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A NOTE ON PROJECTIVE CAPACITY

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Introduction. In [**1**] we defined a capacity in $\mathbf{C}^n$. Recently Molzon, Shiffman and Sibony [**8**] have introduced a different capacity which is useful for certain Bezout estimates. The object of this note is to apply the methods of [**1**] to study the capacity of [**8**]. We shall obtain an equivalent definition of this capacity via Tchebycheff polynomials, along the lines of [**1**]. Half of this equivalence was independently obtained by Sibony [**9**].

To establish the full equivalence of these two approaches to capacity a notion of Jensen measures in a setting more general than uniform algebras is needed. We shall consider Jensen measures for multiplicative semigroups; these are sets of functions in which only the multiplicative structure is postulated. It will also be useful to generalize the notion of polynomial hull in $\mathbf{C}^n$ to a hull with respect to a multiplicative semigroup of polynomials. We can then adapt the approach of [**1**] to these semigroups.

It is central to know when a set has zero capacity. Molzon, Shiffman and Sibony [**8**] showed for their capacity, that if Σ is an irreducible closed subvariety of projective space (hence algebraic by Chow's theorem), then a compact subset of Σ which is not locally pluripolar has positive capacity. As an application of the equivalent definition of capacity we shall generalize this by replacing Σ with a local subvariety which of course need not be algebraic. Finally we give a very short proof of the fact that locally pluripolar implies zero capacity (in the sense of [**1**]); the original proof was inordinately long.

1. Jensen measures and $\mathscr{S}$-hulls.

We shall be using the following notations: For a function f on a set X,

$$\|f\|_X = \sup\{|f(x)| : x \in X\},$$

$C(X)$ will denote the set of all continuous complex-valued functions on a space X, $C(\mathbf{R}, X)$ the real-valued functions. For z in $\mathbf{C}^n$ $\|z\|$ will be the Euclidian norm; for z and w in $\mathbf{C}^n$, $z \cdot w = \sum_1^n z_k w_k$.

Let $\mathscr{S}$ be the set of all homogeneous polynomials in $\mathbf{C}^n$ which split into linear factors. For $n > 2$ this is a proper subclass of the set of all

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homogeneous polynomials. For a compact set X in $\mathbf{C}^n$ we define the $\mathscr{S}$-hull of X by

$$\mathscr{S}\text{-hull}(X) = \{z \in \mathbf{C}^n : |p(z)| \leqq \|p\|_X \text{ for all } p \in \mathscr{S}\}.$$

This is of course completely analogous to the polynomially convex hull where $\mathscr{S}$ is replaced by the algebra of all polynomials. By convention we take $\mathscr{S}$ to contain the constants. It is clear that (i) for X compact, $\mathscr{S}$-hull (X) is compact, (ii) for $\alpha \in \mathbf{C}^n$, α is in $\mathscr{S}$-hull (X) if and only if there is an $M > 0$ such that $|p(\alpha)| \leqq M\|p\|_X$ for each $p \in \mathscr{S}$ (by the usual argument of applying this to p^N) and (iii) $\hat{X} \subseteq \mathscr{S}$-hull (X) where $\hat{X}$ is the polynomially convex hull of X.

Let X be a compact Hausdorff space. We define $\mathscr{G} \subseteq C(X)$ to be a *multiplicative semi-group* (MSG) of continuous functions on X if it satisfies (a) $f, g \in \mathscr{G} \Rightarrow f \cdot g \in \mathscr{G}$ and (b) $\mathscr{G}$ contains the constants. If Γ is a closed subset of X such that $\|f\|_X = \|f\|_\Gamma$ for each $f \in \mathscr{G}$, then we say that Γ is a *boundary* for $\mathscr{G}$.

If $\mathscr{A}$ is a uniform algebra on X with Shilov boundary Γ then $\mathscr{A}$ itself and $\mathscr{G} = \{e^f : f \in \mathscr{A}\}$ are MSG's on X with boundary Γ. Our principal interest is in the MSG $\mathscr{S}X$, the restriction of $\mathscr{S}$ to X, where X is a compact set in $\mathbf{C}^n$. If K is a compact set in $\mathbf{C}^n$ and X is $\mathscr{S}$-hull (K) then $\mathscr{S}X$ is an MSG on X with boundary $K \subseteq X$. This example does not arise from a uniform algebra.

