Forum of Mathematics, Sigma (2025), **Vol. 13**:e80 1–36
doi:10.1017/fms.2025.36

CAMBRIDGE UNIVERSITY PRESS

RESEARCH ARTICLE

On the irreducibility of Hessian loci of cubic hypersurfaces

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Received: 3 July 2024; **Revised:** 6 March 2025; **Accepted:** 23 March 2025

2020 Mathematical Subject Classification: *Primary* – 14J70; *Secondary* – 14M12, 14J17, 14J30, 14J35

Abstract
We study the problem of the irreducibility of the Hessian variety $\mathcal{H}_f$ associated with a smooth cubic hypersurface $V(f) \subset \mathbb{P}^n$. We prove that when $n \leq 5$, $\mathcal{H}_f$ is normal and irreducible if and only if f is not of Thom-Sebastiani type (i.e., if one cannot separate its variables by changing coordinates). This also generalizes a result of Beniamino Segre dealing with the case of cubic surfaces. The geometric approach is based on the study of the singular locus of the Hessian variety and on infinitesimal computations arising from a particular description of these singularities.

Contents

1. Introduction

Let $X = V(f)$ be a hypersurface of the projective space $\mathbb{P}^n$ over an algebraically closed field $\mathbb{K}$ of characteristic zero. In the case where the determinant $h_f = \det(H_f)$ of the associated Hessian matrix H_f is not equivalently zero – for example, for X smooth – it is well known that the associated Hessian hypersurface $\mathcal{H}_f = V(h_f) \subset \mathbb{P}^n$ contains much information of X itself.

A sort of *generic Torelli theorem* for Hessian hypersurfaces is also supposed to be valid (see [8]), up to some known cases. In particular, in [8] the so-called *Hessian map*

$$h_{d,n} : \mathbb{P}(S^d) \dashrightarrow \mathbb{P}(S^{(n+1)(d-2)}) \qquad [f] \mapsto [h_f]$$

where S denotes the ring $\mathbb{K}[x_0, \dots, x_n] = \oplus_{d\geq 0} S^d$, is studied. In the specific case of cubic hypersurfaces, namely when $d = 3$, it is known that $h_{3,1}$ has the generic fiber of dimension 1, $h_{3,2}$ has the generic fiber consisting of 3 points, and $h_{3,3}$ is birational onto its image. As in this last case, the *Ciliberto-Ottaviani conjecture* states that the Hessian map $h_{d,n}$ should be birational onto its image, for higher values of n and even for different values of d. Very recently, this conjecture has been analysed for the case of plane curves in [2] and [9].

One can argue that also the singular locus $\mathrm{Sing}(\mathcal{H}_f)$ of the Hessian hypersurface $\mathcal{H}_f$ must keep track of some crucial aspects of X. Along this line, the aim of this paper is a deep study of the singularities of the hypersurface $\mathcal{H}_f$ associated with a smooth cubic $(n-1)$-fold; in particular, we are interested in the dimension of $\mathrm{Sing}(\mathcal{H}_f)$ and in the irreducibility of $\mathcal{H}_f$ itself. Notice that, for low dimensional hypersurfaces (i.e.,curves or surfaces), this analysis is classical (see, for instance, [19] and [28]). More recently, an approach has been developed in [1] for the case of cubic threefolds, while in [6], the authors have dealt with the higher-dimensional cases.

To explain our main result (Theorem A), let us recall that a polynomial f (or a hypersurface $X = V(f)$) is said to be *of Thom-Sebastiani type*, TS for brevity, if up to a change of coordinates it can be written using two disjoint sets of variables (see Definition 3.1). The name comes from the works [27] of Sebastiani and Thom and [29] of Thom. These polynomials have been extensively studied in several contexts (for example, about their Jacobian ideals in some classical works of Bertini, Longo and Mammana – [3], [22] and [24]) and have appeared with other names too in the literature (for example, they are called *direct sums* in [7] and [13]). The Hessian hypersurface associated with a TS polynomial is not irreducible, and its singular locus has dimension $n-2$. The interesting fact is that this is actually a characterization, as proved in the following.

Theorem A (Theorem 5.1). *Assume that $n \leq 5$ and consider $f \in \mathbb{K}[x_0, \dots, x_n]$ defining a smooth cubic. Then the Hessian hypersurface $\mathcal{H}_f \subset \mathbb{P}^n$ is irreducible and normal if and only if f is not of Thom-Sebastiani type.*

The problem of determining whether a polynomial is of TS type is interesting and investigated in the literature (see, for example, [7] or [13]), also from an algorithmic point of view. As said above, one can apply a sort of 'hessian test': if the Hessian hypersurface associated to a polynomial f is normal, then f cannot be of TS type. Furthermore, with a strong geometric approach, Theorem A guarantees that this 'hessian test' is actually a complete test for (smooth) cubic forms with at most 6 variables.

Beyond the smoothness hypothesis, which is anyway necessary (see Remark 5.2 for details), one could conjecture that the same result still holds for higher dimensions or higher degrees. Even if we strongly believe that Theorem A is valid for smooth cubic hypersurface of any dimension, one can see that this is not the case – for example, in degree 4 (see again Remark 5.2).

Besides the fact that cubic hypersurfaces are classically endowed with interesting and particular properties in relation to their geometry and also, for example, to their associated Hessian variety (see, for example, [12], [26], [15], [20],...), the peculiarity of the case of (smooth) cubics lies in the framework we want to deal with and in the techniques used. Indeed, Hessian loci of cubic hypersurfaces are equipped, among other things, with a special symmetry that will be a key ingredient in the whole article and which makes a crucial difference with the higher degree cases. Indeed, if $X = V(f)$ is a general cubic hypersurface, then its Hessian $\mathcal{H}_f$ is a singular Calabi-Yau variety with a fixed point free rational involution. Indeed, for $d = 3$, one can observe that for all $x, y \in \mathbb{K}^{n+1}$,

$$H_f(x) \cdot y = H_f(y) \cdot x.$$

Such a relation can be easily translated in terms of the associated apolar ring $A_f = \mathcal{L}/\mathrm{Ann}_{\mathcal{L}}(f)$, where $\mathcal{L} = \mathbb{K}[\frac{\partial}{\partial x_0}, \dots, \frac{\partial}{\partial x_0}]$, as $x \cdot y = y \cdot x$. Under the natural identification $\mathbb{P}(A^1) \simeq \mathbb{P}^n$ (which comes from the Gorenstein duality of the apolar ring; see [23] or the comprehensive book [17]), one can define the natural incidence correspondence

$$\Gamma_f = \{([x], [y]) \in \mathbb{P}^n \times \mathbb{P}^n \mid H_f(x) \cdot y = 0\} = \{([x], [y]) \in \mathbb{P}(A^1) \times \mathbb{P}(A^1) \mid x \cdot y = 0\}.$$

https://doi.org/10.1017/fms.2025.36 Published online by Cambridge University Press

Observe that it dominates, via the two projections, the Hessian hypersurfaces $\mathcal{H}_f$. Moreover, Γ_f is also equipped with an involution $\tau(([x],[y])) = ([y],[x])$ which is fixed point free and descends to the above-mentioned fixed point free rational involution defined on the Hessian hypersurface $\mathcal{H}_f$.

Another key point is the determinantal structure of the Hessian hypersurface. Indeed, $\mathcal{H}_f$ is equipped with a natural rank-decreasing filtration

$$\mathcal{H}_f = \mathcal{D}_n(f) \supseteq \mathrm{Sing}(\mathcal{H}_f) \supseteq \mathcal{D}_{n-1}(f) \supseteq \cdots \supseteq D_i(f) \supseteq \cdots \supseteq D_0(f),$$

where $\mathcal{D}_k(f) = \{[x] \in \mathbb{P}^n \mid \mathrm{Rank}(H_f(x)) \leq k\}$. In the case where $V(f)$ is *any* smooth cubic hypersurface, the second inclusion in the above filtration is actually an equality as proved in [1] for cubic threefolds and in [6] for $n \geq 5$. Moreover, the authors proved that for X general, each of the D_i's has the expected dimension, and we computed the basic invariant in some low dimensional cases. In particular, for a smooth cubic $V(f)$, we have

$$\dim(\mathrm{Sing}(\mathcal{H}_f)) \in \{n-3, n-2\}.$$

Indeed, $\mathcal{H}_f$ is reduced, and the expected value (namely, $n-3$) is achieved for f general. Observe that in the other case, we would have that the Hessian locus $\mathcal{H}_f$ is not normal (indeed, see [18, Proposition 8.23]; recall that a hypersurface in $\mathbb{P}^n$ is normal if and only if it is regular in codimension 1). Then, with Theorem A, we partially answer the following natural question:

When, for a smooth cubic hypersurface $V(f)$, is $\mathcal{H}_f$ not normal?

When $n = 2$, given a smooth cubic curve $X = V(f)$, the associated Hessian curve $\mathcal{H}_f$ is singular if and only if X is the Fermat curve. It is remarkable that Beniamino Segre in 1943 proved that a similar result also holds for cubic surfaces in a projective 3-dimensional space. Indeed (see [28]), given a smooth cubic surface $X = V(f) \subset \mathbb{P}^3$, $\mathcal{H}_f$ is reducible if and only if X is cyclic; that is, up to a change of coordinate, we can write $f = z_0^3 + g(z_1, z_2, z_3)$.

It is a real misfortune, due likely to the war and to racial issues, that the book of Beniamino Segre which focuses on non-singular cubic surfaces, is not easy to find. Its analysis is based on the use of *Sylvester's Pentahedral Theorem* which, with a modern terminology, says that the general cubic surface has *Waring rank* equal to 5 (i.e., after a suitable change of coordinates, it can be written as the zero locus of $\sum_{i=1}^5 L_i^3$ where $L_1, \ldots, L_5 \in S^1$). This description has also been recently used, for example, in [10] and [8]. With Theorem A, we extend Segre's result to cubic threefolds and fourfolds.

Let us now explain the main features of our proof. First of all, let us observe that, since no useful 'Sylvester form tool' seems to exist for cubic forms in $\mathbb{P}^n$ with $n \geq 4$, a completely new strategy must be used. In this environment, it is a great pleasure to acknowledge our main source of inspiration: Adler's work. In a remarkable series of appendices to the book [1], among many other results, Adler set up a method to study the singular locus of the Hessian locus $\mathcal{H}_f$ associated with a cubic hypersurface $V(f)$. He considered the correspondence Γ_f introduced above, which can be seen as a partial desingularization of $\mathcal{H}_f$, and moreover, he proved that the singular locus of Γ_f has a 'triangle structure'. More precisely, a point $([x],[y])$ is singular for Γ_f if and only if there exists $[z] \in \mathbb{P}^n$ such that $([x],[y]), ([x],[z]), ([y],[z]) \in \Gamma_f$.

Our crucial observation is that if $\mathrm{Sing}(\mathcal{H}_f)$ contains a component of dimension $n-2$, then the same holds also for $\mathrm{Sing}(\Gamma_f)$. We have then a large amount of triangles to deal with, and moreover, such a description is greatly enlightened by using the *apolar-geometric* method we already exploited in our proof of a Gordan-Noether theorem (see [16], [26], [5]). The whole proof is then devoted to showing that 'too many triangles' for $\mathcal{H}_f$ force f to be of TS type.

Two results can be thought of as the main ingredients to this aim. First of all, we give a characterisation of the cubic polynomials of TS type in terms of the Hessian loci $\mathcal{D}_k$ appearing in the above-mentioned filtration:

Theorem B (Theorem 3.4). *A polynomial $f \in \mathbb{K}[x_0, \ldots, x_n]$ defining a smooth cubic is of TS type of the form $f(x_0, \ldots, x_n) = f_1(x_0, \ldots, x_k) + f_2(x_{k+1}, \ldots, x_n)$ if and only if $\mathcal{D}_{k+1}(f)$ contains a $\mathbb{P}^k$.*

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The second result allows us to make specific assumptions on the general triangle we will deal with. In particular, by considering an irreducible component $\mathcal{F}$ of the variety parametrizing these triangles for $\mathcal{H}_f$ and denoting by π_i its natural projections, we have the following:

Theorem C (Theorem 4.6). *Assume $n \leq 5$ and let $X = V(f)$ be a smooth cubic hypersurface in $\mathbb{P}^n$ not of TS type. If $\mathcal{F}$ is an irreducible family of triangles for $\mathcal{H}_f$ with $\dim(\pi_1(\mathcal{F})) = \dim(\mathcal{F}) = n-2$, then the general element in $\mathcal{F}$ is such that none of its vertices belongs to X.*

Even if, a fortiori, the situation presented in the above theorem cannot be realised, let us stress that this result will allow us to set the right framework on which all the proof is based.

The problem (after some reduction preliminaries) becomes to compute the Zariski tangent space at the general point of $\mathcal{F}$. This approach leads almost immediately to a conclusion in the case of cubic surfaces, and it is reasonably accessible for $n = 4$. In the fourfold case, the computation becomes instead much more complicated: there are really many sub-cases to be considered (this is certainly due also to the fact that for $n \leq 4$ all the cubics of Thom-Sebastiani type are indeed cyclic, which is not true anymore for $n \geq 5$).

It is interesting to notice that there is a family of cubic fourfolds (which is considered in Lemma 6.4), where the infinitesimal methods are not enough in order to conclude. For these hypersurfaces, in the spirit of the possible *Torelli theorem*, we recover the equation of the cubic fourfold $V(f)$, and then, with a direct computation, we show that the dimension of the singular locus of $\mathcal{H}_f$ is actually the expected one (i.e. 2, unless f is of TS type).

Plan of the paper

After setting the notation and proving some preliminary results in Section 2, in Section 3 we deal with polynomials of Thom-Sebastiani type, and we prove Theorem B. In Section 4, we focus on the study of particular families of triangles, and we prove Theorem C. Finally, in Sections 5 and 6, we prove our main result, namely Theorem A, respectively for $n \leq 4$ and for the case of cubic fourfolds.

2. Preliminaries and first results

In this first section, we set the notation and present some preliminary results, some of them proved in [6]. For a complete comprehension of standard Artinian Gorenstein Algebras, which we are going to introduce, one can refer to [17]. Consider $\mathbb{K}$ an algebraically closed field of characteristic 0 and the projective space $\mathbb{P}^n$ for $n \geq 2$. Let us set

$$S = \mathbb{K}[x_0, \dots, x_n] = \bigoplus_{k \geq 0} S^k \qquad \text{and} \qquad \mathcal{L} = \mathbb{K}[y_0, \dots, y_n] = \bigoplus_{k \geq 0} \mathcal{L}^k$$

so that S is the homogeneous coordinate ring of $\mathbb{P}^n$ and $\mathcal{L}$ is the graded algebra of linear differential operators on S, where we define y_i as the first partial derivative with respect to x_i; that is,

$$y_i = \frac{\partial}{\partial x_i} : S^\bullet \to S^{\bullet - 1}. \tag{2.1}$$

If $v \in \mathbb{K}^{n+1}$, we will denote by ∂_v the derivative in the direction of v (i.e., $\sum_{i=0}^n v_i y_i$).

Let us now consider a homogeneous polynomial f of degree d (i.e., an element of S^d). Two objects can then be associated with f in a natural way:

- the *Jacobian ring* of f, defined as the quotient $R_f = S/J_f$, where J_f denotes the Jacobian ideal of f, spanned by the partial derivatives of f;
- the *apolar ring* of f, defined as the quotient $A_f = \mathcal{L}/\mathrm{Ann}_{\mathcal{L}}(f)$, where $\mathrm{Ann}_{\mathcal{L}}(f)$ is the annihilator ideal of f (i.e,. the ideal in $\mathcal{L}$ given by $\{\delta \in \mathcal{L} \mid \delta(f) = 0\}$).

Both the Jacobian and the apolar ring of f are graded Artinian algebras with socle in degree respectively $(n+1)(d-2)$ and d (i.e., $R_f = R^0 \oplus R^1 \oplus \cdots \oplus R^{(n+1)(d-2)}$ and $A_f = A^0 \oplus A^1 \oplus \cdots \oplus A^d$). One can also see that they are standard (i.e., generated in degree 1) and that they satisfy the so-called Poincaré (or Gorenstein duality) – for example, the multiplication map $A^{d-k} \times A^k \longrightarrow A^d$ is a perfect pairing for every suitable positive integer k. In other words, they are both examples of what we call SAGAs, an acronym for standard Artinian Gorenstein algebras.

Finally, given f as above, we can then define the associated *Hessian matrix* and the *hessian polynomial*, respectively the square symmetric matrix whose entries are the second partial derivatives of f with respect to the x_i's and the determinant of such a matrix; that is,

$$H_f = ((y_i y_j)(f))_{i,j=0,\dots,n} \qquad \text{and} \qquad h_f = \det(H_f).$$

Let us observe that if the zero locus of f, $X = V(f) \subset \mathbb{P}^n$ is a smooth hypersurface, then the hessian determinant h_f belongs to $S^{(n+1)(d-2)} \setminus \{0\}$. In this case, one can define the *Hessian hypersurface* $\mathcal{H}_f$ associated with f (or with X) as the zero locus of such a polynomial; that is,

$$\mathcal{H}_f = V(h_f).$$

The smoothness of $X = V(f)$ implies also that the associated apolar ring A_f is such that A^1 has dimension $n+1$: indeed, such a dimension is strictly smaller than $n+1$ if and only if $V(f)$ is a cone. From the natural pairing $S \times \mathcal{L} \longrightarrow S$, one can then deduce an isomorphism

$$\mathbb{P}^n \simeq \mathbb{P}((S^1)^*) \simeq \mathbb{P}(A^1).$$

From now on, let us focus on the case $\mathbf{d} = \mathbf{3}$: the first result we need to recall is the following Proposition (see [6, Proposition 1.2]), which allows us to interpret the cubic hypersurface $X = V(f)$, its singular locus and its associated Hessian variety in terms of the apolar ring A_f.

Proposition 2.1. *Given a cubic hypersurface $X = V(f)$ (not a cone) and the corresponding A_f, we have*

1. *Under the identification $\mathbb{P}^n \simeq \mathbb{P}(A^1)$, $H_f(x) \cdot y = \nabla(xy(f))$, for $x, y \in \mathbb{K}^{n+1}$;*
2. *$\mathcal{H}_f = \{[y] \in \mathbb{P}(A^1) \mid \exists\, [x] \in \mathbb{P}(A^1)$ with $xy = 0$ in $A^2\}$;*
3. *$X = \{[y] \in \mathbb{P}(A^1) \mid y^3 = 0\}$;*
4. $\mathrm{Sing}(X) = \{[y] \in \mathbb{P}(A^1) \mid y^2 = 0\}$.

In this paper, we deal with a homogeneous cubic polynomials whose zero locus is smooth: we will denote by $\mathcal{U} \subset \mathbb{P}(S^3)$, the locus of such elements.

