# 29
# Renormalons

## 29.1 Introduction

The renormalon problem is related to the well-known fact [372,375] (for more complete reviews, see for example [162,154]) that the QCD series is unfortunately divergent (no finite radius of convergence) like $n!$, which is the number of diagrams of $n$th order. Indeed, a given observable can be expressed as a power series of the coupling $g$ as:

$$F(g) = \sum_n f_n g^n \, , \tag{29.1}$$

where the series are divergent:

$$f_n(n \to \infty) \sim K a^n n! n^b \, , \tag{29.2}$$

and where the $n$th order grows like $n!$, such that it is not practicable to have a quantitative meaning of Eq. (29.1). For the approximation to be meaningful, the approximation should asymptotically approach the exact result in the complex $g$-plane, such that:

$$\left| F(g)_{\rm exact} - \sum_{n=0}^{N} f_n g^n \right| < K_{N+1} g^{N+1} \, , \tag{29.3}$$

where the truncation error at order $N$ should be bounded to the order $g^{N+1}$. If $f_n$ behaves like in Eq. (29.2), $K_N$ usually behaves as $a^N N! N^b$. The truncation error behaves similarly as the terms of the series. It first decreases until:

$$N_0 \sim \frac{1}{|a| g} \, , \tag{29.4}$$

beyond which the approximation to $F$ does not improve through the inclusion of higher-order terms. For $N_0 \gg 1$, the approximation is good up to terms of the order:

$$K_{N_0} g^{N_0} \sim e^{-1/|a|g} \, . \tag{29.5}$$

Provided $f_n \sim K_n$, the best approximation is reached when the series is truncated at its minimal term and the truncation error is given by the minimal term of the series.



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One can use the well-known technique (*Borel transform*) for improving the convergence of a power series whose $n$th order grows like $n!$, by considering the related series:

$$B(z) \equiv \sum_n f_n \frac{z^n}{n!} \ . \tag{29.6}$$

If $f_n$ grows not faster than $n!$, then $B(z)$ will at least have a finite radius of convergence. Using the usual formula:

$$\int_0^\infty \exp(-z/g) z^n dz = n! g^{n+1} \ , \tag{29.7}$$

one can see that:

$$\tilde{F} \equiv gF(g) = \int_0^\infty \exp(-z/g) B(z) dz \ . \tag{29.8}$$

The relation in Eq. (29.8) is true order by order in perturbation theory, but there are arguments that it cannot be true for the full Greens functions. From Eq. (29.8), in order to calculate $F(g)$, one needs $B(z)$ only for real positive values of $z$ less than or of the order $g$, which can be obtained from the series in Eq. (29.1) if the singularities of $B(z)$ in the complex plane are all at distances from the origin much greater than $g$. Even if a few poles $z_1, \ z_2, \ldots$ have moduli of order $g$, one can calculate $B(z)$ by using power series for $(z - z_1)(z - z_2) \ldots B(z)$, where we should know the position of the poles. Singularities of $B(z)$ on the positive real axis are much worse, as they invalidate Eq. (29.8). One can distort the contour integral to avoid singularities on the positive real axis, but the ambiguities come from the question of distortion of the contour below or above the singularity? In the following, we will show that some of the singularities of the Borel transform $B(z)$ are associated with terms in the OPE (*renormalons*) and the others with solutions of the classical field equations (*instantons*).

In order to illustrate this discussion, let us assume that:

$$F(g)_{\text{exact}} = K a^n \Gamma(n + 1 + b) \tag{29.9}$$

For positive $b$, its Borel transform is:

$$\mathcal{B}[F](z) = K \frac{\Gamma(1+b)}{(1-az)^{1+b}} \ , \tag{29.10}$$

while for negative integer $b = -l$, one can write:

$$\mathcal{B}[F](z) = \frac{(-1)^l}{\Gamma(l)} (1 - az)^{l-1} \ln(1 - az) \ + \ \text{polynomial in } z. \tag{29.11}$$

In the case of QCD and QED, where one expects $a > 0$ (non-alternating series), one has singularities in the positive $z$ axis, such that the Borel integral does not exist. However, it may still be defined by taking the contour above or below the singularities, where it acquires an imaginary part:

$$\text{Im } \tilde{F}(g) = \mp \pi \frac{K}{a} e^{-1/(ag)} (ag)^b \ , \tag{29.12}$$

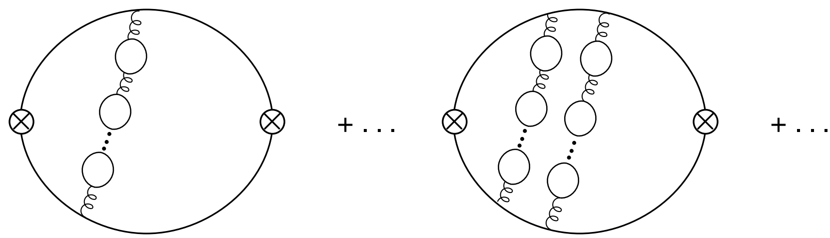


Fig. 29.1. Renormalon chains contributions to the QCD Adler $\mathcal{D}$-function.

where the sign depends on whether the integration is taken in the upper or lower complex plane. The difference in the two definitions is the so-called *ambiguity of the Borel integral*. As it behaves as an exponential in the coupling, it is of non-perturbative origin and induces power corrections.