Our main interest in MSG's comes from the fact that they possess certain Jensen measures. Bishop showed in [**3**] that homomorphisms of uniform algebras can be represented by Jensen measures and in [**1**] this was extended to other functionals on uniform algebras. It turns out that Bishop's argument is true in even greater generality; namely, for MSG's. We shall later apply this fact to the MSG $\mathscr{S}X$ with $X = \mathscr{S}$-hull (K).

For the reader's convenience we shall indicate the proof of the existence of Jensen measures for MSG's; it is merely an adaptation of Bishop's original argument for uniform algebras. A similar extension of Bishop's idea appears in [**4**].

THEOREM 1.1. *Let $\mathscr{S}$ be an* MSG *on a compact Hausdorff space X, $\Gamma \subseteq X$ a boundary and μ a probability measure on X. Then there exists a probability measure ν on Γ such that*

$$\int \log |f| d\mu \leqq \int \log |f| d\nu$$

for all $f \in \mathscr{S}$.

Remark. We shall say that ν is a *Jensen measure for μ.*

Proof. Let

$$N = \{u \in C(\mathbf{R}, \Gamma) : u < 0 \text{ on } \Gamma\}$$

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and set

$$K = \left\{ h \in C(\mathbf{R}, \Gamma) : \exists f \in \mathcal{S} \text{ such that} \right.$$

$$\left. \text{(a) } \int_X \log |f| d\mu \geqq 0 \quad \text{and} \quad \text{(b) } rh > \log |f| \text{ on } \Gamma \text{ for some } r > 0 \right\}.$$

One checks that K is convex and, using the fact that Γ is a boundary for $\mathcal{S}$, that $K \cap N = \emptyset$. By the Hahn–Banach separation theorem, there exists a linear functional $L \in C(\mathbf{R}, \Gamma)^*$ with $\|L\| = 1$ such that

$$\sup\{L(h) : h \in N\} = 0 = \inf\{L(h) : h \in K\}.$$

It follows that L can be represented by a probability measure ν on Γ.

Suppose $f \in \mathcal{S}$ with $\int \log |f| d\mu = 0$. Then if $h > \log |f|$ on Γ, we have $h \in K$ and so

$$\int h d\nu = L(h) \geqq 0.$$

Hence

$$\int \log |f| d\nu \geqq 0.$$

Now in general if $f \in \mathcal{S}$, let $c = \int \log |f| d\mu$ and apply this $e^{-c}f$ to get the theorem.

COROLLARY. *Let X be compact in $\mathbf{C}^n$. Then for $\alpha \in \mathbf{C}^n$, $\alpha \in \mathcal{S}$-hull(X) if and only if there exists a probability measure ν on X such that*

$$\log |\beta \cdot \alpha| \leqq \int \log |\beta \cdot \zeta| d\nu(\zeta) \text{ for all } \beta \neq 0 \text{ in } \mathbf{C}^n.$$

2. Capacity. We shall consider several capacities for subsets of $\mathbf{P}^{n-1}$. Using the natural projection

$$\Pi : \mathbf{C}^n \backslash \{0\} \rightarrow \mathbf{P}^{n-1},$$

we shall identify subsets of $\mathbf{P}^{n-1}$ with circled subsets of the unit sphere ∂B in $\mathbf{C}^n$; namely, $E \subseteq \mathbf{P}^{n-1}$ will be identified with $\Pi^{-1}(E) \cap \partial B$.