As done in [6], given $[f] \in \mathcal{U}$, let us introduce some objects which will be used extensively in what follows. First of all, for $[x] \in \mathbb{P}^n$, we set

$$\iota([x]) = \mathbb{P}(\ker(H_f(x))). \tag{2.2}$$

This is either empty (exactly when $[x] \notin \mathcal{H}_f$) or a projective linear space of dimension $n - \mathrm{Rank}(H_f(x))$. It is then natural to consider the *Hessian loci*

$$\mathcal{D}_k(f) = \{[x] \in \mathbb{P}^n \mid \mathrm{Rank}(H_f(x)) \leq k\}, \tag{2.3}$$

which give a stratification of the projective space $\mathbb{P}^n$ and in particular of the Hessian hypersurface $\mathcal{H}_f$ (for example, we have $\mathcal{D}_{n+1}(f) = \mathbb{P}^n$ and $\mathcal{D}_n(f) = \mathcal{H}_f$). Moreover, in general, for $k \leq n-1$, $\mathcal{D}_{k-1}(f) \subsetneq \mathcal{D}_k(f) \subset \mathcal{H}_f$. We will simply write $\mathcal{D}_k$, when it is clear which polynomial we are referring to in the following. In [6], the authors actually proved that for every integer $k \in \{2, \dots, n\}$, $\mathcal{D}_{k-1}(f) \subseteq \mathrm{Sing}(\mathcal{D}_k(f))$ and that equality holds for $[f] \in \mathcal{U}$ general. For any $[f] \in \mathcal{U}$, let us introduce a useful incidence correspondence:

$$\Gamma_f = \{([x], [y]) \in \mathbb{P}^n \times \mathbb{P}^n \mid H_f(x) \cdot y = 0\} \tag{2.4}$$

and let us denote by pr_i the two natural projections.

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Remark 2.2. By the relation $H_f(x) \cdot y = H_f(y) \cdot x$ (which is equivalent to the relation $xy = yx$ in A_f, by Proposition 2.1), the standard involution

$$\tau : \mathbb{P}^n \times \mathbb{P}^n \to \mathbb{P}^n \times \mathbb{P}^n \qquad ([x],[y]) \mapsto ([y],[x])$$

sends Γ_f to itself. As a consequence, Γ_f dominates $\mathcal{H}_f$ via both pr_1 and pr_2.

By Proposition 2.1, the loci just introduced can be described also in terms of the apolar ring as follows:

$$\iota([x]) = \mathbb{P}(\ker(x\cdot : A^1 \to A^2)) \qquad \Gamma_f = \{([x],[y]) \in \mathbb{P}(A^1) \times \mathbb{P}(A^1) \mid xy = 0\}.$$

Through the article, we will use one description or the other according to the convenience.

We summarise here the main results of [6]:

Theorem 2.3. *The following hold:*

1. *for any* $[f] \in \mathcal{U}$, *we have* $\mathrm{Sing}(\mathcal{H}_f) = \mathcal{D}_{n-1}(f)$;
2. *if* $[f] \in \mathcal{U}$ *is general, then* Γ_f *is smooth and* $\mathrm{pr}_i : \Gamma_f \to \mathcal{H}_f$ *is a desingularization;*
3. *the expected codimension of* $\mathcal{D}_k(f)$ *is* $\binom{n-k+2}{2}$.

In particular, the expected dimension of $\mathrm{Sing}(\mathcal{H}_f)$ *equals* $n-3$.

Hence, by Theorem 2.3, the Hessian variety associated with any smooth cubic hypersurface in $\mathbb{P}^n$ for $n \geq 3$ is always singular, and in the general case, Γ_f is a desingularization for it and we have a lower bound for the dimension of $\mathrm{Sing}(\mathcal{H}_f)$ for *any* $[f] \in \mathcal{U}$. We are now interested in giving an upper bound for this dimension and more generally for $\dim(\mathcal{D}_k(f))$.

Remark 2.4. It is well known that the diagonal $\Delta_{\mathbb{P}^n} \subseteq \mathbb{P}^n \times \mathbb{P}^n$ has decomposition in the Chow group of $\mathbb{P}^n \times \mathbb{P}^n$ given by

$$[\Delta_{\mathbb{P}^n}] = \bigoplus_{p+q=n} pr_1^*[H]^p \cdot pr_2^*[H]^q,$$

where H is a hyperplane in $\mathbb{P}^n$ and π_i are the standard projections. Hence, every effective cycle of dimension at least n intersects $\Delta_{\mathbb{P}^n}$.

Proposition 2.5. *Consider any* $[f] \in \mathcal{U}$. *Then the following hold:*

1. *the variety* Γ_f *is a connected complete intersection in* $\mathbb{P}^n \times \mathbb{P}^n$ *of pure dimension* $n-1$;
2. *for each* k, *one has* $\dim(\mathcal{D}_k(f)) \leq k-1$;
3. *there is a bijective correspondence between irreducible components of* Γ_f *and the irreducible components of the various loci* $\mathcal{D}_k(f)$ *for which the bound in* (b) *is sharp.*

Proof. For (a), first of all, observe that $\Gamma_f \cap \Delta_{\mathbb{P}^n} = \emptyset$ since, otherwise, $V(f)$ would be singular by Proposition 2.1. Hence, by Remark 2.4, we have that Γ_f has dimension at most $n-1$. However, by definition, we have that Γ_f is cut by $n+1$ divisors of $\mathbb{P}^n \times \mathbb{P}^n$ of bidegree $(1,1)$ so each component of Γ_f has dimension at least $n-1$.

Since Γ_f is a complete intersection, its connectedness follows by the Fulton-Hansen-type theorem (see [14], [21, Ch.3] or [25]).

For (b), let us assume by contradiction that there exists an irreducible component W of $\mathcal{D}_k(f)$ of dimension $d \geq k$. Over the general point $[w]$ of W, the fiber of the projection pr_1 from Γ_f is a projective space $\iota([x]) \simeq \mathbb{P}^s$ with $s \geq n-k$. Therefore, there exists a component of Γ_f of dimension at least $d+s \geq n$. This would mean, by Remark 2.4, that $\Gamma_f \cap \Delta_{\mathbb{P}^n}$ is not empty, giving a contradiction.

For (c), assume that W is as in (b) and of dimension $k-1$. Then the same reasoning as above implies the existence of an irreducible component of Γ_f dominating W. For the converse, let G be an irreducible component of Γ_f, set $G' = \mathrm{pr}_1(G)$ and let m be the dimension of G'. If $m = n-1$, then G' is a component of $\mathcal{H}_f = \mathcal{D}_n(f)$, so we are done. Otherwise, $m = n-1-a$ with $a > 0$ so that $\mathrm{pr}_1|_G$

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has the general fiber F of dimension a. Since the whole fiber of pr_1 over a general point of $[x] \in G'$ is a projective space containing a fiber of dimension a, we have that $\mathrm{Rank}(H_f(x)) \leq n-a$ (i.e., the general point of G' lies in $\mathcal{D}_{n-a}(f)$), as claimed. $\square$

As a consequence of Proposition 2.5, recalling that $\mathrm{Sing}(\mathcal{H}_f) = \mathcal{D}_{n-1}(f)$, we have in particular that

$$\dim(\mathrm{Sing}(\mathcal{H}_f)) \in \{n-3, n-2\}$$

and for f general, such dimension coincides with the expected one (i.e., $\dim(\mathrm{Sing}(\mathcal{H}_f)) = n-3$). In this paper, we are interested in answering the following:

Question 2.6. For which $[f] \in \mathcal{U}$ does $\mathrm{Sing}(\mathcal{H}_f)$ have dimension $n-2$?

In this study, the singularities of Γ_f will play a central role.

Definition 2.7. Given $f \in S^3$, we set

$$\mathcal{T} = \{([x],[y],[z]) \in (\mathbb{P}^n)^3 \mid [x] \in \iota(y),\ [y] \in \iota(z) \text{ and } [z] \in \iota(x)\}.$$

Elements in $\mathcal{T}$ are called *triangles for* $\mathcal{H}_f$.

Remark 2.8. Recall that if f is a cubic polynomial, then the incidence variety Γ_f is symmetric with respect to the involution τ. This implies that

$$[x] \in \iota(y) \Longleftrightarrow [y] \in \iota(x)$$

for all $[x],[y] \in \mathbb{P}^n$, so a permutation of the *vertices* of a triangle yields again a triangle. Furthermore, if $[f] \in \mathcal{U}$, two vertices of the same triangle cannot be equal, since we have $\Delta_{\mathbb{P}^n} \cap \Gamma_f = \emptyset$, and so by construction, each vertex of a triangle lies necessarily in $\mathcal{D}_{n-1}(f)$.

The following result links triangles for $\mathcal{H}_f$ and singularities of Γ_f. It has been proved for $n=4$ in [1] and in [6] for the general case.

Lemma 2.9. *For $[f] \in \mathbb{P}(S^3)$, a point $([x],[y])$ is singular for Γ_f if and only if there exists a third point $[z] \in \mathcal{H}_f$ such that the triple $([x],[y],[z])$ is a triangle for $\mathcal{H}_f$.*

To conclude this first section, let us present a couple of technical results which will be useful in what follows.

Lemma 2.10. *Assume that $[f] \in \mathcal{U}$. Consider a point $P \in \Gamma_f$. Then*

1. *the squares of its coordinates are independent;*

Moreover, if $T \in \mathcal{T}$ is a triangle, then

2. *the squares of the vertices of T are independent;*
3. *the vertices of T span a $\mathbb{P}^2$.*

Proof. In order to prove the claims, we will use extensively that $\mathrm{Sing}(V(f))$ can be identified with $\{[x] \in \mathbb{P}(A^1) \mid x^2 = 0\}$ (see Proposition 2.1). More precisely, we will proceed by contradiction by proving that if the conclusion of (a), (b) or (c) is false, then there exists an element whose square is 0 (i.e., a singular point for f), which is impossible by assumption.

Let us start by proving (a). For $P = ([y_1],[y_2]) \in \Gamma_f$, assume, by contradiction, that y_1^2 and y_2^2 are linearly dependent. Then, there exists $\lambda \in \mathbb{K}$ such that $y_1^2 = -\lambda^2 y_2^2$. As $y_1y_2 = 0$ vanishes, we would have that $(y_1 + \lambda y_2)^2 = 0$, which is impossible.

By (a), in order to prove (b), it is enough to show that $z^2 \notin \langle x^2, y^2 \rangle$ in A_f^2: let us assume again by contradiction that it is the case (i.e., there exist α and β in $\mathbb{K}^*$ such that $z^2 = -\alpha^2x^2 - \beta^2y^2$). In the same way as before, we can consider the square $(z + \alpha x + \beta y)^2$, which is zero since $xy = xz = yz = 0$, leading a contradiction.

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For (c), assume, by contradiction, that we could write $z = \alpha x + \beta y$ for some $\alpha, \beta \in \mathbb{K}^*$. Then, by definition of triangle, we would have that x^2, y^2 and z^2 are linearly dependent, which is impossible by (b). □

Assume that $[f] \in \mathcal{U}$. If $\mathcal{F}$ is variety of $(\mathbb{P}^n)^3 \simeq \mathbb{P}(A^1)^3$ whose points are triangles for $\mathcal{H}_f$ (i.e., $\mathcal{F} \subseteq \mathcal{T}$), we will refer to $\mathcal{F}$ as a *family of triangles (for $\mathcal{H}_f$)*. Recall that the tangent space to $(\mathbb{P}^n)^3 \simeq \mathbb{P}(A^1)^3$ at $T = ([x_1], [x_2], [x_3])$ is given by

$$T_{(\mathbb{P}(A^1))^3, T} = \bigoplus_{i=1}^{3} A^1/\langle x_i \rangle. \tag{2.5}$$

Lemma 2.11. *Assume that $[f] \in \mathcal{U}$. Let $T = ([x_1], [x_2], [x_3])$ be a triangle for $\mathcal{H}_f$, and consider a (Zariski) tangent vector $\underline{v} = (v_1, v_2, v_3) \in T_{\mathcal{T},T}$. Then, for $l \in \{1, 2, 3\}$ and for any representatives $x'_l \in A^1$ of the class $v_l \in A^1/\langle x_l \rangle$, one has*

$$x_i x'_j + x_j x'_i = 0 \qquad \text{and} \qquad x'_i \in \mathrm{Ann}_{A^1}(x_j^2, x_k^2)$$

whenever $\{i, j, k\} = \{1, 2, 3\}$. In particular, $v_i \in \mathrm{Ann}_{A^1}(x_j^2, x_k^2)/\langle x_i \rangle$.

Proof. The variety $\mathcal{T}$ can be described in $\mathbb{P}(A^1)^3$ as

$$\{([x], [y], [z]) \in (\mathbb{P}(A^1))^3 \mid xy = xz = yz = 0\}$$

so that the forms xy, xz and yz vanish identically on $\mathcal{T}$. If $\underline{v} = (v_1, v_2, v_3) \in T_{\mathcal{T},T}$ is a tangent vector and $v_l = [x'_l]$ as in the statement, then $(x_1 + tx'_1, x_2 + tx'_2, x_2 + tx'_3)$ satisfies the equations $xy = xz = yz = 0$ at first order:

$$0 = (x_i + tx'_i)(x_j + tx'_j) \mod t^2 = x_i x_j + t(x_i x'_j + x_j x'_i) \mod t^2,$$

so $x_i x'_j + x_j x'_i = 0$, as claimed. Multiplying by x_j, we get $x_j^2 x'_i = 0$ (i.e., we have $x'_i \in \mathrm{Ann}_{A^1}(x_j^2, x_k^2)/\langle x_i \rangle$). □

Notice that in the proof of Lemma 2.11, all the computations do not depend on the choice made for x'_l in v_l. Hence, for brevity, we will use indifferently x'_l and v_l in similar situations.

As a consequence of Lemma 2.10, we can then define a morphism

$$\psi : \mathcal{T} \to \mathrm{Gr}(2, \mathbb{P}^n) \qquad T = ([x], [y], [z]) \mapsto \mathbb{P}(\langle x, y, z \rangle).$$

Proposition 2.12. *The morphism ψ has everywhere injective differential, and it is injective modulo permutations of the vertices.*

Proof. Let us start by proving that the map ψ is injective (up to the permutation of such vertices). Let $T = ([x_1], [x_2], [x_3])$ and $T' = ([y_1], [y_2], [y_3])$ be two triangles which are not equivalent via permutation of the vertices. Assume, by contradiction, that $\psi(T) = \psi(T')$ (i.e., the two triples of vertices span the same projective plane). We can then write $y_i = \sum_{j=1}^{3} a_{ij} x_j$. Recall that $x_k x_l = \delta_{kl} x_k^2$, since $x_k x_l = 0$ for every $k \neq l$, while $x_i^2 \neq 0, y_i^2 \neq 0$ for every i, by the smoothness of $V(f)$. Hence, for $i \neq j$, we get

$$0 = y_i y_j = \sum_k a_{ik} x_k \cdot \sum_l a_{jl} x_l = \sum_{k,l} a_{ik} a_{jl} x_k x_l = a_{i1} a_{j1} x_1^2 + a_{i2} a_{j2} x_2^2 + a_{i3} a_{j3} x_3^2.$$

From Lemma 2.10, we have then that $a_{ik} a_{jk} = 0$ for every $k = 1, 2, 3$ and $i \neq j$. From this, one can easily see that, for every $i = 1, 2, 3$, at least (and at most, by construction) two coefficients among a_{i1}, a_{i2}, a_{i3} are zero. Hence, the vertices of T' and of T are the same up to a permutation.

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Fix a triangle $T = ([x_1], [x_2], [x_3])$ in $\mathcal{T}$. We claim now that the differential

$$d_T\psi : T_{\mathcal{T},T} \longrightarrow T_{\mathrm{Gr}(2,\mathbb{P}(A^1_f)),\langle x_1,x_2,x_3\rangle}$$

of ψ at T is injective. Let us consider a nontrivial vector

$$\underline{v} = (x_1', x_2', x_3') \in T_{\mathcal{T},T} \subset T_{\mathbb{P}(A^1)^3,T} \simeq \bigoplus_{i=1}^{3} A^1/\langle x_i\rangle.$$

Via the isomorphism $T_{\mathrm{Gr}(k,V),W} \simeq \mathrm{Hom}(W, V/W)$, we have that $d_T\psi(\underline{v})$ is the homomorphism

$$d_T\psi(\underline{v}) \in \mathrm{Hom}\Big(\langle x_1,x_2,x_3\rangle, A^1/\langle x_1,x_2,x_3\rangle\Big) \qquad \text{such that} \quad x_i \mapsto x_i' \quad \text{for } i \in \{1,2,3\}.$$

Hence, if we assume that $d_T\psi(\underline{v}) \equiv 0$, we have $x_i' \in \langle x_1,x_2,x_3\rangle$, so we can write

$$x_i' = \sum_{m=1}^{3} a_{im}x_m \tag{2.6}$$

for suitable $a_{im} \in \mathbb{K}$. By Lemma 2.11, we have $x_i x_j' + x_i' x_j = 0$ for $i \neq j$, so using the relations in Equation (2.6), we obtain

$$0 = x_i \sum_{m=1}^{3} a_{jm}x_m + x_j \sum_{m=1}^{3} a_{im}x_m = a_{ji}x_i^2 + a_{ij}x_j^2 \qquad \text{for } i \neq j.$$

By Lemma 2.10, squares of vertices of a triangle are independent, so we obtain $a_{ij} = 0$ for $i \neq j$ and $x_i' = a_{ii}x_i$. This is impossible since $x_i' \in A^1/\langle x_i\rangle$, and we would have $\underline{v} = 0$, whereas $\underline{v}$ is assumed to be nontrivial. □

3. Characterisation of TS Polynomials

In Section 2, we posed a question about a possible description of cubic forms $[f] \in \mathcal{U}$ whose Hessian locus has singularities in codimension 1 (see Question 2.6). First of all, let us notice that the locus in $\mathbb{P}(S^3)$ we are interested in is not empty. Indeed, one can easily exhibit polynomials whose Hessian locus is reducible.

Definition 3.1. Given $f \in S^d \setminus \{0\}$, we say that f is a *Thom-Sebastiani Polynomial* (TS, for brevity) if

$$f = f_1(l_0, \ldots, l_k) + f_2(l_{k+1}, \ldots, l_n) \tag{3.1}$$

for suitable $0 \leq k \leq n-1$, $\{l_0, \ldots, l_n\}$ independent linear forms and f_1, f_2 polynomials of degree d in $k+1$ and $n-k$ variables, respectively.

We will denote by $\mathcal{V}$ the set of smooth hypersurfaces which are not of Thom-Sebastiani type.

Remark 3.2. Notice that the set of Thom-Sebastiani polynomials of degree d is *not* closed in $\mathbb{P}(S^d)$. For instance, one can see Examples 1.3 and 1.4 of [7]. Nevertheless, in Theorem 4.5 of [7], the authors obtain a normal form for polynomials which are limits of TS polynomials and are not themselves of TS type (which are called *direct sums* with their language). All these limits correspond to singular varieties: this proves that $\mathcal{V}$ is an open set in $\mathcal{U}$ (i.e., the open set of polynomials corresponding to smooth varieties) and consequently in $\mathbb{P}(S^d)$.

Examples of TS polynomials are the ones whose zero locus is a cone. These are all singular, clearly. It is easy to see that if f is a TS polynomial as in Equation (3.1), $X = V(f)$ is smooth if and only if

$V(f_1(l_0,\dots,l_k),l_{k+1},\dots,l_n)$ and $V(f_2(l_{k+1},\dots,l_n),l_0,\dots,l_k)$ are smooth. This is also equivalent to ask that both $V(f_1)\subset\mathbb{P}^k$ and $V(f_2)\subset\mathbb{P}^{n-k-1}$ are smooth.

