109), the lemma simplifies to

$$\psi(x) = i \int_{\mathcal{N}} s_m^\wedge(x, y) \psi(y) \psi_{\mathcal{N}}(y) \, d\mu_{\mathcal{N}}(y). \quad (13.111)$$

In preparation, we want to prove that for any $\phi \in C^\infty(\mathcal{M}, S\mathcal{M})$ which has compact support to the past of $\mathcal{N}$ and with the property that $(\mathcal{D} - m)\phi$ has

compact support the equation

$$\phi = s_m^\wedge((\mathcal{D} - m)\phi) \quad (13.112)$$

holds. To this end, we consider the function

$$\Xi(x) := \phi - s_m^\wedge((\mathcal{D} - m)\phi). \quad (13.113)$$

Applying the operator $(\mathcal{D} - m)$ and using the defining equation of the Green's operators, one sees that Ξ is a solution of the Dirac equation. Moreover, Ξ obviously vanishes in the past of the support of ϕ . The uniqueness of the solution of the Cauchy problem implies that Ξ vanishes identically, proving (13.112).

In order to derive equation (13.111), we let $\eta \in C^\infty(\mathcal{M})$ be a function which is identically equal to one at x and on $\mathcal{N}$, but such that the function $\eta\psi$ has compact support to the past. For example, in a foliation $(\mathcal{N}_t)_{t \in \mathbb{R}}$ with $\mathcal{N} = \mathcal{N}_{t_0}$ one can take $\eta = \eta(t)$ as a smooth function with $\eta|_{[t_0, \infty)} \equiv 1$ which vanishes if $t < t_0 - 1$. Then we can apply (13.112) to the wave function $\phi = \eta\psi$. We thus obtain for any x in the future of $\mathcal{N}$ the relations

$$\begin{aligned} \psi(x) &= (\eta\psi)(x) \\ &= \left(s_m^\wedge((\mathcal{D} - m)(\eta\psi)) \right)(x) = \left(s_m^\wedge(i\gamma^j(\partial_j \eta)\psi) \right)(x), \end{aligned} \quad (13.114)$$

where we have used that ψ is a solution of the Dirac equation.

To conclude the proof, for η in (13.114) we choose a sequence η_ℓ which converges in the distributional sense to the function which in the future $\mathcal{N}$ is equal to one and in the past of $\mathcal{N}$ is equal to zero. This yields $\partial_j \eta_\ell \rightarrow \nu$, and thus the right-hand side of (13.114) goes over to the right-hand side of (13.111). $\square$

13.7 Exercises

Exercise 13.1 The *homogeneous Maxwell equations* for the electric field $E : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ and the magnetic field $B : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ read

$$\nabla \times B = \partial_t E, \quad \nabla \times E = -\partial_t B, \quad (13.115)$$

where $\times$ denotes the cross product in $\mathbb{R}^3$. Rewrite these equations as a symmetric hyperbolic system. *Remark:* We here ignore the equations $\operatorname{div} E = \operatorname{div} B = 0$. The reason is that these equations hold automatically if they are satisfied initially.

Exercise 13.2 Consider the scalar wave equation $(\partial_{tt} - \Delta_{\mathbb{R}^m})\phi(t, x) = 0$.

(a) Rewrite the equation as a symmetric hyperbolic system

$$A^0 \partial_t u + \sum_{\alpha=1}^m A^\alpha \nabla_\alpha u + Bu = 0. \quad (13.116)$$

(b) Determine the timelike and future-directed directions. Which directions ξ are characteristic (in the sense that the characteristic polynomial $\det A(x, \xi)$ vanishes)?

(c) Express the “energy”

$$E(t) = \int_{\mathbb{R}^m} \langle u, A^0 u \rangle \, d^m x \quad (13.117)$$

in terms of $\phi(t, x)$. Compare the resulting expression with the conserved physical energy

$$\int_{\mathbb{R}^m} (|\partial_t \phi|^2 + |\nabla \phi|^2) \, d^m x. \quad (13.118)$$

(d) Compute $\frac{dE(t)}{dt}$. Prove the inequality

$$\frac{dE(t)}{dt} \leq E(t) \quad (13.119)$$

and integrate it (Grönwall’s lemma).

Exercise 13.3 Consider the solution of the homogeneous wave equation

$$(\partial_{tt} - \Delta_{\mathbb{R}^n})\phi(t, x) = 0 \quad (13.120)$$

for smooth initial data $\phi(0, x) = f(x)$ and $\partial_t \phi(0, x) = g(x)$.