First recall the definition of projective capacity [**1**] for compact sets K in $\mathbf{P}^{n-1}$; we view K as a compact circled set in $\partial B \subseteq \mathbf{C}^n$. We say that a homogeneous polynomial f of degree k in $\mathbf{C}^n$ is *normalized* if

$$\int \log |f| d\sigma = k \int \log |z_n| d\sigma$$

where σ is unit surface volume on ∂B. Denote by $\mathcal{P}_k$ the set of all normalized homogeneous polynomials of degree k in $\mathbf{C}^n$. Let

$$m_k = m_k(K) = \inf\{\|f\|_K : f \in \mathcal{P}_k\}.$$

Then the projective capacity of K is defined by $\text{cap}(K) = \lim m_k^{1/k}$.

Next we recall the definition of the capacity $\mathscr{C}(K)$, for K compact in $\mathbf{P}^{n-1}$, introduced by Molzon, Shiffman and Sibony [**6**]. For $\mu \in \mathfrak{M}(K)$ (= the set of probability measures on K), let

$$u_\mu(z) = \int \log(\|z\|/|z\cdot w|)d\mu(w) \text{ for } z \in \mathbf{C}^n;$$

here K is a compact circled subset of ∂B. Then

$$\mathscr{C}(K) = \sup_{\mu\in\mathfrak{M}(K)} \frac{1}{\sup_{z\in\partial B} u_\mu(z)}.$$

For $n > 2$, $\mathscr{C}(K)$ and $\operatorname{cap}(K)$ are inequivalent capacities.

We now define a third capacity $\mathscr{S}$-cap(K) by using the elements of $\mathscr{S}$ in the definition of cap in place of the $\mathscr{P}_k$. Namely let $\mathscr{S}_k = \mathscr{P}_k \cap \mathscr{S}$ and set

$$\mathscr{S}m_k(K) = \inf\{\|f\|_K : f \in \mathscr{S}_k\}.$$

Define $\mathscr{S}$-cap $(K) = \lim (\mathscr{S}m_k(K))^{1/k}$. Then $\mathscr{S}$-cap is equivalent to $\mathscr{C}$ in the following sense.

THEOREM 2.1. *For* $K \subseteq \mathbf{P}^{n-1}$,

$$A\cdot\mathscr{S}\text{-cap }(K) \leqq \exp\left(\frac{-1}{\mathscr{C}(K)}\right) \leqq \mathscr{S}\text{-cap }(K)$$

where $A = \exp\left(\int \log |z_1| d\sigma\right)$.

Remark. The definition of $\mathscr{S}$-cap was independently arrived at by Sibony [**9**] who obtained the second inequality of the theorem and also the relation cap $K \leqq \mathscr{S}$-cap (K) (which follows from $\mathscr{S}_k \subseteq \mathscr{P}_k$). He uses the fact, for $f \in \mathscr{P}_k$, that $f \in \mathscr{S}_k$ if and only if $f = \prod_{j=1}^k \alpha_j \cdot z$ where $\|\alpha_j\| = 1$.

For the proof of Theorem 2.1 we shall need the following.

THEOREM 2.2. *Let K be a compact circled subset of ∂B in $\mathbf{C}^n$. Then* $\mathscr{S}$-cap$(K) > 0$ *if and only if* $\mathscr{S}$-hull(K) *contains a neighborhood of the origin. In particular,*

$$\mathscr{S}\text{-hull}(K) \supseteq \{z : \|z\| \leqq \mathscr{S}\text{-cap}(K)\}.$$

Remarks. This is directly analogous to the results of [**1**] where $\mathscr{S}$-cap and $\mathscr{S}$-hull are replaced by cap and polynomial hull respectively. Combining the last two theorems, we can assert that the following are equivalent for $K \subseteq \mathbf{P}^{n-1}$ compact: (i) $\mathscr{C}(K) > 0$, (ii) $\mathscr{S}$-cap$(K) > 0$ and (iii) $\mathscr{S}$-hull(K) contains a neighborhood of the origin.