For brevity, if $\{x_0,\dots,x_n\}$ are linear forms in $\mathbb{P}^n$ and if f_1 and f_2 are polynomials in $k+1$ and $n-k$ variables, respectively, let us define

$$f_1(\underline{x}) := f_1(x_0,\dots,x_k) \qquad \text{and} \qquad f_2(\underline{x}') := f_2(x_{k+1},\dots,x_n).$$

Remark 3.3. Let f be a TS polynomial. Then, we can choose coordinates $\{x_0,\dots,x_n\}$ for $\mathbb{P}^n$ and write $f=f_1(\underline{x})+f_2(\underline{x}')$ for suitable polynomials in $k+1$ and $n-k$ variables of degree d. Set $g_1:=f_1(\underline{x})$ and $g_2:=f_2(\underline{x}')$ so that $g_1,g_2\in S^d$ with g_1 that depends only on the variables $x_0,\dots,x_k$ and g_2 that depends only on the other variables. Then, it is clear that

$$H_f(\underline{x},\underline{x}')=\begin{bmatrix} H_{f_1}(\underline{x}) & 0\\ 0 & H_{f_2}(\underline{x}')\end{bmatrix},$$

so $h_f(\underline{x},\underline{x}')=h_{f_1}(\underline{x})h_{f_2}(\underline{x}')$. In particular, the Hessian variety $\mathcal{H}_f$ associated to a TS polynomial is reducible, and it is the union of the two cones $W_1=V(g_1)$ and $W_2=V(g_2)$. Moreover, $\mathrm{Sing}(\mathcal{H}_f)$ has dimension $n-2$ since it contains the intersection $W_1\cap W_2$.

When $f=x_0^d+f_2(x_1,\dots,x_n)$ (in other words, in Definition 3.1, we are taking $k=0$), one talks about cyclic polynomials (see also Example 3.8 at the end of this section). The name comes from the fact that the projection of $X=V(f)$ from the point $P_0=(1:0:\cdots:0)$ to $V(x_0)\simeq\mathbb{P}^{n-1}$ gives a natural structure of cyclic cover of $\mathbb{P}^{n-1}$ branched along the hypersurface $V(f_2)$. For a smooth cubic $X=V(f)$, being cyclic gives a strong condition both on the associated Hessian locus and on the Jacobian ideal of f. Indeed, in [4], it has been proved that being cyclic is equivalent to having a linear component in the Hessian variety and a point in $\mathcal{D}_1(f)$. This point corresponds to an element in J_f which is a square of some linear form in S^1 (i.e., it gives a nilpotent element of order 2 in the Jacobian ring of f).

The main purpose of this section is to give a characterization of these Thom-Sebastiani polynomials in terms of the existence of suitable linear projective spaces in some Hessian loci. In particular, we will prove Theorem B:

Theorem 3.4. *A polynomial $f\in\mathcal{U}$ is of Thom-Sebastiani type of the form $f(x_0,\dots,x_n)=f_1(x_0,\dots,x_k)+f_2(x_{k+1},\dots,x_n)$ if and only if $\mathcal{D}_{k+1}(f)$ contains a $\mathbb{P}^k$.*

First of all, if we assume that $\mathcal{D}_k(f)\neq\mathcal{D}_{k-1}(f)$, we can define the map

$$\varphi:\mathcal{D}_k(f)\dashrightarrow \mathrm{Gr}(n-k,\mathbb{P}^n)\qquad [x]\notin\mathcal{D}_{k-1}(f)\mapsto\iota([x])$$

whose indeterminacy locus is $\mathcal{D}_{k-1}(f)$.

Proposition 3.5. *Assume that $\mathcal{D}_k(f)\neq\mathcal{D}_{k-1}(f)$. The injectivity of φ can only fail on points along a line contained in $\mathcal{D}_k(f)$ and cutting $\mathcal{D}_{k-1}(f)$. In particular, if $\mathcal{D}_{k-1}(f)=\emptyset$ or if $\mathcal{D}_k(f)$ does not contain lines, φ is injective.*

Proof. If $\mathcal{D}_k(f)\setminus\mathcal{D}_{k-1}(f)$ is a single point, φ is clearly injective. Assume then that $z_1,z_2\in\mathcal{D}_k(f)\setminus\mathcal{D}_{k-1}(f)$ are distinct and that $\varphi([z_1])=\varphi([z_2])$. Then $\iota([z_1])=\iota([z_2])$, so $H_f(z_1)$ and $H_f(z_2)$ have the same kernel. Up to a change of coordinates, we can assume that $\ker(H_f(z_i))=\langle e_0,\dots,e_{n-k}\rangle$, where $\{e_0,\dots,e_n\}$ is the basis corresponding to the basis $\{y_0,\dots,y_n\}$ under the identification $\mathbb{P}^n\simeq\mathbb{P}(A^1)$. Hence, there exist two square matrices A_1 and A_2 of order k with coefficients in $\mathbb{K}$ and maximal rank such that

$$H_f(z_i)=\begin{bmatrix}0&0\\0&A_i\end{bmatrix}.$$

Being $x\mapsto H_f(x)$ linear, $\mathbb{P}(\langle z_1,z_2\rangle)\simeq\mathbb{P}^1$ is clearly contained in $\mathcal{D}_k(f)$.

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Set $p(\lambda, \mu)$ to be the polynomial $\det(\lambda A_1 + \mu A_2)$. Since $\det(A_i) \neq 0$ by assumption, we have that p is homogeneous of degree k and nontrivial. Hence, there exists $[\lambda_0 : \mu_0]$ such that $p(\lambda_0, \mu_0) = 0$ (i.e., $H_f(\lambda_0 z_1 + \mu_0 z_2)$ has rank at most $k-1$). Thus, $\mathbb{P}(\langle z_1, z_2\rangle)$ cuts $\mathcal{D}_{k-1}(f)$, as claimed. □

Proposition 3.6. *Let $f \in \mathcal{U}$ and assume that the $(n-k-1)$-plane $\Pi = \mathbb{P}(V)$ is contained in $\mathcal{D}_{n-k}(f)$. Then there exists $\mathbb{P}(U) \simeq \mathbb{P}^k$ in $\mathcal{D}_{1+k}(f)$. Moreover, for all $[u] \in \mathbb{P}(U)$, one has that $\Pi \subseteq \iota([u])$ with equality holding for $[u]$ general.*

Proof. By assumption, one has $\mathrm{Rank}(H_f(v)) \leq n-k$ for all $v \in V\setminus\{0\}$. One can see ([6]) that the quadric of $\mathbb{P}^n$ given by the vanishing of the polynomial $\partial_v(f)$ is represented by the square symmetric matrix $H_f(v)$. Then the singular locus of the quadric $V(\partial_v(f))$ contains a $\mathbb{P}^k$. By setting $W = \{\partial_v(f)\}_{v\in V}$, we can then observe that $|W|$ is a linear subsystem of dimension $n-k-1$, since the map $v \mapsto \partial_v(f)$ is injective as $V(f)$ is smooth (it would have been enough to ask that $V(f)$ is not a cone).

Let $J = J_f$ be the Jacobian ideal of *f*. Since $|W| \subset |J^2|$ and J^2 is spanned by a regular sequence, we have that $B := \mathrm{BL}(|W|)$ has pure dimension k. Indeed, being B cut by $n-k$ quadrics, we have that $\dim(B) \geq k$. However, if there were a component of B with dimension at least $k+1$, then we would be able to complete a basis of W in such a way that J^2 is not spanned by a regular sequence, against our assumptions. Indeed, if $q_0, \ldots, q_{n-k-1}$ is a basis of W, then for any choice of elements $q_{n-k}, \ldots, q_n$ in J^2, we would have $\bigcap_i V(q_i) \neq \emptyset$. This cannot happen since *f* is smooth by assumption.

By Bertini's theorem, the general element of $|W|$ is smooth away from B, which has pure dimension k. However, as observed before, all quadrics of $|W|$ have a $\mathbb{P}^k$ contained in the singular locus. Therefore, there exists a component of B which is a $\mathbb{P}^k$. Since B has dimension k, it contains at most a finite number of $\mathbb{P}^k$: there exists a component of B which is a $\mathbb{P}^k$ contained in the singular locus of all the elements of $|W|$. Let $\mathbb{P}(U) \simeq \mathbb{P}^k$ be this linear space.

Consider $[u] \in \mathbb{P}(U)$. Since all the quadrics $V(\partial_v(f))$ parametrized by W are singular along $\mathbb{P}(U)$, we have

$$\partial_u\partial_v(f) = 0 \qquad \text{for all } [v] \in \mathbb{P}(V).$$

This implies that $\partial_v(\partial_u(f)) = 0$ for all $[v] \in \mathbb{P}(V)$: $V(\partial_u(f))$ is a quadric whose singular locus contains the $(n-k-1)$-plane $\Pi = \mathbb{P}(V)$. Therefore, $H_f(u)$ has rank at most $k+1$, and thus, $\mathbb{P}(U) \subseteq \mathcal{D}_{k+1}(f)$.

Finally, notice that $\dim(\mathcal{D}_k(f)) \leq k-1$ by Proposition 2.5, so $\mathbb{P}(U) \simeq \mathbb{P}^k$ cannot be contained in $\mathcal{D}_k(f)$. Hence, for the general point $[u] \in \mathbb{P}(U)$, the singular locus of $V(\partial_u(f))$ is exactly the $(n-k-1)$-plane Π. In other terms, we have $\iota([u]) = \mathbb{P}(V)$ for $[u] \in \mathbb{P}(U)$ general. □

Corollary 3.7. *Let $f \in \mathcal{U}$ and assume that there exists $k \geq 1$ such that $\mathcal{D}_{n-k}(f)$ contains a $(n-k-1)$-plane. Then $\mathcal{D}_k(f) \neq \emptyset$.*

Proof. Assume that $\mathbb{P}(V)$ is a $(n-k-1)$-plane in $\mathcal{D}_{n-k}(f)$. By Proposition 3.6, we have that there exist $\mathbb{P}^k \simeq \mathbb{P}(U) \subseteq \mathcal{D}_{k+1}(f)$ such that $\mathbb{P}(V) \subseteq \iota([u])$ for $[u] \in \mathbb{P}(U)$ with equality holding for $[u]$ general. Since, in this case, $\mathcal{D}_{k+1}(f)$ and $\mathcal{D}_k(f)$ do not coincide, for dimensional reason (by Proposition 2.5), we can define the map $\varphi : \mathcal{D}_{k+1}(f) \setminus \mathcal{D}_k(f) \to G(n-k-1, \mathbb{P}^n)$. Then, the injectivity of φ fails on two general points of $\mathbb{P}(U)$. Finally, by Proposition 3.5, we have that $\mathbb{P}(U) \cap \mathcal{D}_k(f) \neq \emptyset$, as claimed. □

We can now prove Theorem 3.4:

Proof. First of all, let us assume that for a fixed $k \geq 0$, there exists $\mathbb{P}^k \simeq \mathbb{P}(V) \subseteq \mathcal{D}_{k+1}(f)$. Then, by Proposition 3.6, there also exists $\mathbb{P}(U) \simeq \mathbb{P}^{n-k-1}$ contained in the locus $\mathcal{D}_{n-k}(f)$. Moreover, we know that for all $[u] \in \mathbb{P}(U)$, the projective space $\mathbb{P}(V)$ is contained in $\iota([u])$. This means that for every $[u] \in \mathbb{P}(U)$ and $[v] \in \mathbb{P}(V)$, we have $uv = 0$, with the identification $\mathbb{P}^n = \mathbb{P}(A^1)$. Let us notice that the spaces $\mathbb{P}(U)$ and $\mathbb{P}(V)$ are skew in $\mathbb{P}^n$ and of complementary dimension. Indeed, if their intersection was nontrivial, we could find a point $[x] \in \mathbb{P}(V) \cap \mathbb{P}(U)$: from above, we would obtain $x^2 = 0$, against the smoothness of $V(f)$. We can then consider for $\mathbb{P}^n$ a coordinate system $x_0, \ldots, x_n$ where $\mathbb{P}(V) = V(x_0, \ldots, x_k)$ and $\mathbb{P}(U) = V(x_{k+1}, \ldots, x_n)$. Up to a change of coordinates, we can then write the polynomial f with respect to these variables: since, by construction, $x_i x_j = 0$ in A^2 for every $i = 0, \ldots, k$ and $j = k+1, \ldots, n$, we get the claim.

Let us now assume that f is TS: as in Remark 3.3, we can write it (up to a possible change of coordinates) as $f = f_1(\underline{x}) + f_2(\underline{x}')$. As already observed, the Hessian matrix of f is of the form

$$H_f(\underline{x}, \underline{x}') = \begin{bmatrix} H_{f_1}(\underline{x}) & 0 \\ 0 & H_{f_2}(\underline{x}') \end{bmatrix}. \tag{3.2}$$

Hence, by defining $\mathbb{P}(V) := V(x_0, \ldots, x_k) \simeq \mathbb{P}^k$, one easily sees that $\mathrm{Rank}(H_f(v)) \leq k+1$ for every $[v] \in \mathbb{P}(V)$ (i.e., $\mathbb{P}(V) \subseteq \mathcal{D}_{k+1}(f)$, as claimed). □

To end this section, let us present some key examples of TS polynomials.

Example 3.8 (Cyclic cubics). The simplest examples of TS polynomials are the cyclic polynomials. We recall that a polynomial $f \in S^d$, where $S = \mathbb{K}[x_0, \cdots, x_n]$ is cyclic if, up to a change of coordinates, it can be written as $f = x_0^d + g(x_1, \ldots, x_n)$, where $g \in \mathbb{K}[x_1, \ldots, x_n]_d$.

As observed before, $X = V(f)$ is smooth exactly when $V(g) \subset \mathbb{P}^{n-1}$ is smooth and $h_f = d(d-1)x_0^{d-2} \cdot h_g(x_1, \ldots, x_n)$, so the Hessian variety splits as the union of a hyperplane and a hypersurface of degree $n(d-2)$, namely

$$H = V(x_0) \qquad \text{and} \qquad W = V(h_g(x_1, \ldots, x_n)).$$

Notice that W does not need to be irreducible, but this is the case if g is general (and $n \geq 3$). Under the identification $\mathbb{P}^{n-1} \simeq V(x_0)$, we can say that the Hessian loci $\mathcal{D}_k(g)$ live in $H = V(x_0)$. We denote by $\hat{\mathcal{D}}_k(g)$ the cone over $\mathcal{D}_k(g) \subseteq V(x_0)$ with vertex the coordinate point $P_0 = (1 : 0 : \cdots : 0)$. For example, one has $W = \hat{\mathcal{D}}_{n-1}(g)$. Then, using the explicit description of the Hessian matrix of f as block matrix such as in Equation (3.2), one can easily prove that

$$\mathcal{D}_k(f) = \mathcal{D}_k(g) \cup \hat{\mathcal{D}}_{k-1}(g) \cup \{P_0\}. \tag{3.3}$$

It is well known that the general cubic surface $S = V(g) \subseteq \mathbb{P}^3$ has an irreducible Hessian variety which is a quartic with 10 nodes as the only singularities. This was known already by B. Segre (see [28]), but one can also refer to the more recent [10]. In particular,

$$\mathcal{D}_3(g) = \mathcal{H}_g \qquad \mathcal{D}_2(g) = \mathrm{Sing}(\mathcal{H}_g) = \{Q_1, \ldots, Q_{10}\} \qquad \mathcal{D}_1(g) = \emptyset.$$

Using this observation and Equation (3.3), one can describe the stratification given by the Hessian loci of a general cyclic cubic threefold $X = V(f)$:

$$\mathcal{D}_5(f) = \mathbb{P}^4 \qquad \mathcal{D}_4(f) = \mathcal{H}_f = H \cup W \qquad \mathcal{D}_3(f) = \mathcal{H}_g \cup \bigcup_{i=1}^{10} \langle P_0, Q_i \rangle$$
$$\mathcal{D}_2(f) = \{P_0\} \cup \{Q_1, \ldots, Q_{10}\} \qquad \mathcal{D}_1(f) = \{P_0\}.$$

Among these, only $H, W \subset \mathcal{D}_4(f)$, $H \cap W = \mathcal{H}_g \subset \mathcal{D}_3(f)$ and $\{P_0\} \subset \mathcal{D}_1(f)$ give irreducible components of Γ_f (this will be clear from Lemma 4.1).

Since smooth binary cubic forms can be written as sum of 2 cubes, every TS polynomial in $n+1$ variables, with $n \leq 4$, is necessarily cyclic (see [4] for details). Let us now describe a new phenomenon arising in $\mathbb{P}^5$.

Example 3.9 (A TS cubic which is not cyclic). Let $g_1, g_2 \in S_w = \mathbb{K}[w_0, w_1, w_2]$ be such that $V(g_1)$ and $V(g_2)$ are smooth cubic curves in $\mathbb{P}^2$ which are not projectively equivalent to the Fermat curve. A classical result implies that this is equivalent to ask that $V(g_i)$ is a cubic whose Hessian $V(h_{g_i})$ is

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irreducible. This equivalence also follows easily from the main result of this paper (see Theorem 5.1 or, more specifically, Proposition 5.6) and can be found in [8].

A smooth cubic fourfold X of TS type is not cyclic if and only if, up to a change of coordinates, it is defined by a polynomial $f = g_1(x_0, x_1, x_2) + g_2(x_3, x_4, x_5)$. From the point of view of moduli, such fourfolds form a dimension 2 variety in the moduli space of smooth cubic fourfolds.

Consider the subvarieties of $\mathbb{P}^5$ defined by

$$W_1 = V(h_{g_1}(x_0, x_1, x_2)) \quad W_2 = V(h_{g_2}(x_3, x_4, x_5)) \quad \Pi_1 = V(x_0, x_1, x_2) \quad \Pi_2 = V(x_3, x_4, x_5)$$

$$C_1 = \Pi_2 \cap W_1 \qquad C_2 = \Pi_1 \cap W_2 \qquad J = J(C_1, C_2),$$

where $J(C_1, C_2)$ is the joint variety of C_1 and C_2 (namely, the union of all lines joining a point of C_1 and a point of C_2). Notice that, by construction, for $\{i, j\} = \{1, 2\}$, the variety W_i is a cone over C_j with vertex Π_i and C_i is isomorphic to the curve $V(h_{g_i})$. Moreover, by the assumptions on the curves $V(g_1)$ and $V(g_2)$, W_i is irreducible. Being f a TS polynomial, one has that $\mathcal{H}_f$ is indeed reducible. More precisely, since $h_f = h_{g_1}(x_0, x_1, x_2)h_{g_2}(x_3, x_4, x_5)$, one has that

$$\mathcal{D}_5(f) = \mathcal{H}_f = W_1 \cup W_2.$$

The other strata of the stratification induced by f are

$$\mathcal{D}_4(f) = \operatorname{Sing}(\mathcal{H}_f) = J \qquad \mathcal{D}_3(f) = \Pi_1 \cup \Pi_2 \qquad \mathcal{D}_2(f) = C_1 \cup C_2,$$

whereas $\mathcal{D}_1(f) = \emptyset$ as X is not cyclic (by the results in [4]).

It is worth highlighting two facts. First of all, Π_1 and Π_2 are two 2-planes contained in $\mathcal{D}_3(f)$. These are exactly the k-planes contained in $\mathcal{D}_{k+1}(f)$ whose existence is guaranteed by Theorem 3.4 since f is a TS polynomial. Moreover, note that for all $k \in \{2, 3, 4, 5\}$, the dimension of $\mathcal{D}_k(f)$ equals $k - 1$ (i.e., the maximum predicted by Proposition 2.5). In particular, Γ_f splits as the union of 7 irreducible fourfolds (this follows from Lemma 4.1).