In the following, we shall discuss for definiteness, the Adler $\mathcal{D}$-function in QCD:

$$\mathcal{D}(Q^2 = -q^2) \equiv -4\pi^2 Q^2 \frac{d\Pi(Q^2)}{dQ^2} , \tag{29.13}$$

built from the electromagnetic current $J_\mu(x) = \bar{\psi}\gamma_\mu\psi$ and which governs the $e^+e^- \to$ hadrons total cross-section. For the $\mathcal{D}$-function, the unnecessary $\nu$-dependence appearing in the two-point correlator $\Pi(q^2)$ from the leading-log term is not there, i.e., $\mathcal{D}$ is RGI. Therefore, its perturbative expansion reads:

$$\mathcal{D}\left(a_s \equiv \frac{\alpha_s}{\pi}\right) = \sum_n K_n a_s^n , \tag{29.14}$$

where $a_s(Q^2)$ is the running coupling and $K_n$ are pure numbers which are, however, RS-dependent.

Renormalon effects are associated to the insertion of n bubbles of quark loops into gluon lines (gluon chains) exchanged between the two quark lines in the $\mathcal{D}$-function built from the quark current as shown in Fig. 29.1.

It is well-known that they induce a n! growth into the perturbative series. This difficulty can be (in principle) cured by working with the Borel transform $\tilde{\mathcal{D}}$ of the correlator $\mathcal{D}(s)$:

$$\mathcal{D}(a_s) - \mathcal{D}(0) = \int_0^\infty db\ \tilde{\mathcal{D}}(b)\ \exp(-b/a_s) , \tag{29.15}$$

which possesses an explicit $1/n!$ suppression factor. However, life is not so simple because of the features described in the following.

## 29.2 Convergence of the Borel integral

The $b$-integral does not converge for $b \to \infty$. This can be seen from the fact that, in the chiral limit, hadrons have a non-zero mass in QCD, such that $\tilde{\mathcal{D}}$ should have singularities at $Q = M_0$, where $M_0$ is the mass of any hadrons having the quantum number of the photon

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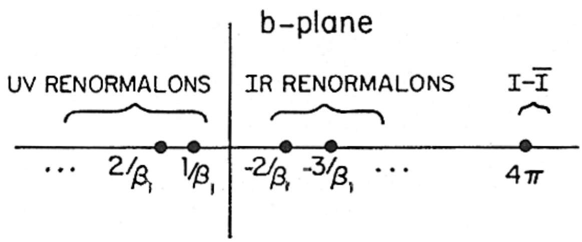


Fig. 29.2. Singularities in the Borel plane of the QCD Adler $\mathcal{D}$-function.

(or gluons). As for large $Q$, $\tilde{\mathcal{D}}$ is only function of:

$$a_s(Q^2) = \frac{1}{(-\beta_1/2)[\log Q^2/\Lambda^2 + (2n+1)i\pi]}, \tag{29.16}$$

one can see that it has an infinite number of singularities in $\alpha_s$, where $\alpha_s = 0$, corresponding to $M_0 = \infty$, is an accumulation point of these singularities. However, the singularities at $\alpha_s = \alpha_s(M_0)$ can arise through the behaviour:

$$\lim_{b\to\infty} \tilde{\mathcal{D}} \sim \exp -b\beta_1[\log Q^2/\Lambda^2 + (2n+1)i\pi], \tag{29.17}$$

which indicates that the $b$-integral does not converge for $b \to \infty$. However, a large $b$-region corresponds to large $Q^2$ where $\tilde{\mathcal{D}}$ decreases rapidly like $\alpha_s$, such that the $\alpha_s$-singularities are very weak and justify the uses of the Borel integral for studying, without any ambiguities, the asymptotic behaviour of QCD at large $Q^2$. In general, $\tilde{\mathcal{D}}$ develops singularities at $b = kb_0 \equiv 2\pi k/(-\beta_1)$ in the real $b$-axis, where the integral is also ambiguous.

## 29.3 The Borel plane in QCD

There are three known types of singularities in the Borel plane of QCD as shown in Fig. 29.2.