LEMMA 2.3. *For K compact and circled in ∂B and $f \in \mathscr{S}_k$,*

$$(\mathscr{S}\text{-cap } K)^k \leqq (\mathscr{S}m_k(K)) \leqq \|f\|_K.$$

Proof. As in [**1**], one shows that

$$\mathcal{S}\text{-cap}(K) \equiv \lim(\mathcal{S}m_k(K))^{1/k} = \inf(\mathcal{S}m_k(K))^{1/k}.$$

Proof of Theorem 2.2. First suppose that $\mathcal{S}$-cap$(K) = \rho > 0$. Observe that for $z \in \mathbf{C}^n$, $z \in \mathcal{S}$-hull(K) if and only if $|f(z)| \leqq \|f\|_K$ for all $f \in \mathcal{S}_k$ for all k. Now if $f \in \mathcal{S}_k$, Lemma 2.3 gives

$$\|f\|_{\partial B} \leqq 1 \leqq \|f\|_K/\rho^k.$$

Hence

$$\|f\|_{\rho B} = \rho^k\|f\|_B \leqq \|f\|_K;$$

i.e., $\rho B \subseteq \mathcal{S}$-hull$(K)$.

Conversely if $\rho B \subseteq \mathcal{S}$-hull$(K)$ for $\rho > 0$, we get for any $f \in \mathcal{S}_k$,

$$\rho^k\|f\|_B = \|f\|_{\rho B} \leqq \|f\|_K.$$

Hence

$$\text{cap}\ (\mathbf{P}^{n-1}) \leqq \|f\|_B{}^{1/k} \leqq \frac{1}{\rho}\|f\|_K{}^{1/k}.$$

This implies that $\mathcal{S}$-cap$(K) \geqq \rho\cdot\text{cap}(\mathbf{P}^{n-1}) > 0$.

Proof of Theorem 2.1. We shall first derive the second inequality. Let $\mu \in \mathfrak{M}(K)$ with $u_\mu(z) \leqq q < \infty$ for all $z \neq 0$. Thus for $\|\alpha\| = 1$,

$$\int \log|\alpha\cdot w|d\mu(w) = -u_\mu(\alpha) \geqq -q.$$

For $p \in \mathcal{S}_k$, since $p(w) = \Pi(\alpha_j\cdot w)$ with $\|\alpha_j\| = 1$, we get

$$\log\|p\|_K \geqq \int \log|p|d\mu = \sum_{j=1}^k \int \log|\alpha_j\cdot w|d\mu(w) \geqq -kq.$$

Hence $\|p\|_K{}^{1/k} \geqq e^{-q}$ and $\mathcal{S}$-cap$(K) \geqq e^{-q}$. As $1/q$ can be chosen arbitrarily close to $\mathcal{C}(K)$, the second inequality follows.

Let $\rho = \mathcal{S}$-cap(K), we may assume that ρ is strictly positive. Then by Theorem 2.2, $\mathcal{S}$-hull$(K) \supseteq \bar{B}\rho$. Now apply Theorem 1.1 on Jensen measure for the MSG $\mathcal{S}(X)$, where X is $\mathcal{S}$-hull(K), with boundary K. For the measure μ take unit surface measure on $\partial B_\rho \subseteq X$; namely

$$\int_{\partial B\rho} g d\mu = \int_{\partial B} g(\rho z)d\sigma(z).$$

Let ν be the associated Jensen measure on $K \subseteq \partial B$.

Now take z with $\|z\| = 1$. Viewing $P(z) \equiv z\cdot w$ as an element of $\mathcal{S}_1$ we have

$$\int\log|P|d\mu \leqq \int\log|P|d\nu.$$

This yields

$$\log \rho + \int_B \log |z \cdot w| d\sigma(w) \leqq \int \log |z \cdot w| d\nu.$$

The integral on the left is equal to $\log A \equiv \int \log |w_1| d\sigma(w)$. We get

$$\int \log \frac{\|z\|}{|z \cdot w|} d\nu(w) \leqq -\log (A \cdot \rho).$$

This implies $\mathscr{C}(K) \geqq -1/\log(A \cdot \rho) > 0$ which gives the first inequality of (2.1).