4. Families of triangles of high dimension

In this section, we focus on the study of suitable families of triangles for $\mathcal{H}_f$ arising naturally, as we will see in a moment, when $\operatorname{Sing}(\mathcal{H}_f)$ exceeds the expected dimension. Moreover, we prove Theorem C.

Let us now set some notations and prove some technical results.

We recall that, given a smooth cubic $V(f) \subseteq \mathbb{P}^n \simeq \mathbb{P}(A^1)$, a family of triangles for $\mathcal{H}_f$ is a subvariety $\mathcal{F}$ of

$$\mathcal{T} = \{([x], [y], [z]) \in (\mathbb{P}(A^1))^3 \mid xy = yz = xz = 0\}.$$

We will denote by π_i the natural projections from $\mathcal{F}$ on the factors. For simplicity, if $\mathcal{F}$ is a family of triangles for $\mathcal{H}_f$, we will set $Y_i = \pi_i(\mathcal{F})$ for $i \in \{1, 2, 3\}$. Notice that $\dim(Y_i) \leq n-2$ since $Y_i \subseteq \mathcal{D}_{n-1}(f)$ which has dimension at most $n - 2$ by Proposition 2.5.

Moreover, recall that if $[f] \in \mathcal{U}$, by Proposition 2.5, all components of Γ_f come from the Hessian loci of f. More precisely, if Z is an irreducible component of $\mathcal{D}_k(f)$ of dimension $k - 1$, there exists a unique irreducible component of Γ_f which dominates Z by first projection. We will denote by $\tilde{Z}$ such component.

Lemma 4.1. *Let $[f] \in \mathcal{U}$ and assume $k \in \{1, \ldots, n-1\}$. Consider an irreducible component W of $\mathcal{H}_f$ and an irreducible component Z of $\mathcal{D}_k(f)$ of dimension $k - 1$ which is contained in W. Then $\tilde{Z} \cap \tilde{W}$ dominates Z via the first projection pr_1. In particular, there exists a family of triangles $\mathcal{F}$ for $\mathcal{H}_f$ of dimension at least $k - 1$. Moreover, if $k = n - 1$, then every family of triangles as above has dimension exactly $n - 2$.*

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Proof. Notice that Z is not contained in $\mathcal{D}_{k-1}(f)$ by Proposition 2.5, as we are assuming $\dim(Z) = k-1$. Hence, the general point $z \in Z$ lies in $\mathcal{D}_k(f) \setminus \mathcal{D}_{k-1}(f)$, and thus, $\iota([z]) \simeq \mathbb{P}^{n-k}$. Since the general fiber of $\mathrm{pr}_1|_{\tilde{Z}} : \tilde{Z} \to Z$ has dimension $n-k$ by construction, one has that the whole fiber $\mathrm{pr}_1^{-1}([z]) = \{[z]\} \times \iota([z])$ is contained in $\tilde{Z}$. However, $\mathrm{pr}_1|_{\tilde{W}} : \tilde{W} \to W$ is surjective and $Z \subseteq W$ by assumption, so there exists at least a point $p = ([z],[y])$ of the whole fiber $\pi_1^{-1}([z])$ with $p \in \tilde{W}$. Then $p \in U = \tilde{W} \cap \tilde{Z}$ and $\mathrm{pr}_1|_U : U \to Z$ is such that $\mathrm{pr}_1|_U(p) = z$. In particular, $\mathrm{pr}_1|_U$ dominates Z.

By the above argument, we have that $\tilde{W}$ and $\tilde{Z}$ are irreducible components of Γ_f which meet in a variety U of dimension at least $k-1$. Then, we have a family of dimension $k-1$ since this variety is contained in $\mathrm{Sing}(\Gamma_f)$ by construction and each point yields (at least) a triangle by Lemma 2.9.

Let $Z \subseteq \mathcal{D}_{n-1}(f)$ be an irreducible component of dimension $n-2$ and let $\mathcal{F}$ be a family of triangles dominating Z via π_1, so that $\dim(\mathcal{F}) \geq n-2$. By construction, the general point $[x]$ of Z is such that $\iota([x]) \simeq \mathbb{P}^1$ so $\pi_1^{-1}([x]) \subset \{[x]\} \times \mathbb{P}^1 \times \mathbb{P}^1$. If the general fiber $\pi_1^{-1}([x])$ has positive dimension, we would have that $\pi_1^{-1}([x]) \cap \{[x]\} \times \Delta_{\mathbb{P}^1}$ is not empty, thus giving rise to a singular point of $V(f)$. Then the general fiber of π_1 has dimension 0, and thus, $\dim(\mathcal{F}) = n-2$. □

Remark 4.2. If $\mathcal{H}_f$ is not normal, we have that there exists at least a family of triangles of dimension $n-2$. Indeed, we have that the singular locus of $\mathcal{H}_f$ has dimension $n-2$ and equals $\mathcal{D}_{n-1}(f)$ by Theorem 2.3. Hence, given an irreducible component Z of $\mathrm{Sing}(\mathcal{H}_f)$ of dimension $n-2$, we have that Z yields a family of triangles of dimension $n-2$ as a consequence of Lemma 4.1.

Lemma 4.3. *Let $X = V(f) \subset \mathbb{P}^n$ be a smooth cubic hypersurface and let $\mathcal{F}$ be an irreducible family of triangles. Then the following hold:*

1. *If $\dim(Y_i) \geq 1$ and $Y_i \subset X$, then $\dim(Y_i) \leq n-3$;*
2. *If $n \geq 3$ and $\dim(Y_i) = n-2$ for some i, then no projection has dimension 0 unless $V(f)$ is of Thom-Sebastiani type.*

Proof. For (a), w.l.o.g. we can assume $\dim(Y_1) \geq 1$ and $Y_1 \subset V(f)$. If $T = ([x],[y],[z]) \in \mathcal{F}$ is a general triangle, then the differential $d\pi_{1,T} : T_{\mathcal{F},T} \to T_{Y_1,[x]}$ is surjective and it sends a tangent vector (x',y',z') to x'. By Lemma 2.11, $x' \in \mathrm{Ann}_{A^1}(y^2,z^2)/\langle x\rangle$. Moreover, since $Y_1 \subset X$, by Proposition 2.1, we have $X = \{[x] \mid x^3 = 0\}$ so $T_{X,[x]} = \mathrm{Ann}_{A^1}(x^2)/\langle x\rangle$. Hence,

$$T_{Y_1,[x]} \subseteq \mathrm{Ann}_{A^1}(x^2,y^2,z^2)/\langle x\rangle.$$

Since T is a triangle, by Lemma 2.10, one has that $\langle x^2,y^2,z^2\rangle$ has dimension 3, and thus, by the Gorenstein duality, $\dim(\mathrm{Ann}_{A^1}(x^2,y^2,z^2)) = n+1-3 = n-2$. Moreover, being $[x] \in X$, one has $x \in \mathrm{Ann}(x^2,y^2,z^2)$, so $\mathrm{Ann}_{A^1}(x^2,y^2,z^2)/\langle x\rangle$ has dimension $n-3$.

For (b), w.l.o.g. assume that Y_3 is of dimension $n-2$. By contradiction, let us assume that $Y_1 = \{[x]\}$ and that f is not of TS type. Hence, $Y_3 \subseteq \iota([x])$, and this implies that $\iota(x) = \mathbb{P}^s$ with $s \in \{n-2, n-1\}$.

Then, we would get $\iota([x]) = \mathbb{P}^{n-2} = Y_3 \subseteq \mathcal{D}_{n-1}(f)$ and $[x] \in \mathcal{D}_1(f)$, respectively. Both cases yield a contradiction by Theorem 3.4. □

Before proving Theorem C (Theorem 4.6), let us focus on (families of) triangles with all vertices on the cubic X. These are linked to families of 2-planes in the cubic hypersurface:

Remark 4.4. First of all, recall that if $T = ([x],[y],[z])$ is a triangle for $\mathcal{H}_f$, then $\langle [x],[y],[z]\rangle = \mathbb{P}^2$, by Lemma 2.10. If we assume, moreover, that all the vertices of T belong to the cubic hypersurface X, we have $x^3 = y^3 = z^3 = 0$ and also $xy = yz = zx = 0$; this implies that the 2-plane is actually contained in X. Hence, a triangle with three vertices on X cannot exist if X is a smooth cubic hypersurface of dimension at most 3. Furthermore, since on smooth cubic fourfolds one has at most a finite number of 2-planes (see, for example, [11]), by Proposition 2.12, we can have at most a finite number of triangles with all the vertices on the cubic X.

Lemma 4.5. *Assume that $\mathcal{F}$ is a family of triangles for $\mathcal{H}_f$ with $\dim(Y_1) = \dim(\mathcal{F}) > \dim(Y_3)$. Then, none of the fibers of the projection π_3 can be contracted to points via π_1.*

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Proof. This follows from a more general fact: if $g : X \to Z$ is a surjective morphism between irreducible varieties and $f : X \to Y$ is a morphism, the locus

$$A = \{z \in Z \mid \dim(f(g^{-1}(z))) = 0\}$$

is an open Zariski set of Z. In order to prove this, let us consider the map $F = (f, g) : X \to Y \times Z$ and denote by $X' \subseteq Y \times Z$ the image of X under the morphism F. Moreover, let p_1 and p_2 be the two projections from X' to Y and Z, respectively. It is then clear that

$$A = \{z \in Z \mid \dim(p_1(p_2^{-1}(z))) = 0\} = \{z \in Z \mid \dim(p_2^{-1}(z)) = 0\},$$

and thus, it is an open subset of Z. In our situation, if we assume that a fiber of π_3 is contracted to points via π_1, then the same is true for the general one, contradicting the assumption $\dim(Y_1) = \dim(\mathcal{F})$. □

We are now ready to prove Theorem C, one of the main ingredients in the proof of Theorem A.

Theorem 4.6. *Assume $n \leq 5$ and consider a smooth cubic $V(f)$ not of TS type. If $\mathcal{F}$ is an irreducible family of triangles for $\mathcal{H}_f$ with $\dim(\mathcal{F}) = \dim(\pi_1(\mathcal{F})) = n - 2$, then the general element in $\mathcal{F}$ is such that none of its vertices belongs to $X = V(f)$.*

Proof. Notice that the proof follows easily if none of the three projections of $\mathcal{F}$ is contained in X. Indeed, in this case, $\pi_i^{-1}(Y_i \cap X)$, the locus where the triangles of $\mathcal{F}$ have the i-th vertex on the cubic X, is a proper closed subset of $\mathcal{F}$. We would like to show that also for $n \leq 5$, this is indeed the only possible case (i.e., no projection of $\mathcal{F}$ can be contained in X). For $n = 3$, This is an easy consequence of Lemma 4.3, so we can assume $n \geq 4$.

By hypothesis, π_1 is generically finite (and thus, by Lemma 4.3, Y_1 is not contained in X) and, by contradiction, $Y_3 \subset X$. Notice that, under these assumptions, Y_2 is not contained in X. Otherwise, $\pi_1^{-1}(Y_1 \cap X)$ would be an $(n-3)$-dimensional family of triangles with all the vertices contained in X. Then, we would have a contradiction as observed in Remark 4.4. For brevity, set $\mathcal{F}_c = \mathcal{F} \cap (X \times X \times X)$ (i.e., $\mathcal{F}_c$ is the locus of the triangles of $\mathcal{F}$ with all 3 vertices on the cubic hypersurface X).

Assume that $n = 4$. By Lemma 4.3, since $Y_3 \subset X$ and Y_1 has dimension $2 = n - 2 = \dim(\mathcal{F})$, we have that $\dim(Y_3) = 1$. Then, since $\mathcal{F}$ is irreducible, all the fibers of π_3 have pure dimension 1. The dimension of Y_2 is either 1 or 2. If the dimension of Y_2 is 1, all the fibers of π_2 are curves too and $Y_2 \cap X$ is not empty. Consider $[y_0] \in Y_2 \cap X$ and its fiber $C = \pi_2^{-1}([y_0])$. By Lemma 4.5, C cannot be contracted by π_1, so $\pi_1(C)$ is a curve. Then, $\pi_1(C) \cap X$ is not empty, and we can consider a point $[x_0]$ in this intersection. Hence, any element in $\pi_1^{-1}([x_0]) \cap C \neq \emptyset$ is a triangle in $\mathcal{F}_c$. This is impossible by Remark 4.4. Then we have necessarily $\dim(Y_2) = 2$.

Let Y be an irreducible component of $Y_2 \cap X$. Being $Y_2 \nsubseteq X$ and of dimension 2, Y is a curve, and there exists an irreducible component C in $\pi_2^{-1}(Y)$ of dimension 1 (since $\mathcal{F}$ is irreducible) dominating Y via the second projection. If either $\pi_1(C)$ is a curve or $\pi_1(C) = [x_0]$ with $[x_0] \in X$, we have an element in $\mathcal{F}_c$, so the only possible case is $\pi_1(C) = [x_0]$ with $[x_0] \notin X$.

Looking at the third projection, we have that $\pi_3(C)$ either is a point $[z]$ or it coincides with Y_3. Moreover, let us observe that $\iota([x_0]) \simeq \mathbb{P}^s$ with $s \in \{1, 2\}$ such that Y and $\pi_3(C)$ are contained in $\iota([x_0])$. To rule out the first case, namely $\pi_3(C) = [z]$, first of all, observe that $[z] \notin Y$; otherwise, we would have a singular point for the cubic X. Then s is forced to be 2. Moreover, by construction, we have $yz = 0$ and $y^3 = z^3 = 0$ for any $[y] \in Y$: by reasoning as in Remark 4.4, all the lines $\langle [y], [z] \rangle$ lie in X. This implies that the whole $\iota([x_0]) \simeq \mathbb{P}^2$ is contained in the smooth threefold X, but this is not possible, as observed in Remark 4.4.

For the remaining case, we have that $\pi_3(C) = Y_3$, and by construction, both the curves Y and Y_3 are contained in $\iota([x_0])$. Then, if $s = 1$, we necessarily have $Y = Y_3 \simeq \mathbb{P}^1$ and C is a family of triangles of dimension 1 in $\{[x_0]\} \times Y \times Y$. This yields a contradiction since C has to meet $\{[x_0]\} \times \Delta_Y$, thus giving a singular point for X. Hence, we have necessarily $s = 2$, and we can assume $Y \neq Y_3$ or $Y = Y_3 \neq \mathbb{P}^1$.

In both cases, as done above, considering the lines $\langle[y],[z]\rangle$ with $[y]\in Y$, $[z]\in Y_3$ and $yz=0$, we get that the 2-plane $\iota([x_0])$ is contained in the cubic threefold X, which is not possible.

Assume now that $n=5$. We are working in the following framework: $\mathcal{F}$ is an irreducible 3-dimensional family of triangles with π_1 generically finite, Y_1,Y_2 not contained in X and $Y_3\subseteq X$ (this will lead to a contradiction). Being $Y_3\subseteq X$ by assumption, $\mathcal{F}_c$ is cut out from $\mathcal{F}$ by two divisors, so its expected dimension is $n-4=1$. Then, either $\mathcal{F}_c$ is empty or $\dim(\mathcal{F}_c)\geq 1$. However, as observed in Remark 4.4, under the above-mentioned hypotheses, we have that $\dim(\mathcal{F}_c)\leq 0$, so $\mathcal{F}_c$ is necessarily empty. We will now prove that $\mathcal{F}_c$ is not empty, thus leading to a contradiction.

First of all, notice that $\dim(Y_2)\in\{1,2,3\}$ by Lemma 4.3. If $\dim(Y_2)\leq 2$, the general fiber of π_2 has positive dimension and cannot be contracted to a point by π_1 by Lemma 4.5. Then, its image meets X, and thus, we produce a triangle in $\mathcal{F}_c$ as the analogous case for the threefold. We can then assume $\dim(Y_2)=3$ so that π_2 is generically finite as π_1.

Denote by Y an irreducible component of $Y_2\cap X$. The preimage $\pi_2^{-1}(Y)$ has dimension 2, and we can consider an irreducible component $S\subset\pi_2^{-1}(Y)$ dominating Y. If $\pi_1(S)$ is not a point or is a point on the cubic fourfold, as in the threefold case, one can easily construct an element in $\mathcal{F}_c$: we can assume $\pi_1(S)=[x_0]\notin X$. As done in the previous case, we have that $\iota([x_0])\simeq\mathbb{P}^s$ containing Y and $\pi_3(S)$ (thus, $s\in\{2,3\}$ since X is not of TS type), where $\pi_3(S)\subseteq Y_3$ can be either a point $[z]$, a curve $C\not\subset Y_3$ (with $\dim(Y_3)=2$) or the whole Y_3 (with $\dim(Y_3)\in\{1,2\}$). To conclude the proof, let us study these distinguished cases.

- $\boldsymbol{\pi_3(S)=[z_0]}$: since $[z_0]\notin Y$ (otherwise we would have a singular point in X), s is forced to be equal to 3. Considering again the lines $\langle[y],[z_0]\rangle$ with $[y]$ varying in Y, we have that the whole 3-space $\iota([x_0])$ is contained in the smooth cubic fourfold X, which is clearly not possible.
- $\boldsymbol{\pi_3(S)=C}$: In this case, one has $\iota([x_0])\simeq\mathbb{P}^3$ since, otherwise, we would have $C\subseteq Y=\mathbb{P}^2$ and $S\subseteq\{[x_0]\}\times Y\times Y$ and then a singular point for X as in a previous case. If we assume $C\not\subset Y\subset\iota([x_0])=\mathbb{P}^3$, then one easily sees that the lines $\langle[y],[z]\rangle$ with $[y]\in Y,[z]\in C$ and $yz=0$ cover $\iota([x_0])$: we have a contradiction since we would have a projective 3$-$space in X. The only remaining case to analyse is then the one where $C\subset Y\subset\mathbb{P}^3$ with Y surface which is not a $\mathbb{P}^2$. In this case, S is a surface in $\{[x_0]\}\times Y\times C$ with the projections $p_2=\pi_2|_S$ and $p_3=\pi_3|_S$ which are surjective. Then, for all $[z]\in C$, $p_3^{-1}([z])$ has pure dimension 1. Let $[z]$ be a point in C and let D be an irreducible component of one of those fibers. For all $[y]\in p_2(D)$, we have $yz=y^3=z^3=0$ and $[z]\notin p_2(D)$, so the joint variety $J(p_2(D),[z])$ has dimension 2, is a cone with vertex $[z]$ and is completely contained in $\iota([x_0])\cap X$. Since $\iota([x_0])\simeq\mathbb{P}^3$ cannot be contained in X, these cones have to vary at most discretely, when $[z]$ moves in C. Notice that Y lies, by construction, in the union of these cones, so $[z]$ is in the vertex $\mathrm{Vert}(Y)$ of Y. Then $C\subseteq\mathrm{Vert}(Y)$, and this forces Y to be a $\mathbb{P}^2$, which is against our assumptions.
- $\boldsymbol{\pi_3(S)=Y_3}$: As in the previous case, we necessarily have $\iota([x_0])\simeq\mathbb{P}^3$ containing both the surface Y and $\pi_3(S)$, which can be either a curve or a surface. We can assume, moreover $Y_3=\pi_3(S)\subseteq Y$, since otherwise, proceeding as above, we would have $\iota([x_0])\subset X$. In particular, Y is not a 2-plane. If Y_3 is a curve, we can obtain a contradiction as in the previous case by considering the cones with vertex $[z]\in Y_3$ spanned by the curves in Y whose elements annihilate $[z]$. Hence, we can assume $Y=Y_3$ (and thus, π_3 is generically finite). By construction, for any element $[y]\in Y$, there exists at least one element $[z]\in Y$ such that $yz=0$; hence, again, the line $\langle[y],[z]\rangle$ is contained in $\iota([x_0])\cap X$. If, for $[y]$ general, at least one of these lines is not contained in Y, one can see that the whole 3-space $\iota([x_0])$ is contained in X, yielding a contradiction. We can then assume that for $[y]\in Y$ general, the above-mentioned lines are contained in Y. Our aim is now to show that $Y\simeq\mathbb{P}^2$, against our assumptions. First of all, let us show that the union Λ of these lines as subset in the Grassmannian $\mathrm{Gr}(1,\mathbb{P}^3)$ has dimension 2. If, by contradiction, $\dim(\Lambda)=1$, this would mean that for all $\ell\in\Lambda$ and for all $[y]\in\ell$, there exists $[z]\in\ell$ with $yz=0$. In other words, this would yield a correspondence in $\mathbb{P}^1\times\mathbb{P}^1$, which intersects the diagonal $\Delta_{\mathbb{P}^1}$ nontrivially: the cubic X would be singular, which is not possible. Let us finally consider the incidence variety

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$$\Psi := \{(y,\ell) \mid y \in \ell \in \Lambda\} \subset Y \times \mathrm{Gr}(1,\mathbb{P}^3)$$

and denote by ψ_1 and ψ_2 the two projections. We have just shown that $\mathrm{Im}(\psi_2) = \Lambda$, and moreover, it is clear that if $\ell \in \Lambda$, then $\psi_2^{-1}(\ell)$ is described by ℓ itself; hence, such a fiber has dimension 1. Then $\dim(\Psi) = 3$, and looking at the first projection ψ_1, we have that there exist infinitely many lines in Λ contained in Y and passing through the general point $[y] \in Y$. Hence, Y has to be a cone with $[y] \in \mathrm{Vert}(Y)$: from the generality of $[y]$, it follows that $Y \simeq \mathbb{P}^2$, as claimed. □

Remark 4.7. Observe that if $n = 2$, the hypotheses of Theorem cannot be realized since the existence of a triangle implies that the locus $\mathcal{D}_1(f)$ is nonempty. Hence, by Theorem 3.4, the cubic f is of TS type, against our assumption.