- UV renormalons occur in the negative real axis ($\beta_1$ is negative in our notation.), and are harmless since the integration contour in Eq. (29.15) is along the positive $b$-axis. At the $n$th order of perturbation theory, integrand of the form:

$$\frac{d^4p}{p^6} \ln^n p^2, \tag{29.18}$$

  gives a $n!$ factor and reflects the fact that such integrals are less convergent for large $n$.
- IR renormalons are singularities in the positive $b$-axis, which are due to the IR region of the Feynman integrals.
- Instanton–anti-instanton singularities occur because far separated instanton–anti-instanton pairs which can exist cannot be properly treated in a perturbative expansion around $\alpha_s = 0$.

## 29.4 IR renormalons

The IR renormalons correspond to the singularities at $k = +2, \ +3, \dots$, and are generated by the low-energy behaviour of these higher-order diagrams, where fermion bubbles are

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inserted into the internal gluon line exchanged between the two fermion lines. In order to illustrate this feature, let us consider the one-gluon exchange diagram with a gluon of momentum $p$, where we shall focus on the low-$p$ region $R$.[1] Then:

$$\mathcal{D}\left(a_s(Q)\right) \sim \int_R \frac{d^4 p}{p^2} a_s(p^2) F(p) \,. \tag{29.19}$$

where $F(P^2 \equiv -p^2)$ behaves as $P^2$. Using:

$$a_s(p) \simeq \frac{a_s(Q)}{1 - \beta_1 a_s(Q) \log\left(P^2/Q^2\right)} \,, \tag{29.20}$$

and:

$$\tilde{\mathcal{D}}(b) = \frac{1}{2i\pi} \int_{a_s - i\infty}^{a_s + i\infty} d(1/a_s) e^{b/a_s} \mathcal{D}(a_s) \,, \tag{29.21}$$

one obtains:

$$\tilde{\mathcal{D}}(b) \sim \int_R P^2 dP^2 \left(\frac{P^2}{Q^2}\right)^{-b/b_0} , \tag{29.22}$$

which gives the singularity near $b = -4\pi/\beta_1$:

$$\tilde{\mathcal{D}}(b) \sim \left(1 + \frac{b\beta_1}{4\pi}\right)^{-1} , \tag{29.23}$$

or to two loops, i.e., for two gluons-exchange:

$$\tilde{\mathcal{D}}(b) \sim \left(1 + \frac{b\beta_1}{4\pi}\right)^{-1+2\beta_2/\beta_1^2} , \tag{29.24}$$

for $b > 2b_0 \equiv -4\pi/\beta_1$. This indicates that the pole at $b = 2b_0$ gives rise to an IR ambiguity, if one tries to reconstruct $\mathcal{D}(a_s)$ from $\tilde{\mathcal{D}}(b)$ taken from perturbation theory. Converting the $a_s$-dependence into a $Q$-one, one can expect that the non-perturbative corrections to perturbation theory are of the size $1/Q^4$. More generally, diagrams with one chain of gluons contribute as:

$$\mathcal{D}(a_s) \sim \sum_n n \left(\frac{\alpha_s}{kb_0}\right)^n \quad \Longrightarrow \quad \mathcal{B}(\mathcal{D}) \equiv \tilde{\mathcal{D}}(b) \sim -\frac{kb_0}{b - kb_0} \,, \tag{29.25}$$

for $b > kb_0$, which indicates that the pole at $b = kb_0$ gives rise to an IR ambiguity:

$$\delta\mathcal{D}(a_s) \sim \exp\left(-\frac{kb_0}{\alpha_s}\right) \sim \left(\frac{\Lambda^2}{-q^2}\right)^k , \tag{29.26}$$

if one tries to reconstruct $\mathcal{D}(a_s)$ from $\tilde{\mathcal{D}}(b)$ taken from perturbation theory. However, different prescriptions for defining $\mathcal{D}$ in perturbation theory for $b > kb_0$ can be compensated by the changes in the value of the non-perturbative condensates introduced via the SVZ

[1] IR renormalons have been studied in the $O(N)$ non-linear $\sigma$ model [374] and in QCD [376,377]. Here, we shall limit ourselves to the QCD case.

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expansion, which *one must add* to perturbation theory in order to obtain a reliable result [162,342].

The absence of a $k = 1$ singularity is related to the absence of any gauge invariant operator of dimension 2. The absence of this singularity has been proved [331] from an explicit calculation in the limit of large $n_f$-number of flavours, where it has been shown that the relation:

$$\mathcal{B}\,(\mathrm{Im}\Pi)\,(b) \sim \frac{\sin(\pi b/b_0)}{(\pi b/b_0)}\tilde{\mathcal{D}}(b)\,, \tag{29.27}$$

implies that $\mathcal{B}\,(\mathrm{Im}\Pi)$ has only a zero at $k = 1$ but not the other alternative where $\tilde{\mathcal{D}}$ has a pole at this point, which follows from the simple factorization of the $Q^2$ dependence in the Borel transform of the $D$-function in the large $\beta$-limit. Then, one can conclude that no $1/Q^2$-ambiguity can be generated by the IR renormalons and they are intimately connected to the gauge invariant condensates in the SVZ-expansion. Restricting to the lowest IR renormalon pole, one can derive the perturbative contribution to the gluon condensate [378]:

$$\frac{\langle 0|\alpha_s G^2|0\rangle_{\mathrm{ren}}}{24\pi Q^4} = \sum_{n\ \mathrm{large}} \frac{3}{2\pi}\left(\frac{\alpha_s(Q^2)}{\pi}\right)^{n+1}\left(\frac{-\beta_1}{4}\right)^n . \tag{29.28}$$