Show by a suitable choice of lens-shaped regions that $\phi(t_0, x_0)$ depends only on the initial data in the closed ball $\{x \in \mathbb{R}^n : |x - x_0| \leq t_0\}$.

Exercise 13.4 We consider the system

$$\partial_t u_1(t, x) + \partial_x u_1(t, x) + 4\partial_x u_2(t, x) = 0 \quad (13.121)$$

$$\partial_t u_2(t, x) + 4\partial_x u_1(t, x) + \partial_x u_2(t, x) = 0 \quad (13.122)$$

(a) Write the system in symmetric hyperbolic form.

(b) Compute the solution of the Cauchy problem for initial data given by $u_1(0, x) = \sin x$ and $u_2(0, x) = \cos x$.

Exercise 13.5 (*The Euler equations*) The evolution equation for an isentropic compressible fluid reads

$$\begin{cases} \partial_t v + \nabla_v v + \frac{1}{\rho} \text{grad}(p) &= 0 \\ \partial_t \rho + \nabla_v \rho + \rho \text{div}(v) &= 0. \end{cases} \quad (13.123)$$

Here $v : \mathbb{R}^+ \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is the velocity vector field, $\rho : \mathbb{R}^+ \times \mathbb{R}^3 \rightarrow (0, \infty)$ the density and $p = A\rho^\gamma$ the pressure (where $A > 0$ and $\gamma > 1$).

(a) Show that (13.123) is equivalent to a quasilinear symmetric hyperbolic system, provided that ρ is bounded away from zero.

(b) Show that for smooth solutions, the system (13.123) is equivalent to

$$\begin{cases} \partial_t v + \nabla_v v + \text{grad}(h(\rho)) &= 0 \\ \partial_t \rho + \text{div}(\rho v) &= 0, \end{cases} \quad (13.124)$$

where $h \in C^\infty(\mathbb{R})$ satisfies the equation $h'(\rho) = \rho^{-1}p'(\rho)$.

(c) Let (v, ρ) be a solution of (13.124) with $v(t, x) = \nabla_x \varphi(t, x)$ for a real-valued potential φ . Prove *Bernoulli’s law*: If φ and ρ decay at infinity sufficiently

fast and if $h(0) = 0$, then

$$\partial_t \varphi + \frac{1}{2} |\nabla_x \varphi|^2 + h(\rho) = 0. \quad (13.125)$$

(d) Show that (13.123) can also be rewritten as a system for (p, v) ,

$$\begin{cases} \partial_t v + \nabla_v v + \rho(p)^{-1} \text{grad}(p) &= 0 \\ \partial_t p + \nabla_v p + (\gamma p) \text{div}(v) &= 0. \end{cases} \quad (13.126)$$

Rewrite this system in symmetric hyperbolic form.

Exercise 13.6 Let $\lambda > 0$. A symmetric hyperbolic system of the form

$$\partial_t u + A^\alpha(u) \partial_\alpha u + \lambda u = 0, \quad (13.127)$$

where the matrices A^α are smooth, uniformly bounded and uniformly positive, is an example of a so-called *dissipative system*.

(a) Prove that for spatially compact solutions, the following energy estimate holds:

$$\frac{d}{dt} \|u(t)\|_{H^p}^2 \leq \left(-2\lambda + c \|u(t)\|_{C^1} \right) \|u(t)\|_{H^p}^2. \quad (13.128)$$

(b) Prove: If the initial data u_0 is sufficiently small in the C^1 -norm, then there exists a global solution.

Hint: Choose p sufficiently large and use the Sobolev embedding theorem.

Exercise 13.7 (*Causality in the setting of symmetric hyperbolic systems*) The Dirac equation $(i\not{\partial} - m)\psi = 0$ can be rewritten as a *symmetric hyperbolic system*, that is, in the form ($c > 0$)

$$(A^0(x) \partial_0 + A^\alpha(x) \partial_\alpha + B(x))\psi = 0, \quad \text{with } (A^i)^\dagger = A^i \text{ and } A^0(x) \geq c\mathbb{I}. \quad (13.129)$$

For such systems a notion of *causality* can be introduced: a vector $\xi \in \mathbb{R}^4$ is said to be *timelike* or *lightlike* at $x \in \mathbb{R}^4$, if the matrix $A(x, \xi) := A^i(x) \xi_i$ is definite (either positive or negative) or singular, respectively.

Find the matrices A^i and B for the Dirac equation and show that the above-mentioned notions of timelike and lightlike vectors coincide with the corresponding notions in Minkowski space.