5. Proof of main theorem: the cubic threefold case

In this section, we state and begin to prove the main result of this article – namely, the following.

Theorem 5.1 (Theorem A). *Assume that $2 \le n \le 5$ and consider $f \in \mathbb{K}[x_0,\dots,x_n]$ defining a smooth cubic. Then, the singular locus of the Hessian hypersurface $\mathcal{H}_f \subset \mathbb{P}^n$ has the expected dimension if and only if f is not of TS type. In particular, $f \in \mathcal{V}$ if and only if $\mathcal{H}_f$ is irreducible and normal.*

As we observe now, the assumptions on the degree and the smoothness of the hypersurface $X = V(f)$ are essential.

Remark 5.2. Let us stress that the result stated in Theorem 5.1 is false for smooth hypersurfaces of degree $d \ge 4$ and for non-smooth cubics. We provide here two simple examples proving these claims.

- Let $f(x,y,z) = x^4 + y^4 + z^4 + x(y^3 + z^3)$ and consider $C = V(f)$. Then one easily sees that C is a smooth quartic plane curve and that

$$h_f = 54 \cdot yz\Big(8x^4 + 16x^3(y+z) + 32x^2yz - x(y^3+z^3) - 2yz(y^2+z^2)\Big).$$

  The quartic factor in the above factorization of h_f yields a smooth quartic by the Jacobian criterion and thus an irreducible one. This also implies that f is not of TS type since, otherwise, we would have the Hessian polynomial which is product of linear factors: there are smooth hypersurfaces of degree $d \ge 4$, which are not of TS type, with reducible Hessian variety.
- Let $f(x_0,x_1,x_2,x_3) = x_0x_1^2 + x_1x_2^2 + x_2x_3^2$ and consider the cubic surface $S = V(f)$. One can see that S is an irreducible cubic surface whose singular locus coincides with the point $p_0 = (1:0:0:0)$, which is a singularity of type D_5. Its associated Hessian variety is a reducible and non-reduced quartic surface $V(x_1^2(x_1x_2 - x_3^2))$. Notice that the quadratic factor of the Hessian polynomial h_f is irreducible so, reasoning as in the previous example, one can see that $f \neq f_1(z_0,z_1) + f_2(z_2,z_3)$ for suitable coordinates $\{z_0,\dots,z_3\}$. With a direct and easy computation, also the cyclic case is ruled out as follows. If we assume that f is cyclic, then $V(f)$ would be projectively equivalent to $V(g)$ where $g = x_0^3 + m(x_1,x_2,x_3)$. Notice that $h_g = x_0 \cdot h_m(x_1,x_2,x_3)$. Since a linear form has to appear in the factorization of h_g with multiplicity 2 (since x_1^2 divides h_f) we have that h_m is the product of 3 linear forms, which is incompatible with the above description. Hence, f is not of TS type although its Hessian is reducible (and thus non-normal) and non-reduced. The same phenomenon happens for the cuspidal cubic curve (see, for example, [8]).

Nevertheless, not every type of singularity gives the same behaviour as in the last example above. Indeed, the nodal cubic curve and the 1-nodal cubic surface $V(x_0(x_1^2 + x_2^2 + x_3^2) + x_1^2x_3 + x_2x_3^2)$ have irreducible and normal associated Hessian variety (and thus, they are not of TS type). One can easily construct examples of 1-nodal cubic threefolds and fourfolds with the same property.

Going back to the case of smooth cubic hypersurfaces, we make the following:

Conjecture 5.3. *The same result stated in Theorem 5.1 holds for smooth cubic hypersurfaces in $\mathbb{P}^n$ for every $n \geq 2$.*

The techniques used in the proof of Theorem 5.1 do not seem to adapt to an argument that could be valid in any dimension: already for cubic fourfolds, one can see the large amount of cases one has to consider. For this reason, we will give the proof of the Theorem for $n \leq 4$ at the end of this section, while the case of cubic fourfolds is treated in the subsequent one, since it is more involved, even if the techniques are similar.

We stress that, for a smooth cubic $V(f)$, the implication

$$\mathcal{H}_f \text{ irreducible and normal} \Longrightarrow f \text{ not of TS type}$$

is always true for all $n \geq 2$ as we have seen in Remark 3.3: the hard part of the conjecture is to prove the other implication.

Let us now explain the strategy that will be used for the proof of the Theorem for the various values of $n \leq 5$.

Framework 5.4. Assume that f defines a smooth cubic $X = V(f)$ which is not of TS type (i.e., $[f] \in \mathcal{V}$) such that $\mathcal{H}_f$ is not normal. Then by Lemma 4.1 and Remark 4.2, we have a family $\mathcal{F}$ of dimension $n-2$ of triangles for $\mathcal{H}_f$ dominating, via the first projection, a component Y_1 of $\mathcal{D}_{n-1}(f)$ of the same dimension. As we have done in the previous sections, we denote by Y_i the images of the projections π_i. As just observed, $\pi_1 : \mathcal{F} \to Y_1$ is generically finite.

If $T = ([x],[y],[z]) = ([x_1],[x_2],[x_3])$ is a general point of $\mathcal{F}$, by generic smoothness, we can assume that the differentials $d_T\pi_i : T_{\mathcal{F},T} \to T_{Y_i,[x_i]}$ are surjective; in particular, $d_T\pi_1$ is an isomorphism. Moreover, since we are assuming $n \leq 5$, by Theorem 4.6, we have that none of the vertices of T belongs to the cubic $V(f)$ (i.e., $x_i^3 \neq 0$ for $i \in \{1,2,3\}$). If we set

$$V_1 = \langle x, y, z\rangle \subset A^1 \qquad \text{and} \qquad V_2 = \operatorname{Ann}_{A^1}(x^2, y^2, z^2) \subset A^1, \tag{5.1}$$

by Lemma 2.10 and by Gorenstein duality, we have that $\dim_{\mathbb{K}}(V_1) = 3$ and $\dim_{\mathbb{K}}(V_2) = n-2$. Moreover, since $x^3, y^3, z^3 \neq 0$, we also have $V_1 \cap V_2 = \{0\}$. Hence, by dimension reason, one has

$$A^1 = V_1 \oplus V_2. \tag{5.2}$$

If $\{i,j,k\} = \{1,2,3\}$, by Lemma 2.10 and Lemma 2.11, one has also

$$\dim_{\mathbb{K}} \operatorname{Ann}_{A^1}(x_j^2, x_k^2) = n-1 \qquad \dim_{\mathbb{K}} \operatorname{Ann}_{A^1}(x_j^2, x_k^2)/\langle x_i\rangle = n-2. \tag{5.3}$$

By dimension reason and since $x_i \notin V_2$, we have a canonical isomorphism $V_2 \simeq \operatorname{Ann}_{A^1}(x_j^2, x_k^2)/\langle x_i\rangle$ induced by the inclusion $V_2 \hookrightarrow \operatorname{Ann}_{A^1}(x_j^2, x_k^2)$ followed by the quotient by $\langle x_i\rangle$. By Lemma 2.11 and since we have that $d_T\pi_i$ is surjective, we have

$$T_{Y_i,[x_i]} \subseteq \operatorname{Ann}_{A^1}(x_j^2, x_k^2)/\langle x_i\rangle \simeq V_2,$$

so we can interpret $d_T\pi_i$ as maps $T_{\mathcal{T},T} \to V_2$. Being $d_T\pi_1$ an isomorphism, we have then the endomorphisms

$$\psi_m = d_T\pi_m \circ (d_T\pi_1)^{-1} : V_2 \to T_{Y_m,[x_m]} \hookrightarrow V_2 \qquad \text{for } m \in \{2,3\}. \tag{5.4}$$

Our approach is to analyse obstructions for such a configuration, by studying the Zariski tangent spaces. These information can be naturally codified by looking at the operators ψ_m; this strategy can be followed a priori in every dimension and in the cases of our interest yield to a direct and conclusive computation.

In the proof of Theorem 5.1, we will start by analyzing these specific maps, ruling out both the cases where ψ_2 (or ψ_3) can be diagonalized or not and ultimately proving that a family of triangles of dimension $n-2$ cannot exist.

As we have done in the proof of Lemma 2.11, to a tangent vector $\underline{v} = (x', y', z') \in T_{\mathcal{T},T}$, we can associate the 'first-order deformation' of T in the direction of $\underline{v}$ (for brevity, $\underline{v}$-deformation of T), which we write in a compact way as

$$T + \underline{v} = (x + tx', y + ty', z + tz'). \tag{5.5}$$

Lemma 5.5. *Let $\mathcal{F}$ be a family of triangles. Assume furthermore that both $\mathcal{F}$ and Y_1 have dimension $n-2$. If the general element $T = ([x], [y], [z]) \in \mathcal{F}$ has no vertices on $X = V(f)$, then $x\cdot : V_2 \to A^2$ is injective.*

Proof. Since Y_1 has dimension $n-2$, its general point $[x]$ is in $\mathcal{D}_{n-1}(f) \setminus \mathcal{D}_{n-2}(f)$ (i.e., the kernel of the multiplication map $x\cdot : A^1 \to A^2$ has dimension 2). The point $[x]$ is also the vertex of an element $T = ([x], [y], [z])$ of $\mathcal{F}$, and thus, $\ker(x\cdot) = \langle y, z\rangle \subseteq V_1$. However, by the assumptions, one has $V_1 \cap V_2 = 0$, as observed above. □

Let us now show the Theorem 5.1 in the first cases.

Proposition 5.6. *Theorem 5.1 is true for $n \in \{2, 3\}$.*

Proof. By contradiction, let us assume that $[f] \in \mathcal{V}$ and $\dim(\mathrm{Sing}(\mathcal{H}_f)) = n-2$. Notice that if $n = 2$, since $\mathrm{Sing}(\mathcal{H}_f) = \mathcal{D}_1(f)$, we have a contradiction by Theorem 3.4. Hence, we can assume $n = 3$.

Fix the notation explained in the framework (see 5.4). Since $n = 3$, we have that

$$\mathrm{Ann}_{A^1}(x^2, y^2, z^2) = V_2 = \langle u\rangle$$

for suitable $u \in A^1 \setminus \{0\}$. All projections of $\mathcal{F}$ have dimension exactly 1 by Lemma 4.3, so $T_{\mathcal{F},T} = \langle (Au, Bu, Cu)\rangle$ with $A, B, C \in \mathbb{K}^*$. By Lemma 2.11, the associated first-order deformation $T + u(A, B, C)$ has to satisfy

$$Bux + Auy = 0, \qquad Cux + Auz = 0, \qquad Cuy + Buz = 0.$$

We can then observe that the three independent points $Bx + Ay$, $Cx + Az$, $Cy + Bz$ belong to the kernel of the multiplication by u. In other words, $\iota(u) \supseteq \mathbb{P}^2$, so that $[u] \in \mathcal{D}_1(f)$. Hence, as before, we have a contradiction. □

Proposition 5.7. *Theorem 5.1 is true for $n = 4$: for a smooth cubic threefold $X = V(f)$, the Hessian quintic threefold $\mathcal{H}_f$ is normal if and only if f is not of TS type.*

Proof. By contradiction, let us assume that $[f] \in \mathcal{V}$ and $\dim(\mathrm{Sing}(\mathcal{H}_f)) = 2$. Then we are in the situation described in 5.4: $T = ([x], [y], [z])$ will denote a general triangle in $\mathcal{F}$ (recall that by Theorem 4.6 none of its vertices belongs to the cubic $X = V(f)$). Since $n = 4$, we have that $\dim(V_2) = 2$. Recall that we have the endomorphisms

$$\psi_i = d_T\pi_i \circ (d_T\pi_1)^{-1} : V_2 \simeq T_{Y_1,[x]} \to V_2 \qquad i \in \{2, 3\},$$

which have image $T_{Y_2,[y]}$ and $T_{Y_3,[z]}$, respectively. We have that either one of the two is diagonalizable or that none is. We treat differently the two cases.

Case (I): Let us suppose that at least one of the two above endomorphisms is diagonalizable. W.l.o.g. we can assume that $\{u, w\}$ is a basis of V_2 whose elements are eigenvectors for ψ_2. Then two independent tangent vectors to $\mathcal{F}$ in T are given as

$$\underline{v} = (u, Au, Cu + Dw) \qquad \text{and} \qquad \underline{v}' = (w, Bw, Eu + Fw) \tag{5.6}$$

for suitable A, B, C, D, E, F scalars depending on the triangle, where, in particular, A and B are the eigenvalues corresponding respectively to u and w.

We recall that, for a given tangent vector $\underline{v} = (x', y', z')$ at $T = ([x], [y], [z])$, we have the relation $xy' + yx' = 0$ by Lemma 2.11. For brevity, we refer to this relation with the notation $(\underline{v})_{xy}$. We denote by $(\underline{v})_{xz}$ and $(\underline{v})_{yz}$ the analogous relations.

For example, using the vectors in (5.6), we have the corresponding first-order deformations

$$T + t\underline{v} = (x + tu, y + tAu, z + t(Cu + Dw)) \qquad T + s\underline{v}' = (x + sw, y + sBw, z + s(Eu + Fw)),$$

which yield

$$\begin{array}{ll} (\underline{v})_{xy} : Aux + yu = 0 & (\underline{v}')_{xy} : Bwx + yw = 0 \\ (\underline{v})_{xz} : Cux + Dwx + zu = 0 & (\underline{v}')_{xz} : Eux + Fwx + zw = 0 \\ (\underline{v})_{yz} : Cuy + Dwy + Auz = 0 & (\underline{v}')_{yz} : Euy + Fwy + Bwz = 0. \end{array} \tag{5.7}$$

By elementary operations, one gets two new relations:

$$2ACux + D(A + B)wx = 0 \qquad E(A + B)ux + 2FBwx = 0.$$

For example, the former relation is obtained by substituting in $(\underline{v})_{yz}$ of Equation (5.7), the products uy, wy, and uz obtained respectively from $(\underline{v})_{xy}$, $(\underline{v}')_{xy}$ and $(\underline{v})_{xz}$ of Equation (5.7).

By Lemma 5.5, the multiplication map $x\cdot : V_2 \to A_2$ is injective, so

$$AC = 0 \qquad D(A + B) = 0 \qquad E(A + B) = 0 \qquad FB = 0. \tag{5.8}$$

Let us now observe that A and B cannot be simultaneously zero when evaluated in a general triangle T; otherwise, the second projection of $\mathcal{F}$ would be zero-dimensional, contradicting Lemma 4.3.

Lemma 5.8. *In this situation, for T general, none of the two eigenvalues of the endomorphism ψ_2 can be zero.*

Proof. Let $T = ([x], [y], [z])$ be a general triangle of $\mathcal{F}$. W.l.o.g. we can assume by contradiction that $A \neq 0$ and $B = 0$. We claim that $Y_2 \subseteq \iota(z)$ and $Y_3 \subseteq \mathcal{D}_2(f)$.

Since $A \neq 0$, we also get that $C = D = E = 0$ by Equation (5.8). Hence, the first-order deformations of T are given by

$$T + t\underline{v} = (x + tu, y + tAu, z) \quad \text{and} \quad T + s\underline{v}' = (x + sw, y, z + sFw). \tag{5.9}$$

Since the differential maps of the projections from $\mathcal{F}$ are surjective, we have $\dim(Y_1) = 2$ and $\dim(Y_2) = \dim(Y_3) = 1$.

Consider the curve $C_T = \pi_3^{-1}([z])$. By construction, the tangent to C_T in T is spanned by $\underline{v}$ which is projected to u and Au via $d_T(\pi_1|_{C_T})$ and $d_T(\pi_2|_{C_T})$, respectively. Hence, $\pi_1(C_T)$ is a curve in Y_1 and $\pi_2(C_T) = Y_2$. In particular, $\pi_1(C_T) \cup Y_2 \subseteq \iota([z])$, as claimed. Moreover, since $\mathcal{D}_1(f) = \emptyset$, by assumption (by Theorem 3.4), we get that $\iota([z]) \simeq \mathbb{P}^2$ up to the case where it is a projective line coinciding both with Y_2 and $\pi_1(C_T)$. But in this last case, we would have an involution on $Y_2 \simeq \mathbb{P}^1$, which yields a fixed point and then a singular point for $V(f)$. Hence, $Y_3 \subseteq \mathcal{D}_2(f)$, as claimed.

Notice that the same argument can be used to prove that $Y_3 \subseteq \iota([y])$ and thus that $Y_2 \subseteq \mathcal{D}_2(f)$. By Proposition 3.5, one can see that for two general points $[z]$ and $[z']$ in Y_3, we have that $\iota([z]) \neq \iota([z'])$. Hence, $Y_2 \subseteq \iota([z]) \cap \iota([z']) = \mathbb{P}^1$, and thus, $Y_2 = \mathbb{P}^1 \subseteq \mathcal{D}_2(f)$. This is impossible by Theorem 3.4 since f is not of TS type. □

From the above Lemma 5.8, since T is general, we have that both A and B are not zero; hence, by Equation 5.8, we also get $B = -A$. Indeed, if $A + B \neq 0$, we would obtain $C = D = E = F = 0$, which

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is not possible by Lemma 4.3. Since $A = -B \neq 0$, from Equations (5.8), we have $C = F = 0$. Then, the first-order deformation, in this case, can be written as

$$T + t\underline{v} = (x + tu, y + tAu, z + tDw) \qquad T + s\underline{v}' = (x + sw, y - sAw, z + sEu), \tag{5.10}$$

and the conditions (5.7) are equivalent to

$$u(y + Ax) = 0 \qquad zw + Exu = 0 \qquad w(y - Ax) = 0 \qquad zu + Dxw = 0. \tag{5.11}$$

Moreover, we cannot have $D = E = 0$ as observed above, so we can assume $D \neq 0$.