One should notice that renormalons are target-blind:

$$\langle p|\alpha_s G^2|p\rangle_{\mathrm{ren}} = \langle 0|\alpha_s G^2|0\rangle_{\mathrm{ren}} . \tag{29.29}$$

They cannot also produce a non-vanishing quark condensate $\langle \bar{q}q\rangle$ as they respect the symmetries of the QCD Lagrangian, and cannot bring some insights on confinement due to their 'perturbative' origin.

However, at the one-loop level, renormalons are not the only way to probe the IR regions perturbatively. Another possibility is the introduction of the gluon mass $\lambda$ [478] as a fit parameter, while an IR perturbative contribution to the gluon condensate has been obtained in [479]:

$$\langle 0|\alpha_s G^2|0\rangle_{\mathrm{pert}} = \frac{3\alpha_s}{\pi^2}\lambda^4 \ln\lambda^2 . \tag{29.30}$$

A similar result has been obtained in a QCD-like model [369,374], which is an alternative to the renormalon contribution for massless gluon. Phenomenology using gluon mass has been developed [366], while in [162], a one-to-one correspondence between the two approaches has been proposed. Keeping only IR-sensitive contributions, a one-loop calculation with a gluon mass $\lambda$ can be translated as:

$$\begin{aligned} &C_0\alpha_s \ln\lambda^2 + C_1\alpha_s\frac{\sqrt{\lambda^2}}{Q} + C_2\alpha_s\frac{\lambda^2\ln\lambda^2}{Q^2} + \cdots \to \\ &C_0'\alpha_s \ln\Lambda^2 + C_1'\alpha_s\frac{\Lambda}{Q} + C_2'\alpha_s\frac{\Lambda^2}{Q^2} + \cdots . \end{aligned} \tag{29.31}$$

where $C_i,\ C_i'$ are coefficients.

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## 29.5 UV renormalons

The UV renormalon singularities correspond to $k = -1, -2, \ldots,$ and are generated by the high-energy behaviour of the virtual momenta. They lead to a Borel-summable series thanks to the asymptotic freedom property of the theory. After a Borel sum, they cannot limit the applicability of perturbation theory [377,379], although they can induce an uncertainty in the truncated perturbative series when the Borel sum is not done. Their contributions are dominated by the leading singularity at $k = -1$:

$$K_n \sim \frac{n!}{(-b_0)^n} , \tag{29.32}$$

which gives rise to an asymptotic series:

$$|K_1 a| > |K_2 a^2| \cdots > K_{N-1} a^{N-1} \sim |K_N a^N| < |K_{N+1} a^{N+1}| < \cdots , \tag{29.33}$$

where the successive terms decrease like $N \sim b_0/a$, at which the minimum value is attained, while the series explodes afterwards. The alternating sign of $K_n$ guarantees that the series is Borel summable. For a truncated series, the accuracy is limited by the size of the minimum term:

$$4\pi^2 \delta\mathcal{D}(a) \equiv |K_N a^N| \sim N!\, n^N \sim \sqrt{2\pi N} e^{-N} \sim \exp(-b_0/a) \sim \Lambda^2/-q^2 , \tag{29.34}$$

which indicates that the UV renormalon can contribute as $1/Q^2$ [161,162,297–300].

However, this result is subtraction-scale dependent [162], as a more careful analysis shows that the ambiguity scales as:

$$A\sqrt{\alpha_s(\nu)}\left(\frac{\Lambda^2 Q^2}{\mu^4}\right) , \tag{29.35}$$

where $A$ and $\Lambda$ absorb this renormalization scheme (RS)-dependence, whilst $\mu$ is an arbitrary UV cut-off. However, it can be shown that the results obtained in the limit of infinite numbers of flavours within the one-chain approximation, can be strongly affected by the UV renormalon induced by the two-, three-,... chains of gluons [342,343], such that, it is premature to deduce any reliable quantitative estimates from this approach. However, some more optimistic authors have considered a more refined version of the one-chain of gluons approximation, involving next-to-leading $\beta$ functions and RS-invariant quantities. The analysis indicates that the UV renormalon effect is much smaller [339,340] than naïvely expected [344,331], and than that of the perturbative error based on the last calculated coefficient term of the series (theorem of divergent series [337]) [338,323]. Taking into account the different existing (qualitative) estimates of UV renormalon effects [331–340], one can conclude that the estimate of the perturbative errors based on the last calculated term of the QCD series [338,323] gives a reasonable or presumably an overestimate of the true error. It is also clear that the UV renormalon contribution *cannot* be considered as a *new source* of uncertainty, but it is of the same nature as the perturbative error. An independent extraction of such a contribution is needed. The only available alternative attempt for doing this, is its phenomenological extraction from the $e^+e^- \to I = 1$ hadrons data [341,329](Section 52.10)], from QSSR. It has been noticed from the analysis of [329], that the obtained constraint is strongly correlated to the value of the gluon condensate. Postulating that a new term of dimension-two exists in the QCD series, the OPE is modified as:

$$\mathcal{D}(Q^2) = 1 + \left(\frac{\alpha_s}{\pi}\right) + \cdots + \frac{d_2}{Q^2} + \cdots \tag{29.36}$$

one obtains [329]:

$$d_2 \approx (0.03 \sim 0.05)\ \text{GeV}^2\ , \tag{29.37}$$

if one uses the value of the gluon condensate $\langle\alpha_s G^2\rangle \simeq 0.08\,\text{GeV}^4$. This term would induce an effect of about 1% in the QCD expression of the $\tau$-width [325], which is a negligible effect.

## 29.6 Some phenomenology in the large $\beta$-limit

The large $\beta$-limit corresponds to the case where one takes large numbers of quark flavours and neglect the remainder terms of the $\beta$-function:

$$\beta_1(n_f \to \infty) \simeq n_f/3\ , \tag{29.38}$$

and then corresponds to the abelianisation of QCD.

### *29.6.1 The D-function*

In this limit the $D$-function can be expressed as:

$$\mathcal{D}(Q^2) = 1 + \left(\frac{\alpha_s}{\pi}\right) \sum_{n=0} \alpha_s^n \left[ d_n \left(\frac{\beta_1}{2\pi}\right)^n + \delta_n \right]\ , \tag{29.39}$$

where $d_0 = 1$ and $\delta_0 = 0$. The coefficient $d_n$ comes from the bubble diagrams. Its Borel transform reads:

$$\mathcal{B}(\mathcal{D})(b) = \sum_{n=0} \frac{d_n}{n!} b^n = \frac{32}{3}\left(\frac{Q^2}{\nu^2} e^C\right)^{-b} \frac{b}{1-(1-b)^2} \sum_{j=2}^{\infty} \frac{(-1)^j j}{(j^2-(1-b)^2)^2}\ , \tag{29.40}$$

where in the $\overline{MS}$ scheme $C = -5/3$. The UV renormalon poles at $b = -1, -2, \ldots$ are double poles, while the IR renormalon poles at $b = 3, 4, \ldots$ are double poles and a single pole at $b = 2$. It is informative to decompose the Borel transform into the sum of leading poles:

$$\begin{aligned}\mathcal{B}(\mathcal{D})(b) = e^{-5/3}\left[\frac{4}{9}\frac{1}{(1+u)^2} + \frac{10}{9}\frac{1}{(1+u)}\right] + e^{10/3}\frac{2}{(2-u)} \\ e^{-10/3}\left[-\frac{2}{9}\frac{1}{(2+u)^2} - \frac{1}{2(2+u)}\right] + \cdots\end{aligned} \tag{29.41}$$

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Working in the $\overline{MS}$ scheme, the ambiguity in summing the series can be quantified as (see e.g. [154]):

$$\delta\mathcal{D}(Q^2)_{\rm ren} = \left(\frac{4}{-\beta_1}\right)\frac{e^{10/3}}{\pi}\frac{\Lambda^4}{Q^4} \approx \frac{0.06\ {\rm GeV}^4}{Q^4}\ . \tag{29.42}$$

This effect is smaller than the non-perturbative gluon condensate contribution:

$$\delta\mathcal{D}(Q^2)_{\rm cond} \simeq \frac{2\pi}{3}\frac{\langle 0|\alpha_s G^2|0\rangle}{Q^4} \simeq \frac{0.14\ {\rm GeV}^4}{Q^4}\ , \tag{29.43}$$

where we have used the most recent QSSR determination [313,329]. This result is not significant for raising doubts on the existence of the non-perturbative gluon condensate in the SVZ expansion [1], although it can contribute to its perturbative component.

### 29.6.2 Semi-hadronic inclusive $\tau$ decays

Semi-hadronic tau decays have been discussed in details in BNP [325]. We shall be interested here in its asymptotic perturbative expansion, which have been discussed by many authors [331–345]. In the large $\beta$-function limit, one can write, the branching ratio [154]:

$$R_\tau = 3(|V_{ud}|^2 + |V_{us}|^2)\left\{1 + \left(\frac{\alpha_s}{\pi}\right)\sum_{n=0}\alpha_s^n\left[d_n^\tau\left(\frac{\beta_1}{2\pi}\right)^n + \delta_n^\tau\right]\right\}\ , \tag{29.44}$$

where one can neglect the remainder $\delta_n^\tau$. The Borel transform is [154]:

$$\mathcal{B}(\mathcal{R}_\tau)(b) = \mathcal{B}(\mathcal{D})(b)\sin(\pi b)\left[\frac{1}{\pi b} + \frac{2}{\pi(1-b)} - \frac{2}{\pi(3-b)} + \frac{1}{\pi(4-b)}\right], \tag{29.45}$$

where the sinus attenuates all renormalon poles except those at $b = 3, 4$. The point $b = 1$ is regular but will not be suppressed by a factor $\alpha_s$ when one uses the Cauchy contour integral for evaluating $R_\tau$.

Taking the leading renormalon poles, one can approximately have:

$$\mathcal{B}(\mathcal{R}_\tau)(b) \simeq e^{-5/3}\frac{2}{15(1+u)} + e^{-10/3}\frac{2}{135(2+u)} + e^5\left[\frac{8}{3(3-u)^2} - \frac{8}{9(3-u)}\right] + \cdots \tag{29.46}$$

Expressing the rate as in BNP:

$$R_\tau = 3(|V_{ud}|^2 + |V_{us}|^2)S_{EW}[1 + \delta_{PT} + \delta_{EW} + \delta_{NP}]\ , \tag{29.47}$$

one can compare the measured value of $\delta_{PT}$ with the one obtained from the large $\beta$-limit prediction. One can notice that the value of $\alpha_s(M_\tau)$ can reduce by 15% compared to the one from the truncated series but this effect is smaller than one obtained by adding the $\alpha_s^3$ correction. Another point is that the error induced by the $\Lambda^2/M_\tau^2$ term which arises when the series is truncated at the onset of UV renormalon divergence is numerically very small due to the smallness of its coefficient. Therefore, the induced uncertainty is negligible in the $\overline{MS}$ scheme.

### 29.7 Power corrections for jet shapes

The phenomenology of power corrections in jets and DIS has been developed [162,366], while numerous experimental studies have been devoted for measuring these contributions [480,481]. Renormalons are most useful in these frameworks as they can fix unambiguously the power $n$ of the corrections $(\Lambda/Q)^n$. However, in order to find relations between various corrections, models are still needed as one expects [342,162] that any number of renormalon chains gives power corrections of the same order, and there is no way to evaluate all of them. Some other reservations to be made in the renormalon approach are also the extrapolation of the small QCD coupling expansion in the UV regime down to the IR domain where the QCD coupling is of order one, and where, terms which dominate in the UV region do not necessarily dominate in the IR region.

For definiteness, let us consider the thrust variable defined in the previous chapters dedicated to jets:

$$T = {}^{\max}_{\mathbf{n}} \frac{\sum_i |\mathbf{p_i} \cdot \mathbf{n}|}{\sum_i |\mathbf{p_i}|} , \tag{29.48}$$

where $\mathbf{p}$ is the momenta of particles produced and $\mathbf{n}$ is a unit vector. From perturbation theory $T \neq 1$ due to the emission of gluons from quarks. The contribution due to a soft gluon emission can be quantified as [162]:

$$\langle 1 - T \rangle_{\text{soft}} \sim \int_0^{\Lambda} \frac{d\omega}{\omega} \frac{\omega}{Q} \alpha_s \sim \frac{\Lambda}{Q} , \tag{29.49}$$

where $d\omega/\omega$ is the standard factor for the gluon emission; $\omega/Q$ comes from the definition of $T$ while $\alpha_s$ is of the order one. Alternatively, if one attributes to the gluon an intrinsic invariant mass squared $\zeta Q^2$, and evaluate the thrust mean value, one obtains [366,154]:

$$\langle 1 - T \rangle = C_F \left(\frac{\alpha_s}{\pi}\right) [0.788 - k\sqrt{\zeta} + \cdots] , \tag{29.50}$$

where $\sqrt{\zeta} \sim \Lambda/Q$, and its coefficient depends on the definition of thrust used ($k = -7.32$ with the previous definition, while it is 4 for the definition used in [366]). One can generalize the previous result by using an *universality picture*. That can be done by keeping terms which contributes perturbatively as $\alpha_s^n \ln^k Q$ and extrapolating such terms in the IR region where, however, they no longer dominate! In this way, the $1/Q$ correction can be expressed in terms of the *universal factor* [162]:

$$E_{\text{soft}} = \int dk_\perp \gamma_{eik}(\alpha_s(k_\perp^2)) , \tag{29.51}$$

where $\gamma_{eik}$ is the so-called eikonal anomalous dimension, and the integral over the Landau pole is understood as the principal value. In this way, on gets the different relations