Since these equations hold, by assumption, for the general point $T \in \mathcal{F}$, by deforming T at the first order in the direction of $\underline{v}$, that is, by considering a curve

$$T(t) = ([x(t)], [y(t)], [z(t)]) = T + t\underline{v} + t^2(\cdots),$$

also the corresponding eigenvectors of ψ_2 'move'. More precisely, we have two curves

$$\gamma_u : U \longrightarrow A^1 \qquad \gamma_w : U \longrightarrow A^1$$

defined in a neighbourhood of 0 such that $\gamma_u(0) = u$, $\gamma_w(0) = w$ and $\{\gamma_u(t), \gamma_w(t)\}$ is a basis of eigenvectors of $d_{T(t)}\pi_2 \circ d_{T(t)}\pi_1^{-1}$. These eigenvectors satisfy equations analogous to the ones in (5.7) where the coefficients depend on t. As observed above, the sum of the two eigenvalues of ψ_2 is 0 also in a neighbourhood of T, so that the Equations (5.11) hold also locally.

We can then consider an expansion of the curves

$$\gamma_u(t) = u + tu' + t^2(\cdots) \qquad \gamma_w(t) = w + tw' + t^2(\cdots)$$

and substitute them in the Equations (5.11) in order to get new relations. We write $A(t) = A + A't + t^2(\cdots)$ for the curve following the eigenvalue relative to $\gamma_u(t)$ with an analogous notation for the coefficients that appear in Equations (5.11).

For example, from the condition $u(y + Ax) = 0$, one has

$$0 \equiv \gamma_u(t)(y(t) + A(t)x(t)) = (u + tu' + t^2(\cdots))(y + Ax + t(2Au + A'x) + t^2(\cdots)),$$

so we get $2Au^2 + A'xu + u'(Ax + y) = 0$. One can do the same reasoning for the $\underline{v}'$-deformation, and we also can use two 'parameters' to take into account in a compact description the deformation of T in the direction of $t\underline{v} + s\underline{v}'$. In this way, u and w 'deform' at first order as

$$u + tu' + su'' \qquad \text{and} \qquad w + tw' + sw'',$$

respectively. Moreover, we can assume that u', u'' and w', w'' do not depend on u and w, respectively. This argument yields the relations

$$2Au^2 + A'xu + (Ax + y)u' = 0 \qquad u''(y + Ax) + A''xu = 0 \tag{5.12}$$

$$w'z + Dw^2 + Eu^2 + E'ux + Eu'x = 0 \qquad 2Euw + w''z + E''ux + Eu''x = 0 \tag{5.13}$$

$$w'(y - Ax) - A'xw = 0 \qquad w''(y - Ax) - A''xw - 2Aw^2 = 0 \tag{5.14}$$

$$2Duw + D'xw + Dw'x + u'z = 0 \qquad Dw^2 + Dxw'' + D''xw + Eu^2 + zu'' = 0. \tag{5.15}$$

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First of all, observe that multiplying by x Equation $(5.14)_I$, since $xy = 0 = x^2w$, we get

$$x^2w' = 0.$$

Hence, multiplying by x Equation $(5.15)_I$, we obtain

$$xuw = 0. \tag{5.16}$$

Since $xuw = 0$, from Equation (5.11), one can easily see that also

$$zu^2 = zw^2 = yuw = 0. \tag{5.17}$$

We claim now that $A' = A'' = 0$ and $xu^2, xw^2 \neq 0$. Indeed, let us observe that the product xu vanishes if multiplied by $x,\ y,\ z,\ w$. If $xu^2 = 0$ too, then by the Gorenstein duality in the apolar ring A_f, we would get that $xu = 0$, which is not possible as observed with Lemma 5.5.

In the same way, one sees that $xw^2 \neq 0$. Then, recalling that $u(Ax + y) = 0 = w(y - Ax)$, we can multiply by u and by w, respectively, Equations $(5.12)_{II}$ and $(5.14)_I$, getting $A''xu^2 = 0$ and $A'xw^2 = 0$, and so the claim:

$$A' = A'' = 0 \qquad xu^2, xw^2 \neq 0. \tag{5.18}$$

Lemma 5.9. *The tangent vectors w' and u'' are trivial.*

Proof. Let us prove it for w'. Since we can assume that w' does not depend on w, we can write it as $w' = \alpha x + \beta y + \gamma z + \delta u$. Multiplying Equation $(5.14)_I$ by x and y, we get $x^2w' = 0$ and $y^2w' = 0$ respectively. Moreover, multiplying by z Equation $(5.13)_I$, we have also $z^2w' = 0$. These last conditions yield

$$\alpha x^3 = \beta y^3 = \gamma z^3 = 0,$$

but since no vertex for the general triangle T belongs to $V(f)$, we have that $\alpha = \beta = \gamma = 0$. Finally, since we have just shown that $A' = 0$, from Equation $(5.14)_I$, we get

$$\delta u(y - Ax) = 0.$$

Since, from Equations (5.11), we get $uy = -Aux$, we would have $-2\delta Axu = 0$, which implies that $\delta = 0$, by Lemma 5.5.

Then $w' = 0$, as claimed. The same reasoning can also be used to prove that $u'' = 0$. □

Remark 5.10. From the above lemma, one can see that we can assume that also $E \neq 0$. Indeed, if $E \equiv 0$ locally (and thus, we can simply set $E' = E'' = 0$ in the above equations), from Equation $(5.13)_I$ we would have $w^2 = 0$. Hence, $[w]$ would be a singularity for $V(f)$, which is not possible.

We claim now that $wu^2 = 0$ and $w^2u = 0$. Multiplying by u Equation $(5.15)_I$, one gets

$$2Du^2w + zuu' = 0. \tag{5.19}$$

Since we have just shown that for T general, also the condition $zu^2 = 0$ is satisfied (see Equation (5.17)), we can deform it at the first order:

$$0 \equiv (z + tDw + sEu)(u + tu')^2 \mod \langle t, s\rangle^2 \qquad \text{and so} \qquad Du^2w + 2zuu' = 0. \tag{5.20}$$

Putting together Equations (5.19) and (5.20), one gets

$$u^2w = 0.$$

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Let us now do the same for the second claim. Multiplying by w the Equation (5.13)$_{II}$, we get $2Euw^2 + zww'' = 0$. Moreover, by deforming the condition $zw^2 = 0$ (see Equation (5.17)), we get $Euw^2 + 2zww'' = 0$. As before, putting together these last conditions, one gets

$$uw^2 = 0$$

by using $E \neq 0$ (see Remark 5.10).

We claim now that $zuw = 0$. In order to show this last claim, let us deform the condition just obtained (i.e., $u^2w = 0$):

$$0 \equiv (u + tu')^2(w + sw'') \mod \langle t, s\rangle^2 \qquad \text{and so} \qquad uwu' = 0.$$

Write u' as $\alpha x + \beta y + \gamma z + \delta w$ for simplicity. Since $xuw = yuw = uw^2 = 0$ and since $z^2w = 0$ by definition of w, one has

$$0 = uwu' = \gamma zuw \qquad z^2u' = \gamma z^3.$$

If $\gamma \neq 0$ we have done, let us then assume $\gamma = 0$: we get $z^2u' = 0$, and multiplying by z Equation (5.15)$_I$ we get $2Dzuw = 0$, as desired.

Finally, having $zuw = 0$ yields a contradiction: in this case, from Equations (5.11), we would have $xw^2 = 0$, which is not possible by Equation (5.18). Hence, neither ψ_2 nor ψ_3 can be diagonalizable for T general.

Case(II): For T general, the map ψ_2 is not diagonalizable. We can choose a basis $\{u, w\}$ of V_2 in such a way that ψ_2 is written in its Jordan normal form

$$\begin{bmatrix} A & 1 \\ 0 & A \end{bmatrix}.$$

As done before, the corresponding first-order deformations are

$$T + t\underline{v} = (x + tu, y + tAu, z + t(Cu + Dw)) \qquad T + s\underline{v}' = (x + sw, y + s(u + Aw), z + s(Eu + Fw)),$$

with C, D, E, F not all simultaneously zero (by Lemma 4.3).

These, using Lemma 2.11, yield

$$\begin{array}{ll} (\underline{v})_{xy} : Axu + yu = 0 & (\underline{v}')_{xy} : xu + Axw + yw = 0 \\ (\underline{v})_{xz} : Cxu + Dxw + zu = 0 & (\underline{v}')_{xz} : Exu + Fxw + zw = 0 \\ (\underline{v})_{yz} : Cyu + Dyw + Azu = 0 & (\underline{v}')_{yz} : Eyu + Fyw + zu + Azw = 0. \end{array} \tag{5.21}$$

Again, by elementary operations, one gets:

$$(2AC + D)ux + (2AD)wx = 0 \qquad (2AE + C + F)ux + (2AF + D)wx = 0$$

By Lemma 5.5, the multiplication map $x\cdot : V_2 \longrightarrow A_2$ is injective so

$$AD = 0 \qquad 2AC + D = 0 \qquad 2AF + D = 0 \qquad 2AE + C + F = 0. \tag{5.22}$$

Notice that in the case where $A \neq 0$, from the above relations, one easily sees that also $D = C = F = E = 0$, which is not possible, as we have stressed before, so we can assume

$$A = 0, \qquad D = 0 \qquad \text{and} \qquad F = -C.$$

One can then observe that the matrix associated to the endomorphism ψ_3 with respect to the basis u, w is of the form

$$\begin{bmatrix} C & E \\ 0 & -C \end{bmatrix}$$

If $C \neq 0$, the map ψ_3 would be diagonalizable: this is not possible for T general as proved in Case (I). We can then assume that

$$C = F = 0 \qquad \text{and} \qquad E \neq 0.$$

We are then considering the first-order deformations

$$T + t\underline{v} = (x + tu, y, z)) \qquad T + s\underline{v}' = (x + sw, y + su, z + sEu),$$

with $E \neq 0$, and the relations (5.21) are then equivalent to

$$yu = 0 \qquad zu = 0 \qquad xu + yw = 0 \qquad Eux + zw = 0. \tag{5.23}$$

By considering the $\underline{v}$-deformation and the $\underline{v}'$-deformation of the first two equations, we obtain

$$yu' = 0 \qquad yu'' + u^2 = 0 \qquad zu' = 0 \qquad zu'' + Eu^2 = 0. \tag{5.24}$$

We claim now that

$$\mathrm{Ann}_{A^1}(u^2) = \langle x, y, z, u\rangle \qquad u^2 w \neq 0. \tag{5.25}$$

Conditions $yu^2 = zu^2 = 0$ and $xu^2 = 0$ follow easily by multiplying by u or by x the equations in (5.23). One obtains $u^3 = 0$ from Equation $(5.24)_{II}$ after multiplying by u and by remembering that $yu = 0$. Since u^2 annihilates x, y, z and u, it cannot annihilate w too, since, otherwise, from the perfect pairing induced by the Gorenstein duality in A_f, $[u]$ would give a singular point for $V(f)$.

As a consequence of the above relation, notice that $([y], [z], [u])$ is a triangle for $\mathcal{H}_f$ since $yz = yu = zu = 0$. Hence, by Lemma 2.10, we have $\dim(\langle y^2, z^2, u^2\rangle) = 3$. Being $([x], [y], [z])$ a triangle and by Equation (5.25), we can conclude

$$\mathrm{Ann}_{A^1}(y^2, z^2, u^2) = \langle x, u\rangle. \tag{5.26}$$

We claim now that $u'' \in \langle x, u\rangle$. By the above relation, it is enough to show that $y^2u'' = z^2u'' = u^2u'' = 0$. The first relation comes from $(5.24)_{II}$ if we multiply both terms by y and use (5.26). One gets the second relation working on the Equation $(5.24)_{IV}$ and using $E \neq 0$. To get the third and last relation, let us simply observe that we have shown that the equation $u^3 = 0$ holds for the general triangle T in $\mathcal{F}$ and so, we can write its $\underline{v}'$-deformation:

$$0 \equiv (u + su'')^3 \mod s^2 \qquad \text{which yields} \qquad u^2u'' = 0,$$

as claimed.

As a consequence of the last claim, we can write $u'' = \alpha x + \beta u$ for suitable $\alpha, \beta \in \mathbb{K}$. Now consider Equation $(5.24)_{II}$ and recall that $yu = 0$ by Equation (5.23). By substituting, one obtains

$$0 = yu'' + u^2 = y(\alpha x + \beta u) + u^2 = u^2,$$

which is impossible by Proposition 2.1, since $V(f)$ is smooth. This concludes the analysis of Case (II) and, consequently, the proof of the main theorem for the case of cubic threefolds. □

6. Proof of main theorem: the cubic fourfold case

In this section, we prove Theorem A in the last remaining case: given *any* smooth cubic fourfold $X = V(f)$, the Hessian variety $\mathcal{H}_f$ is normal and irreducible if and only if f is not of TS type.

We set ourselves in the framework described in 5.4. We assume by contradiction that given $X = V(f)$, a smooth cubic fourfold, with f which is not of TS type the associated variety $\mathcal{H}_f$ is not normal. Then there exists an irreducible 3-dimensional family $\mathcal{F}$ of triangles for $\mathcal{H}_f$ with the first projection dominating a 3-dimensional component of $\mathrm{Sing}(\mathcal{H}_f)$. Fixing a general triangle $T = ([x],[y],[z]) \in \mathcal{F}$ for $\mathcal{H}_f$, let us now study the behaviour of ψ_2 and ψ_3 as endomorphisms of V_2. We distinguish the following mutually exclusive cases:

1. for the general T, ψ_2 (or ψ_3) has a Jordan decomposition with one Jordan block;
2. for the general T, ψ_2 (or ψ_3) has a Jordan decomposition with two Jordan blocks;
3. for the general T, ψ_2 and ψ_3 are diagonalizable.

We will rule out all the possibilities, by proving the following Lemmas 6.1 (for the case (a)), 6.2 (for (b)), 6.3 and 6.4 (both of them dealing with the case (c)). The last one, concerning a particular subcase of (c), is proved in the dedicated subsection 6.1.

Let us start by ruling out case (a).

Lemma 6.1. *For T general, neither the map ψ_2 nor ψ_3 can have a Jordan decomposition with only one block.*

Proof. Let us suppose, w.l.o.g, that ψ_2 has a Jordan decomposition with one Jordan block. We can choose a basis $\{u,v,w\}$ of $V_2 = \mathrm{Ann}_{A^1}(x^2,y^2,z^2)$ in such a way that ψ_2 is written in its Jordan normal form with w as eigenvector. Then we have a basis $\{x,y,z,u,v,w\}$ of A^1 with the first three vectors such that $x^3, y^3, z^3 \neq 0$. Then three independent tangent vectors to $\mathcal{F}$ in T are given as

$$\underline{v} = (u, Au+v, Bu+Cv+Dw), \quad \underline{v}' = (v, Av+w, Eu+Fv+Gw), \quad \underline{v}'' = (w, Aw, Hu+Iv+Lw)$$

for suitable scalars depending on the triangle.

As done in the case of threefolds, one uses Lemma 2.11 in order to obtain conditions from the first-order deformations associated to $\underline{v}$, $\underline{v}'$ and $\underline{v}''$. By elementary operations between these equations and by using Lemma 5.5, one gets the following relations on the coefficients appearing in the above description of the tangent vectors:

$$\begin{gathered} 2AB+E = 2AC+B+F = 2AD+C+G = 2AE+H = 0 \\ 2AF+E+I = 2AG+F+L = 2AH = 2AI+H = 2AL+I = 0. \end{gathered}$$

Note that if $A \neq 0$, then one has that all the other coefficients have to be 0, which is not possible since the second and third projections cannot send $\mathcal{F}$ to a point by Lemma 4.3. Hence, $A = 0$ and then $E = H = I = C+G = B+F = F+L = 0$. We can then write the above tangent vectors as

$$\underline{v} = (u, v, Lu - Gv + Dw), \quad \underline{v}' = (v, w, -Lv + Gw), \quad \underline{v}'' = (w, 0, Lw).$$

Then, the above-mentioned equations can be reduced to the following system of equations:

$$\begin{array}{ll} (\underline{v})_{xy} : xv + yu = 0 & (\underline{v})_{xz} : Lxu - Gxv + Dxw + zu = 0 \\ (\underline{v}')_{xy} : xw + yv = 0 & (\underline{v}')_{xz} : -Lxv + Gxw + zv = 0 \\ (\underline{v}'')_{xy} : yw = 0 & (\underline{v}'')_{xz} : Lxw + zw = 0. \end{array} \tag{6.1}$$

As usual, if T is deformed in the direction of $t\underline{v} + s\underline{v}' + r\underline{v}''$, we have the corresponding deformation $u + tu' + su'' + ru'''$ of u (and analogously the ones for v and w).

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Claim: $uw^2 \neq 0$, $L = 0$ and $zw = 0$.

First of all, let us study $\ker(w^2\cdot : A^1 \to A^3)$. Clearly, $yw^2 = 0$ by Equation $(\underline{v}'')_{xy}$. Moreover, if we multiply by w Equation $(\underline{v}')_{xy}$ and use Equation $(\underline{v}'')_{xy}$, we get $xw^2 = 0$. Similarly, one gets $zw^2 = 0$ upon multiplying by w Equation $(\underline{v}'')_{xz}$. Since Equation $(\underline{v}'')_{xy}$ holds for T general, one can deform it in the direction of $t\underline{v} + s\underline{v}' + r\underline{v}''$ and obtain

$$0 = (y + tv + sw)(w + tw' + sw'' + rw''') \mod (t, s, r)^2.$$

This yields

$$yw' + wv = 0 \qquad yw'' + w^2 = 0.$$

If we multiply by w these relations, we get $w^2v = 0$ and $w^3 = 0$. Hence, we have

$$\langle x, y, z, v, w\rangle \subseteq \ker(w^2\cdot : A^1 \to A^3).$$

Observe now that $uw^2 \neq 0$. Indeed, if $uw^2 = 0$, then we would also have $w^2 \cdot A^1 = \{0\}$ so, by Gorenstein duality, this would imply $w^2 = 0$, which contradicts the smoothness of the cubic fourfold $V(f)$.

Finally, by deforming Equation $(\underline{v}'')_{xz}$ and multiplying by w, using the various vanishings obtained before, we get $2Luw^2 = 0$, and thus, $L = 0$. Then one has the claim by Equation $(\underline{v}'')_{xz}$.

Claim: $zv = 0$, $G = 0$ and $D \neq 0$.