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among the event-shape variables (see the definitions in the jet chapters from Eqs. (24.32) to (24.37))[162]:

$$\langle 1-T\rangle_{1/Q} = \frac{2}{3\pi}\langle C\rangle_{1/Q} = \left(\frac{\langle M_h^2 + M_l^2\rangle}{Q^2}\right)_{1/Q} \tag{29.52}$$

and:

$$\left(\frac{\langle M_h^2\rangle}{Q^2}\right)_{1/Q} \approx \left(\frac{\langle M_l^2\rangle}{Q^2}\right)_{1/Q} . \tag{29.53}$$

These relations are well verified experimentally [480]:

$$\begin{aligned}\frac{1}{2}\langle 1-T\rangle_{1/Q} &= \frac{C}{Q}(0.511 \pm 0.009)\\ \frac{1}{3\pi}\langle C\rangle_{1/Q} &= \frac{C}{Q}(0.482 \pm 0.008)\\ 2\left(\frac{\langle M_h^2\rangle}{Q^2}\right)_{1/Q} &= \frac{C}{Q}(0.616 \pm 0.018)\,,\end{aligned} \tag{29.54}$$

where $C$ is a constant.

## 29.8 Power corrections in deep inelastic scattering

Power corrections in deep inelastic scattering have been developed at the single renormalon chain level [482], and an alternative derivation using Landau pole of the power corrections has been given in [162].

### 29.8.1 Drell–Yan process

The inclusive cross-section can be expressed in terms of the moments:

$$\int d\tau\ \tau^{n-1}\frac{d\sigma(Q^2,\tau)}{dQ^2} = M_n\left[1 + \alpha_s C_\lambda \frac{\sqrt{\lambda}^2}{Q}\right] , \tag{29.55}$$

where $Q$ is the invariant mass of the lepton pair; $\sqrt{s}$ is the invariant mass of the $\bar{q}q$ from the initial hadrons $h_{1,2}$ and $\tau = Q^2/s$; $\lambda$ is the gluon mass. To one loop, one finds [154]:

$$C_\lambda = 0 \quad \text{for} \quad n\cdot\Lambda/\sqrt{s} \ll 1 . \tag{29.56}$$

An understanding of this result from general arguments based on the inclusive nature of momenta has been given in [162].

### 29.8.2 Non-singlet proton structure functions $F_2$

A systematic measurement of power corrections in DIS for the moments of the non-singlet structure functions $F_2$ has been performed [481]. The moments:

$$\mathcal{M}_2^{(n)}(Q^2) \equiv \int_0^1 dx\, x^{n-2} F_2(x, Q^2)\,, \tag{29.57}$$

have been parametrized as:

$$\mathcal{M}_2^{(n)}(Q^2) = \mathcal{M}_{2,PT}^{(n)} \left(1 + C_2^{(n)} \frac{\tau^2}{Q^2} + C_4^{(n)} \frac{\lambda^4}{Q^4} \cdots\right)\,, \tag{29.58}$$

where $\mathcal{M}_{2,PT}^{(n)}$ is the perturbative QCD prediction. The $n$-dependence of the power corrections has been included into $C_{2,4}^{(n)}$. In the range of $Q^2$ values from 5 to 260 GeV$^2$, and studying different figures, the analysis leads to a non-vanishing contribution:

$$\tau^2 \simeq (0.2 \pm 0.1)\ \text{GeV}^2\,, \tag{29.59}$$

if $\lambda^4 = 0$ and $(0.25 \pm 0.2)$ GeV$^2$ if $\lambda^4 \neq 0$. This result and the $n$-dependence agree with the renormalon-based result [162].

### 29.8.3 Gross–Llewellyn Smith and polarized Bjorken sum rules

Power corrections to other DIS sum rules (Gross–Llewellyn Smith (GLS), polarized Bjorken (PBj) sum rules) have been also analysed from the renormalon approach [331]. In the large $\beta$-limit, one can approximately assume them to be the same because the perturbative contributions differ only by light-by-light scattering starting at $\alpha_s^3$. Let's remind ourselves of the GLS sum rule given in Eq. (29.60):

$$\int_0^1 \frac{dx}{x}[F_3^{\bar{\nu}p}(x, Q^2) + F_3^{\nu p}(x, Q^2)] = 3\left\{1 - a_s(Q^2) - 3.58\, a_s^2(Q^2) - 19.0\, a_s^3(Q^2) + \delta_{GLS}\right\}, \tag{29.60}$$

to which we add the power correction (twist-4) term $\delta_{GLS}$. In the large $\beta$-limit, one obtains in the $\overline{MS}$ scheme [154]:

$$\delta_{GLS} \simeq e^{5/3}\left(-\frac{16}{9\beta_1}\right)\frac{\Lambda^2}{Q^2} \approx \frac{0.1\ \text{GeV}^2}{Q^2}\,, \tag{29.61}$$

which is comparable in strength but differs in sign with the twist-4 QSSR estimate [483] and fit using the CCFR data [249]:

$$\delta_{HT} \approx -\frac{(0.10 \pm 0.05)\ \text{GeV}^2}{Q^2}\,. \tag{29.62}$$

However, an extraction of this power correction from the polarized Bjorken sum rule [260] leads to an inaccurate value consistent with zero as given in Eq. (19.8).

### 29.9 Power corrections to the heavy quark pole mass

We have defined in the chapter on perturbation theory the notion of pole mass, which is defined at the pole of the propagator. We have seen that this definition is not renormalized [141,147,133], independent of the regularization procedure [148] and free from IR singularities [133]. However, when this mass is related to the short distance $\overline{MS}$ running mass, one can notice its sensitivity to long distances. In the renormalon approach, this difference is given by the self-energy diagram with one-gluon chain:

$$M(p^2=\bar{m}^2)=\bar{m}(\nu)+(-i)\int\frac{d^nk}{(2\pi)^n}\alpha_s(ke^{-5/6})\frac{\gamma^\mu(\hat{p}+\hat{k}+m)\gamma_\mu}{k^2[(p-k)^2-m^2]}\bigg|_{p^2=m^2} . \tag{29.63}$$

It shows that for $p^2=m^2$, the integral behaves for small $k$ like $d^4k/k^3$, which implies that the series gives an IR renormalon singularity at $b=-\pi/\beta_1$. The asymptotic behaviour of the series expansion reads [331,154]:

$$M(p^2=\bar{m}^2)=\bar{m}(\nu)+C_F\frac{e^{5/6}}{\pi}\nu\sum_n\left(\frac{-\beta_1}{\pi}\right)^n n!\alpha_s^{n+1} . \tag{29.64}$$

Writing:

$$\delta m\equiv M_{\rm pole}\equiv M(p^2=\bar{m}^2)-\bar{m}(\bar{m})=\bar{m}(\bar{m})\frac{C_F}{4}\left(\frac{\alpha_s}{\pi}\right)\sum_{n=0}[d_n(-\beta_1/\pi)^n+\delta_n]\alpha_s^{n+1} , \tag{29.65}$$

its Borel transform reads, in the large $\beta$-limit, [331]:

$$\mathcal{B}[\delta m/\bar{m}]=\left(\frac{\bar{m}^2}{\nu^2}\right)^{-u}e^{5u/3}6(1-u)\frac{\Gamma(u)\Gamma(1-2u)}{\Gamma(3-u)}+\cdots , \tag{29.66}$$

where $\cdots$ indicates subtraction terms which are rapidly convergent and give negligible contributions to the coefficients $d_n$ for increasing $n$. Comparisons of the values of $d_n$ with the available calculations [151,153] show that the asymptotic series reproduce approximately the first two coefficients [331]. One can also notice that the series is rapidly dominated by the IR renormalon contributions and the series start to diverge to order $\alpha_s^3$ for the charm quark mass, and to order $\alpha_s^4$ for the bottom. An intuitive derivation of this IR effect can be obtained from the Coulomb potential. In this way, the IR correction to the heavy quark mass is [162]:

$$\frac{\delta m}{\bar{m}}=-\frac{1}{2\bar{m}}\int_{|\vec{q}|<\mu}\frac{d^3\vec{q}}{(2\pi)^3}V(\vec{q})\simeq -C_F\alpha_s\frac{\mu}{\bar{m}} , \tag{29.67}$$

where:

$$V(\vec{q})=-4\pi C_F\frac{\alpha_s(\vec{q})}{\vec{q}^2} . \tag{29.68}$$

https://doi.org/10.1017/9781009290296.040 Published online by Cambridge University Press

It has also been noticed [158,159] that the IR singularity of the Borel transform for the pole mass in Eq. (29.66) is cancelled by that of the potential [486,162]:

$$\mathcal{B}[V(\vec{r})] = -(4\pi C_F)(\nu^2 e^{-C})^u \int \frac{d^3\vec{q}}{(2\pi)^3} \frac{e^{i\vec{q}\cdot\vec{r}}}{(\vec{q}^2)^{1+u}}$$
$$= -\frac{C_F}{r}(\nu^2 r^2 e^{-C})^u \frac{\Gamma((1/2)+u)\Gamma((1/2)-u)}{\pi\Gamma(1+2u)} \, . \qquad (29.69)$$

This leads to the proposal of a new mass definition that is less IR sensitive than the pole mass in this approximation (see Section 11.13). In, for example, the derivations of the inclusive $B$-decays using a $1/m_b$ expansion, which behaves to leading order as $m_b^5$, it has been noticed that the use of the pole mass definition introduces an ambiguity of the order of $\Lambda/m_b$ when summing the series, which does not match with any non-perturbative parameters of the heavy quark expansion. This problem does not appear when one expresses the width in terms of the short distance $\overline{MS}$-mass, where a cancellation of the leading divergence between that of the width and of the relation between the pole and running mass occurs.