By deforming equations $(\underline{v})_{xy}$ in the direction of $t\underline{v} + s\underline{v}'$ we get, respectively,

$$xv' + 2uv + yu' = 0 \qquad \text{and} \qquad xv'' + v^2 + yu'' + uw = 0.$$

If one multiplies these by z, one obtains $zuv = zv^2 = 0$. One can now observe that $\ker(zv\cdot : A^1 \to A^3) = A^1$, so by Gorenstein duality, one has $zv = 0$, as claimed. Since $xw \neq 0$ (by Lemma 5.5), from Equation $(\underline{v}')_{xz}$ one obtains $G = 0$ and, consequently, by Lemma 4.3, also $D \neq 0$.

Claim: $uw^2 = 0$.

Since $zv = 0$ for T general, we can deform this equation in the direction of $t\underline{v}$. We get $zv' + Dvw = 0$, and so $Dv^2w = 0$, if we multiply by v. Being $D \neq 0$, one has also $v^2w = 0$. Let us now deform $(\underline{v})_{xz}$ and $(\underline{v}')_{xy}$ in the direction of $t\underline{v}$ in order to get

$$D'xw + Dxw' + 2Duw + zu' = 0 \qquad \text{and} \qquad xw' + wu + yv' + v^2 = 0.$$

Upon multiplying by w, one gets $xww' + 2uw^2 = xww' + uw^2 = 0$, which yields $xww' = uw^2 = 0$. This is impossible as observed in the first claim above. □

Let us now prove that case (b) cannot be realised.

Lemma 6.2. *For T general, neither the map ψ_2 nor ψ_3 can have a Jordan decomposition with two blocks.*

Proof. Let us suppose, w.l.o.g, that ψ_2 has a Jordan decomposition with two Jordan blocks. As done in Lemma 6.1, we can choose a basis $\{u, v, w\}$ of $V_2 = \mathrm{Ann}_{A^1}(x^2, y^2, z^2)$ in such a way that ψ_2 is written in its Jordan normal form with v and w as eigenvectors. Then, in this case, we can write three independent tangent vectors to $\mathcal{F}$ in T as

$$\underline{v} = (u, Au + v, Cu + Dv + Ew), \quad \underline{v}' = (v, Av, Fu + Gv + Hw), \quad \underline{v}'' = (w, Bw, Iu + Lv + Mw)$$

for suitable scalars depending on the triangle.

As done in the previous cases, one gets the following relations involving the coefficients appearing in the above description of the tangent vectors:

$$2AC + F = 2AD + C + G = AF = 2AG + F = BM = 0$$
$$E(A+B) + H = H(A+B) = I(A+B) = L(A+B) + I = 0.$$

We distinguish four cases, depending on the vanishing of the two eigenvalues.

Case (I): A = B = 0.

Since $A = B = 0$, we also have $F = H = I = C + G = 0$. Among the various equations obtained by deforming at first order the general triangle T, one gets

$$yv = 0 \qquad yw = 0. \tag{6.2}$$

We claim now that $uv^2 \neq 0$. By deforming at first order the equation $yv = 0$ in the direction of $t\underline{v}$, one gets $yv' + v^2 = 0$, which implies that

$$\langle x, y, z, v, w\rangle \subseteq \ker(v^2). \tag{6.3}$$

However, this has to be an equality; otherwise, we would have $v^2 = 0$ by Gorenstein duality. In particular, $uv^2 \neq 0$.

From the Equation $yv' + v^2 = 0$, one can also see that $yv' \neq 0$; otherwise, we would contradict the smoothness of $V(f)$. We claim now that $yv' = 0$, so we conclude Case (I).

Since T is a triangle and by Equations (6.2), we have $\langle x, z, v, w\rangle \subseteq \ker(y\cdot : A^1 \to A^2)$, so in order to prove $yv' = 0$, it is enough to show that v' does not depend on y and u. One easily sees that $0 = y(yv' + v^2) = y^2v'$. Since, by assumption, $\langle x, z, u, v, w\rangle = \ker(y^2\cdot : A^1 \to A^3)$, we get that v' does not depend on y. Moreover, since we have just shown in Equation (6.3) that for T general, also the equation $v^3 = 0$ holds, we can deform it and in the same way, one proves that v' does not depend on u.

Case (II): A = 0, B ≠ 0.

In this case, we also have $E = F = C + G = H = I = L = M = 0$ so that

$$\underline{v} = (u, v, Cu + Dv), \quad \underline{v}' = (v, 0, -Cv), \quad \underline{v}'' = (w, Bw, 0).$$

Among the equations deduced by deforming the general triangle, one gets the conditions

$$yv = 0 \qquad zw = 0 \qquad xv + yu = 0 \qquad Bxw + yW = 0 \qquad -Cxv + zv = 0 \qquad Cxu + Dxv + zu. \tag{6.4}$$

We claim now that $C = 0$. First of all, notice that

$$\langle x, y, z, v, w\rangle \subseteq \ker(xv).$$

Indeed, we have $v \in V_2$ by assumption so x^2v, and since T is a triangle, we also have $xyv = xzv = 0$. The last two vanishing can be easily obtained form Equations (6.4) upon a multiplication by v and w (and by recalling that we are assuming $B \neq 0$). In particular, we have $xuv \neq 0$, by Lemma (5.5).

Then by taking the fifth and sixth equations in (6.4) multiplied by u and v, respectively, one has

$$-Cxuv + zuv = Cxuv + zuv = 0$$

so $Cxuv = 0$. Then, since $xuv \neq 0$, we have necessarily $C = 0$ for the general triangle, and so $D \neq 0$ by Lemma 4.3. In particular, we have $\psi_3(u) = Dv \neq 0$ and $\psi_3(v) = \psi_3(w) = 0$. Hence, for the general

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triangle T, ψ_3 is not diagonalizable and has two Jordan blocks with eigenvalues both equal to 0. This is impossible, as seen in Case (I).

Case (III): $\mathbf{A \neq 0, B = 0}$.

This case can be treated in a 'geometric' way as done in Lemma 5.8. Indeed, since $A \neq 0$ and $B = 0$, we have also all the other variables, besides M, are zero. Moreover, by Lemma 4.3, $M \neq 0$. In particular, the tangent vectors to $\mathcal{F}$ in T are spanned by

$$\underline{v} = (u, Au + v, 0), \quad \underline{v}' = (v, Av, 0), \quad \underline{v}'' = (w, 0, Mw),$$

and the varieties $\pi_i(\mathcal{F}) = Y_i$ have dimension $3, 2$ and 1, respectively. As done in the other cases, by studying the relations coming from the deformation at the first order, one can easily see that Y_3 is contained in $\mathcal{D}_2(f)$ and Y_2 is a surface living in $\mathcal{D}_3(f)$. Since Y_2 cannot be contained in $\mathcal{D}_2(f)$ (otherwise, we would have singular points for $V(f)$ by Proposition 2.5), for the general $[y] \in Y_2$, we have that $\iota([y]) \simeq \mathbb{P}^2$.

For a general triangle $T = ([x], [y], [z])$, consider the curve $C_T = \pi_2^{-1}([y])$. The tangent to C_T in T is generated by $\underline{v}''$ which is projected to w and Mw via $d_T(\pi_1|_{C_T})$ and $d_T(\pi_3|_{C_T})$, respectively. Hence, $\pi_1(C_T)$ is a curve in Y_1 and $\pi_3(C_T) = Y_3$.

In particular, $\pi_1(C_T) \cup Y_3 \subseteq \iota([y]) \simeq \mathbb{P}^2$. By varying the point $[y]$, the kernel has to move, since the curve $\pi_1(C_T)$ has to cover the threefold Y_1. Hence, Y_3 lies in the intersection of distinct projective planes: we have $\mathbb{P}^1 \simeq Y_3$ and thus a line in $\mathcal{D}_2(f)$. This implies, by Theorem 3.4, that f is of TS type, against our assumptions.

Case (IV): $\mathbf{A, B \neq 0}$.

First of all, notice that assuming $A, B \neq 0$ implies that $A + B = 0$. Indeed, if we assume also $A + B \neq 0$, we would obtain that all the other coefficients are equal to 0. This is impossible by Lemma 4.3. Then

$$\underline{v} = (u, Au + v, Ew), \quad \underline{v}' = (v, Av, 0), \quad \underline{v}'' = (w, Bw, Lv),$$

with E, L not both zero. In particular, ψ_3 is not diagonalizable for the general triangle T and all its eigenvalues are zero. This is impossible as seen in the previous cases. □

As a consequence of Lemmas 6.1 and 6.2, the maps ψ_2 and ψ_3 have to be diagonalizable. In what follows, we rule out this remaining case, splitting it up into two lemmas, the second of which is postponed in the following subsection.

Lemma 6.3. *For T general, neither the map ψ_2 nor ψ_3 can be diagonalizable.*

Proof. As a consequence of Lemma 6.1 and 6.2, we have that ψ_2 and ψ_3 are both diagonalizable for the general triangle. We can choose a basis $\{u, v, w\}$ of $V_2 = \mathrm{Ann}_{A^1}(x^2, y^2, z^2)$ in such a way that ψ_2 is in diagonal form. Thus, three independent tangent vectors to $\mathcal{F}$ in T are

$$\underline{v} = (u, Au, Du + Ev + Fw), \quad \underline{v}' = (v, Bv, Gu + Hv + Iw), \quad \underline{v}'' = (w, Cw, Lu + Mv + Nw)$$

for suitable coefficients depending on the triangle.

As done so far, one gets the following relations:

$$\begin{aligned} AD = BH = CN = 0 \qquad & E(A+B) = G(A+B) = 0 \\ F(A+C) = L(A+C) = 0 \qquad & I(B+C) = M(B+C) = 0. \end{aligned}$$

Notice that A, B and C cannot be all equal to zero by Lemma 4.3. Hence, we distinguish three cases, depending on the vanishing of the three eigenvalues.

Case (I): $\mathbf{A = B = 0}$ and $\mathbf{C \neq 0}$.

In this case, one can easily see that three tangent vectors to $\mathcal{F}$ at T can be written as

$$\underline{v} = (u, 0, Du + Ev), \quad \underline{v}' = (v, 0, Gu + Hv), \quad \underline{v}'' = (w, Cw, 0)$$

for suitable coefficients so that $\dim(Y_1) = 3$, $\dim(Y_2) = 1$ and $\dim(Y_3) \in \{1, 2\}$.

- **Claim:** $\dim(Y_3) = 1$ and $Y_2 \subseteq \mathcal{D}_2(f)$.

  Among the equations obtained by deforming the general triangle T, one has $yu = yv = 0$. Moreover, by definition of triangle, we clearly have also $yx = yz = 0$: the general $[p] \in Y_2$ lives in $\mathcal{D}_2(f)$; that is, for $[p] \in Y_2$, one has $\iota([p]) \simeq \mathbb{P}^3$ (since $\mathcal{D}_1(f)$ is empty by hypothesis). In the same way, since one gets $zw = 0$, one also has $Y_3 \subseteq \mathcal{D}_3(f)$. Notice that the surface $\pi_2^{-1}([p]) = S_T$ projects onto a surface in Y_1 via π_1 and dominates Y_3 via π_3. This means that $Y_3 \subset \iota([p]) \simeq \mathbb{P}^3$. Observe that if $[q_1]$ and $[q_2]$ are distinct points of Y_2, then their kernels have to be distinct: indeed, if $\Lambda := \iota([q_1]) = \iota([q_2])$, then by symmetry, $\iota(\Lambda) \supset \mathbb{P}(\langle [q_1], [q_2] \rangle$ (i.e., $\Lambda \subseteq \mathcal{D}_4(f)$), which is not possible by Theorem 3.4. Then, for $[q_1]$ and $[q_2]$ two distinct general points of Y_2, one gets that $Y_3 \subseteq \iota([q_1]) \cap \iota([q_2]) \simeq \mathbb{P}^s$ with $s \in \{1, 2\}$. From this, one can see that $\dim(Y_3) = 1$: indeed, if Y_3 is a surface, then we necessarily have $s = 2$ and $Y_3 \simeq \mathbb{P}^2$, but this means there exists a projective plane in $\mathcal{D}_3(f)$, which is impossible by Theorem 3.4
- **Claim:** $Y_3 \subseteq \mathcal{D}_2(f)$.

  Since Y_3 is a curve, the endomorphism $\psi_3|_{\langle u,v \rangle}$ has necessarily rank 1 (i.e., there exists a vector $au + bv$ that is sent to 0 by ψ_3). Among the first-order conditions given by the tangent vectors above, one has

$$(Du + Ev)x + zu = (Gu + Hv)x + zv = 0.$$

  Then, since $\psi_3(au+bv) = a(Du+Ev)+b(Gu+Hv) = 0$, one has $z(au+bv) = 0$. Hence, $\iota([z]) \simeq \mathbb{P}^3$, and we have that Y_3 is contained in $\mathcal{D}_2(f)$ too.

  Being f not of TS type, and being Y_3 a curve in $\mathcal{D}_2(f)$, we have that Y_3 is not a line. However, for $[q_1]$ and $[q_2]$ distinct general points in Y_2 as above, $Y_3 \subseteq \iota([q_1]) \cap \iota([q_2]) \simeq \mathbb{P}^2$, so the general triangle $T = ([x], [y], [z])$ is such that $\iota([y]) \simeq \mathbb{P}^3$ contains a fixed $\mathbb{P}^2$, denoted by Π, which coincides with the projective plane spanned by Y_3. To conclude, let us take a general point $[\eta] \in \Pi$: by symmetry, the general $[p] \in Y_2$ is such that $[p] \in \iota([\eta])$, and so $Y_2 \subset \iota([\eta]) \simeq \mathbb{P}^r$. Since $Y_2 \neq \mathbb{P}^1$, we have that $r \geq 2$: this means that $\Pi \simeq \mathbb{P}^2 \subseteq \mathcal{D}_3(f)$, which yields a contradiction as above.

Let us stress that having a $\mathbb{P}^2$ contained in $\mathcal{D}_3(f)$ is a phenomenon that happens exactly when $V(f)$ is a smooth, non-cyclic cubic of TS type as described in the specific Example 3.9.

Case (II): $\mathbf{A = 0}$ and $\mathbf{B, C \neq 0}$.
This case cannot occur. It will be treated in Lemma 6.4.

Case (III): $\mathbf{A, B, C \neq 0}$.
Being $A, B, C \neq 0$, one has $D = H = N = 0$. Notice that the three values $A + B$, $A + C$ and $B + C$ cannot be simultaneously zero; moreover, at least one of them has to be 0, since otherwise we would get $D = E = F = G = H = I = L = M = N = 0$, which is impossible by Lemma 4.3. W.l.o.g, we distinguish 2 cases: either $A + C = 0$ and $A + B, B + C \neq 0$ or $A + B = A + C = 0$ and $B + C \neq 0$. In the first case, ψ_3 is diagonalizable with one zero eigenvalue, whereas in the second case, one has the same conclusion or that ψ_3 is not diagonalizable and its Jordan normal form has 1 Jordan block with 0 as the only eigenvalue. Both conclusions yield a contradiction as observed in the previous cases or in Lemma 6.1. □

6.1. The end of the proof

To end the proof of Theorem 5.1, we have to rule out a last remaining possibility which could arise in the case where the both the maps ψ_2 and ψ_3 are diagonalizable (case (c), as stated at the beginning of Section 6). This subsection is devoted to this subcase, which is ruled out with the following Lemma 6.4, which yields also the end of the proof of the main theorem. To prove this last Lemma, we start by analyzing the usual framework 5.4 and obtaining different relation that both the vertices of the general triangle and the tangent vectors to it have to satisfy. After that, we will use these conditions to reconstruct

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the cubic fourfolds which the framework is, in this case, associated with, showing that these do not actually satisfy the hypotheses we are setting.

Lemma 6.4. *For T general, neither the map ψ_2 nor ψ_3 can be diagonalizable with dimension of the kernel equal to* 1.

Proof. We refer to the notations introduced at the beginning of Lemma 6.3. W.l.o.g, we can set $A = 0$ so that $B, C \neq 0$ by hypothesis. Then, one has $E = F = G = H = L = N = 0$. First of all, notice that $B + C = 0$. Indeed, otherwise, we would get $I = M = 0$, so v and w would be two eigenvectors for ψ_3 with associated eigenvalue 0. This is impossible as observed in Case (I).

Then, the tangent vectors to $\mathcal{F}$ at T can be written as

$$\underline{v} = (u, 0, Du), \quad \underline{v}' = (v, Bv, Iw), \quad \underline{v}'' = (w, -Bw, Mv)$$

with $B \neq 0$ and $(D, I, M) \neq (0, 0, 0)$ by Lemma 4.3. Moreover, notice that $I, M \neq 0$ since, otherwise, we would have that ψ_3 is not diagonalizable (and this cannot happen for T general by Lemma 6.2).

The first-order conditions obtained as a consequence of Lemma 2.11 are

$$\begin{array}{ll} (\underline{v})_{xy} : yu = 0 & (\underline{v})_{xz} : (Dx + z)u = 0 \\ (\underline{v}')_{xy} : (Bx + y)v = 0 & (\underline{v}')_{xz} : Ixw + zv = 0 \\ (\underline{v}'')_{xy} : (-Bx + y)w = 0 & (\underline{v}'')_{xz} : Mxv + zw = 0 \end{array} \tag{6.5}$$

Consider the following subsets of $\mathcal{L} = \mathbb{K}[x, y, z, u, v, w]$:

$$\mathcal{M}^{nv} = \{x^3, y^3, z^3, xu^2, xv^2, xw^2, yv^2, yw^2, zvw, uv^2\}$$

$$M_0 = \{xy, xz, yz\} \cup \left(\{x^2, y^2, z^2\} \cdot \{u, v, w\}\right) \qquad M_1 = \{yu, xuv, xuw, zuv, zuw\}$$

$$M_2 = \{u^2v, u^2w\} \qquad M_3 = \{xvw, yvw, zv^2, zw^2\} \qquad M_4 = \{v^3, w^3\} \qquad M_5 = \{v^2w, vw^2\}$$

and the elements

$$r_1 = x(Iw^2 - Mv^2) \qquad r_2 = u(Mv^2 - Iw^2) \quad \text{and} \quad r_3 = u(Dw^2 + Mvw).$$

Notice that all the monomials in M_0 are 0 in $A_f = \mathcal{L}/\mathrm{Ann}_{\mathcal{L}}(f)$ by the conditions imposed by our framework. We want to prove that the same holds for all the elements in M_i for $i \in \{1, \ldots, 5\}$ and for r_1, r_2 and r_3, whereas all the monomials in $\mathcal{M}^{nv}$ are *not* 0 (in A_f).

If T is deformed in the direction of $t\underline{v} + s\underline{v}' + r\underline{v}''$, the corresponding first-order deformation of u is written as $u + tu' + su'' + ru'''$ (and analogously the ones for v, w, B, D, I and M).

Claim: All the monomials in M_1 and in M_2 are 0.

One has $yu = 0$ from Equation $(\underline{v})_{xy}$. Upon multiplying by u, the other equations in (6.5), one gets the vanishing for the monomials in M_1. By deforming Equation $(\underline{v})_{xy}$ in the direction of $s\underline{v}' + r\underline{v}''$, we get

$$yu'' + Buv = yu''' - Buw = 0.$$

Multiplying by u these relations and by using the vanishing $yu = 0$, one gets the claim.

Claim: All the monomials in M_3 are 0.

Observe that it is enough to show hat $xvw = 0$: all the other vanishings come from Equations (6.5) after multiplication by v or w and vanishing in M_1 or M_2.

Let us deform Equation $(\underline{v}')_{xy}$ in the direction of $r\underline{v}''$:

$$B'''xv + Bxv''' + yv''' = 0. \tag{6.6}$$

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Multiplying by x, we get the relation $x^2 v''' = 0$. Recalling that $V_2 = \langle u, v, w \rangle = \mathrm{Ann}_{A_1}(x^2, y^2, z^2)$, let us consider the vanishing $x^2 v = 0$: its deformation in the direction of $r\underline{v}''$ yields $x^2 v''' + 2xvw = 0$. Since, as just shown, $x^2 v''' = 0$, we get $xvw = 0$, as claimed.

Claim: All the monomials in $\mathcal{M}^{nv}$ besides uv^2 are not 0 and $B'' = B''' = 0$.

We have $x^3, y^3, z^3 \neq 0$ by assumption, since the general triangle of $\mathcal{F}$ cannot have a vertex on the cubic fourfold. As a consequence of the framework and since the monomials in M_1 and M_3 are 0, we get that $\langle x, y, z, u, w \rangle \subseteq \ker(xv\cdot : A^1 \to A^3)$. Hence, xv^2 cannot be zero; otherwise, by Gorenstein duality, also xv would be 0, which is not possible as observed in Lemma 5.5. In the same way, one gets that also xu^2 and xw^2 are not 0. For the remaining monomials, they have to be different from 0; otherwise, one would get a contradiction multiplying the equations in (6.5) by u, v or w.

For the second claim, observe that multiplying by v Equation (6.6), since $(Bx + y)v = 0$ by Equation $(\underline{v}')_{xy}$, one gets $B'''xv^2 = 0$. Being $xv^2 \neq 0$, as just shown, we have also that $B''' = 0$, as claimed. In order to show that $B'' = 0$, one proceeds in an analogous way by deforming Equation $(\underline{v}'')_{xy}$ in the direction of $s\underline{v}'$:

$$-B''xw - Bxw'' + yw'' = 0. \tag{6.7}$$

One gets the claim by multiplying by w.

Claim: The monomials in M_4 are 0.

Let us consider the first-order deformation of $(\underline{v}')_{xy}$ and $(\underline{v}'')_{xy}$ in the direction of $s\underline{v}'$ and $r\underline{v}''$, respectively:

$$B''xv + Bxv'' + 2Bv^2 + yv'' = 0 \qquad -B'''xw - Bxw''' - 2Bw^2 + yw''' = 0. \tag{6.8}$$

Since $B'' = B''' = 0$ as shown in the previous claim, if one multiplies the above Equations (6.8) by v and w, respectively, one gets the claim.

Claim: One has $w'' = v''' = 0$ as tangent vectors.

Let us start by proving that $w'' = 0$. Since, by construction, we have that $w \in V_2$, one can deform in the direction of $s\underline{v}'$ the relations $x^2 w = y^2 w = z^2 w = 0$. Recalling that $xvw = yvw = zw^2 = 0$ by the previous claims, one obtains that $w'' \in V_2$, so we can write $w'' = \alpha u + \beta v + \gamma w$. By substituting this expression in Equation (6.7), one has

$$-\alpha Bxu + \beta(-Bx + y)v = 0,$$

which, if multiplied by u, gives $\alpha Bxu^2 = 0$. Since $Bxu^2 \neq 0$, one has $\alpha = 0$.

Being $\alpha = 0$, it follows $\beta(-Bx + y)v = 0$ from the above equation. However, one has $(Bx + y)v = 0$ (see Equation $(\underline{v}'_{xy})$), so $\beta = 0$. Indeed, otherwise, we would get $xv = 0$, which is not possible by Lemma 5.5. This means that $w'' = 0$ in $A^1/\langle w \rangle$.

In order to get $v''' = 0$, one proceeds in a similar way: first of all, one proves $v''' \in V_2$ starting from $v \in V_2$ and by using previous vanishings. Then, by substituting in Equation (6.6) and by using Equation $(\underline{v}')_{xy}$, one concludes as above.

Claim: The monomials in M_5 are 0.

We have shown that $zv^2 = 0$ and $xvw = 0$ for the general triangle T, so we can deform these equations in the direction of $s\underline{v}'$. By using Equation $(\underline{v})_{xz}$ and $w'' = 0$, one can write these relations as

$$Iv^2 w + 2vzv'' = I(v^2 w - 2xwv'') = 0 \qquad v^2 w + xwv'' = 0.$$

As observed above, I is not 0; thus, we deduce $v^2 w = 0$.

For the vanishing $vw^2 = 0$, one works in a similar way by deforming $zw^2 = 0$ and $xvw = 0$ in the direction of $r\underline{v}''$, and by using Equation $(\underline{v}'')_{xz}$, $v''' = 0$ and $M \neq 0$.

Claim: $v' \in \langle v, w\rangle$ and u' does not depend on y.

Proceeding as we have done above for proving $v''', w'' \in V_2$, one can also obtain that $v' \in V_2 = \langle u, v, w\rangle$. Consider the first-order deformation of the Equation $(\underline{v}')_{xy}$ in the direction of $t\underline{v}$, namely

$$B'xv + Bxv' + Buv + yv' = 0.$$

Upon multiplication by u, using $B \neq 0$ and the various vanishing shown above, one gets $xuv' = 0$. Since $xu^2 \neq 0$ and $xu \cdot \langle x, y, z, v, w\rangle = 0$, one has that $v' \in \langle v, w\rangle$.

For the second claim, by deforming $(\underline{v})_{xy}$ in the direction of $t\underline{v}$, one gets $yu' = 0$ so $y^2u' = 0$, and this implies that u' does not depend on y, since $y^3 \neq 0$ and $y^2 \cdot \langle x, z, u, v, w\rangle = 0$.

Claim: $uv^2 \neq 0$ in A_f.

Assume, by contradiction, that $uv^2 = 0$. We claim that $v' = 0$ as tangent vector. Consider the first-order deformation of the Equation $zv^2 = 0$ in the direction of $t\underline{v}$ – that is,

$$0 = Duv^2 + 2zvv' = 2zvv'.$$

Since $v' \in \langle v, w\rangle$ (by the previous claim), $zv^2 = 0$ and $zvw \neq 0$, one has that $v' = 0$ as tangent vector.

Since we are assuming that $uv^2 = 0$ for the general triangle in $\mathcal{F}$, we can deform this equation in the direction of $t\underline{v}$. This operation yields the relation $0 = v^2u' + 2uvv' = v^2u'$. Now recall that u' does not depend on y, $v^2 \cdot \langle z, v, w\rangle$ by previous vanishings and $uv^2 = 0$, by assumption. Since $xv^2 \neq 0$, from $v^2u' = 0$, one has that u' does not depend on x.

This yields a contradiction by deforming $x^2u = 0$ in the direction of $t\underline{v}$. Indeed, one has $0 = x^2u' + 2xu^2 = 2xu^2$ but $xu^2 \neq 0$.

Claim: Elements r_1, r_2 and r_3 are 0.

The relation $r_1 = x(Iw^2 - Mv^2) = 0$ is easily obtained from Equations $(\underline{v}')_{xz}$ and $(\underline{v}'')_{xz}$ upon multiplication by w and v, respectively.

We prove now that $r_2 = u(Mv^2 - Iw^2) = 0$. Consider the first-order deformation in the direction of $s\underline{v}'$ of the Equations $zuw = 0$ and $xuv = 0$, together with Equation $(6.8)_I$ multiplied by u, namely

$$Iuw^2 + zuw'' + zwu'' = 0 \quad uv^2 + xuv'' + xvu'' = 0 \quad Bxuv'' + 2Buv^2 + yuv'' + B''xuv = 0. \tag{6.9}$$

Now, since $w'' = zw + Mxv = 0$, and $B'' = yu = 0 \neq B$, we have

$$Iuw^2 - Mxvu'' = 0 \qquad xuv'' + 2uv^2 = 0,$$

which give the desired relation, if substituted into Equation $(6.9)_{II}$.

The other relation, namely $r_3 = u(Dw^2 + Mvw) = 0$, is obtained in a similar way from the first-order deformation in the direction of $r\underline{v}''$ of the Equations $(\underline{v})_{xy}$ and $(\underline{v})_{xz}$ upon multiplication by suitable elements (more precisely, the first one by w and Mv and the second one by w, respectively).

To sum up, we have proved that if we define

$$\mathcal{R} = \{r_1, r_2, r_3\} \cup \left(\bigcup_{i=0}^{5} M_i\right) \cup \{\text{LHS of relations in (6.5)}\},$$

then

$$\mathcal{R} \subseteq \mathrm{Ann}_{\mathcal{L}}(f) \qquad \text{and} \qquad \mathcal{M}^{nv} \cap \mathrm{Ann}_{\mathcal{L}}(f) = \emptyset. \tag{6.10}$$

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Now we would like to partially reconstruct the cubic fourfold f from the information about its apolar ring A_f obtained so far. For simplicity, we are using the same symbols for the indeterminates in $S = \mathbb{K}[x_0, \dots, x_5]$ and in $\mathcal{L} = \mathbb{K}[y_0, \dots, y_5] = \mathbb{K}[x, y, z, u, v, w]$. Consider the following cubics in S^3:

$$s_0 = x^3 \quad s_1 = y^3 \quad s_2 = z^3 \quad s_3 = (x - Dz)u^2 \quad s_6 = u^3$$

$$s_4 = x(Iv^2 + Mw^2) + yB(Mw^2 - Iv^2) - 2IMzvw \quad s_5 = -2Duvw + u(Iv^2 + Mw^2).$$

It is easy to see that

$$W = \langle s_i \rangle_{i=0}^5 = \{ f \in S^3 \mid \mathcal{R} \subseteq \mathrm{Ann}_D(f) \}.$$

This can be checked directly by hand by writing $f = \sum \alpha_m \cdot m$, where m runs over the set of monomials of degree 3 in S. Each element in $\mathcal{R}$ is a linear differential equation satisfied by f and thus gives a linear closed condition on the vector space S^3. For example, since $r_2 = u(Mv^2 - Iw^2) \in \mathcal{R}$, we have the corresponding condition $2M\alpha_{uv^2} - 2I\alpha_{uw^2} = 0$ on the coefficients of f.

Hence, *any* cubic polynomial that we are analysing in this case can be written as $f = \sum_{i=0}^6 p_i s_i$ for suitable $p_i \in \mathbb{K}$. Having proved that $\mathcal{M}^{nv} \cap \mathrm{Ann}_{\mathcal{L}}(f) = \emptyset$ gives nontrivial open conditions, indeed, it is translated into

$$p_0, p_1, p_2, p_3, p_4, p_5 \neq 0, \tag{6.11}$$

so all the cubic fourfolds satisfying Conditions (6.10) live in a dense open subset of $|W|$. Notice that the base locus of $|W|$ is the line $L = V(x, y, z, u)$. Moreover, as $p_4, B, I, M \neq 0$ and since

$$y_0(f)|_L = p_4(Iv^2 + Mw^2) \qquad y_1(f)|_L = p_4 B(Mw^2 - Iv^2),$$

we have that the general cubic in $|W|$ is indeed smooth on the points of L and thus smooth everywhere by Bertini.

Claim: One has $D \neq 0$.

Consider the first-order deformation of $v^2 w$ and of $z^2 v$ in the direction of $s\underline{v}'$, namely

$$2vwv'' + w''v^2 = 2vwv'' = 0 \qquad z^2 v'' + 2Izvw = 0, \tag{6.12}$$

where we also used $w'' = 0$. Assume, by contradiction, that $D = 0$. Then, from the expression of f and as $p_5 \neq 0$, we have that $uvw = 0$ in A_f. Hence, $vw \cdot \langle x, y, u, v, w \rangle = 0$. Then, by previous vanishings and since $zvw \neq 0$, from Equation $(6.12)_I$, we get that v'' does not depend on z. This implies that $z^2 v'' = 0$, so Equation $(6.12)_{II}$ yields $2Izvw = 0$, and thus, $I = 0$, which is impossible.

Claim: If f satisfies the Conditions in (6.10), then $\mathrm{Sing}(\mathcal{H}_f)$ is of dimension 2 near $[x]$.

By changing coordinates, we can simplify a little the expression of f. Indeed, as B, D, I, M, p_4 and p_5 are not 0, by an easy change of coordinates, and by redefining the p_is, one can write

$$\begin{aligned} 2f = p_0\left(x^3\right) + p_1\left(y^3\right) + p_2\left(z^3\right) + p_3\left((x - z)u^2\right) + p_6\left(u^3\right) + \\ + (x + u)(w^2 + v^2) + y(w^2 - v^2) - 2\left(\lambda z + \lambda^{-1} u\right) vw \end{aligned} \tag{6.13}$$

with $\lambda, p_0, p_1, p_2, p_3 \neq 0$.

By construction, $[x] \in \mathbb{P}(A^1) \leftrightsquigarrow (1 : 0 : 0 : 0 : 0 : 0) \in \mathbb{P}^n$ is a vertex of a triangle for $\mathcal{H}_f$ so $[x] \in \mathrm{Sing}(\mathcal{H}_f)$. We are assuming also that there exists a family of dimension $5 - 2 = 3$ whose general element is a triangle dominating via the first projection a component of dimension 3 of $\mathrm{Sing}(\mathcal{H}_f)$. Then, in order to conclude the proof of the lemma, it is enough to show that the local dimension of $\mathrm{Sing}(\mathcal{H}_f)$ near $[x]$ is actually 2.

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The Hessian matrix of f is

$$H_f = \begin{bmatrix} 3p_0x & 0 & 0 & p_3u & v & w \\ 0 & 3p_1y & 0 & 0 & -v & w \\ 0 & 0 & 3p_2z & -p_3u & -\lambda w & -\lambda v \\ p_3u & 0 & -p_3u & p_3x - p_3z + 3p_6u & v - \lambda^{-1}w & -\lambda^{-1}v + w \\ v & -v & -\lambda w & v - \lambda^{-1}w & x - y + u & -\lambda z - \lambda^{-1}u \\ w & w & -\lambda v & -\lambda^{-1}v + w & -\lambda z - \lambda^{-1}u & x + y + u \end{bmatrix} \tag{6.14}$$

Since f is smooth, we have that $\mathrm{Sing}(\mathcal{H}_f) = \mathcal{D}_4(f)$. In particular, $\mathrm{Sing}(\mathcal{H}_f)$ is cut out by 21 quintic equations corresponding to the minors of order 5 of the Hessian matrix (there are 36 minors but 15 appear twice since H_f is symmetric). Let m_{ij} be the minor obtained by removing the i-th row and the j-th column. We are interested in the local expression of $\mathrm{Sing}(\mathcal{H}_f)$ near $[x]$. Notice that the linear form $(y_iy_j)(f)$ (defined as the second partial derivative of f as in Equation (2.1)), depends on x if and only if $(i,j) \in \{(0,0),(3,3),(4,4),(5,5)\}$, so no term of h_f can have as exponent of x an integer greater than 4: this is a confirmation of the fact that $[x] \in \mathrm{Sing}(\mathcal{H}_f)$. By differentiating m_{ij}, it is easy to see that $[x]$ is singular for $V(m_{ij})$ if $(i,j) \notin \{(1,1),(1,2),(2,1),(2,2)\}$. Consider the variety $Z = V(m_{11}, m_{12}, m_{22})$ and notice that $\mathrm{Sing}(\mathcal{H}_f) \subseteq Z$ by construction. Being defined by 3 equations, one has that $\dim(Z) \geq 2$. We claim that Z has dimension 2 near $[x]$; to do that, we will show that $(1:0:0:0:0:0)$ is isolated in $Z \cap V(u,v) = V(m_{11}, m_{12}, m_{22}, u, v)$.

We compute now the local expression of m_{11}, m_{12} and m_{22} modulo (u,v) in the local ring A_m, where $A = \mathbb{K}[y,z,u,v,w]$ and m is the maximal ideal of the origin in $\mathbb{A}^5$. By the explicit expression of H_f in Equation (6.14), one can easily see that

$$m_{12}(1,y,z,0,0,w) = -3p_0 \cdot w^2 \cdot (w^2 - p_3\lambda^2 z(1-z)) = 0, \tag{6.15}$$

so one between w and $w^2 - p_3\lambda^2 z(1-z)$ is zero.

Assume first that $w = 0$. By a direct computation, one can see that

$$m_{11}(1,y,z,0,0,0) = 9p_0p_2p_3 \cdot z(z-1)(y^2 + \lambda^2 z^2 - 1) \sim z$$

$$m_{22}(1,y,z,0,0,0) = 9p_0p_1p_3 \cdot y(z-1)(y^2 + \lambda^2 z^2 - 1) \sim y$$

since $p_0, p_1, p_3 \neq 0$ by assumption and since both $z-1$ and $y^2 + \lambda^2 z^2 - 1$ are invertible in A_m. This shows that $[x]$ is isolated in $Z \cap V(u,v,w)$.

Assume now that $w^2 = p_3\lambda^2 z(1-z)$. Again, by a direct computation, one can show that w appears only with even powers in $m_{ij}(1,y,z,0,0,w)$ for $i,j \in \{1,2\}$, so one can substitute $p_3\lambda^2 z(1-z)$ to w^2 in order to obtain the two expressions

$$r_{11} = p_3 \cdot z(z-1) \cdot \left(3p_2y + (3p_2 - p_3\lambda^4)(z-1)\right) \cdot \left(3p_0y + (p_3\lambda^2)z^2 - \lambda^2(3p_0 + p_3)z + 3p_0\right)$$

$$r_{22} = 9p_0p_1p_3 \cdot (z-1) \cdot (y+z-1) \cdot \left(y^2 + \frac{p_3\lambda^2}{3p_0} yz^2 - \lambda^2 \frac{3p_0 + p_3}{3p_0} yz + y + \frac{p_3\lambda^2}{3p_1} z(z-1)\right).$$

In A_m, one has

$$r_{11} \sim z \cdot \left(3p_2y + (3p_2 - p_3\lambda^4)(z-1)\right) \qquad r_{22} \sim y \cdot g(y,z) + \frac{p_3\lambda^2}{3p_1} z(z-1)$$

with $g(0,0) \neq 0$. Since $w^2 = p_3\lambda^2 z(1-z)$, if we assume $z = 0$, we also have that $w = 0$, so we can conclude by the previous case. We can then suppose that $3p_2y + (3p_2 - p_3\lambda^4)(z-1) = 0$ in the local ring. This can happen if and only if $3p_2 = \lambda^4 p_3$ and $y = 0$. However, if $y = 0$, from the expression of

r_{11}, one has that $z(z-1) = 0$, and thus again, $w^2 = 0$. This shows that $[x]$ is isolated in $Z \cap V(u, v)$ too and thus that the local dimension of $\mathcal{D}_4(f) = \mathrm{Sing}(\mathcal{H}_f)$ near $[x]$ is 2. □

With this lemma, we also conclude the proof of Theorem 5.1 in the case of cubic fourfolds.

Acknowledgements. The authors want to express their gratitude to Carlos D'Andrea and Giorgio Ottaviani for stimulating discussions and for pointing out some interesting papers related to some of the topics treated here. The authors also thank the anonymous referee for the useful observations.

Competing interest. The authors have no competing interests to declare.

Funding statement. The authors are partially supported by INdAM-GNSAGA and by PRIN 2022 '*Moduli spaces and special varieties*'. The first and second authors are partially supported by the INdAM – GNSAGA Project, '*Classification Problems in Algebraic Geometry: Lefschetz Properties and Moduli Spaces*' (CUP_E55F22000270001). The first author is holder of a research grant from Istituto Nazionale di Alta Matematica (INdAM).

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