The capacity $\mathscr{C}$ in some sense measures the size of a set of hyperplanes while cap measures complex lines. The next result reflects this fact. For $K \subseteq \mathbf{P}^{n-1}$ define

$$\begin{aligned} K^* &= \{z \in \mathbf{P}^{n-1} : z \cdot w = 0 \text{ for some } w \in K\} \\ &= \cup \{H^w : w \in K\} \end{aligned}$$

where

$$H^w = \{z \in \mathbf{P}^{n-1} : z \cdot w = 0\}.$$

Proposition 2.4. *If* $\mathscr{C}(K) > 0$ *then* $\operatorname{cap}(K^*) > 0$.

Remark. The converse is false: take K to be a hyperplane. Then $K^* = \mathbf{P}^{n-1}$ but $\mathscr{C}(K) = 0$.

Proof. We view K as a subset of the unit sphere in $\mathbf{C}^n$. There exists a probability measure μ on K such that

$$\varphi(z) \equiv \int \log |z \cdot w| d\mu(w) \geqq -M > -\infty \text{ for } \|z\| = 1.$$

Arguing by contradiction, suppose that $\operatorname{cap}(K^*) = 0$, so that K^* is locally pluripolar in $\mathbf{P}^{n-1}$. It follows that $\Pi^{-1}(K^*)$ is locally pluripolar in $\mathbf{C}^n$ where $\Pi : \mathbf{C}^n \backslash \{0\} \to \mathbf{P}^{n-1}$ is the natural projection. Set $L = \{0\} \cup \Pi^{-1}(K^*)$.

Now for $z \in \mathbf{C}^n \backslash L$ and $w \in K (\subseteq \partial B)$ we have $z \cdot w \neq 0$. It follows that φ is a pluriharmonic function on $\mathbf{C}^n \backslash L$. Since φ is locally bounded above and below (except at the origin) and L is locally pluripolar, it follows that φ extends to be pluriharmonic on all of $\mathbf{C}^n$, the origin included since $n \geqq 2$. But $\varphi(\lambda z) = \log |\lambda| + \varphi(z)$ implies $\varphi(z) \to -\infty$ as $z \to 0$. This is a contradiction.

Corollary. $\operatorname{cap}(K) > 0 \Rightarrow \mathscr{C}(K^*) > 0$.

Proof. $\operatorname{cap}(K) > 0 \Rightarrow \mathscr{C}(K) > 0 \Rightarrow \operatorname{cap}(K^*) > 0 \Rightarrow \mathscr{C}(K^*) > 0$.

3. An application. The following is a version of the classical Hartogs lemma (see [**6**], p. 21).

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LEMMA 3.1. *Let Ω be a complex manifold and L a compact subset of Ω with non-empty interior L^0. Let $\{\varphi_n\}_1^\infty$ be plurisubharmonic on Ω, uniformly bounded on compact subsets, with* $\limsup \varphi_n \equiv -\infty$ *on L. Then $\{\varphi_n\}$ converges uniformly to $-\infty$ on each compact subset of Ω.*

Proof. Let $\varphi = \limsup \varphi_n$ and let φ^* be its upper semicontinuous regularization, which is known to be plurisubharmonic on Ω. Then $\varphi \equiv -\infty$ on L^0 implies $\varphi^* \equiv -\infty$ on L^0 and so $\varphi^* \equiv -\infty$ on Ω. Hence $\varphi \equiv -\infty$ on Ω and the conclusion now follows from the classical Hartogs lemma in $\mathbf{C}^n$.

LEMMA 3.2. *Let Ω be a complex manifold, L a compact subset of Ω and K a compact subset of Ω which is not pluripolar. Then there exists an α, $0 < \alpha < 1$, such that*

$$\| f \|_L \leqq \| f \|_K{}^\alpha \| f \|_\Omega{}^{1-\alpha}$$

for all holomorphic functions f on Ω.

Remark. This generalizes the Three Regions Lemma of Bishop [**3**] where L and K are taken as the closures of open sets. Although we shall not need the converse, the validity of such inequalities characterizes non-locally pluripolar sets K in Ω. An alternate proof could be based on the work of Gamelin–Sibony [**5**]. Or one can consider the class $\mathscr{F}$ of negative psh functions on Ω which are $\leqq -1$ on K. One shows that the uppersemicontinuous regularization of sup $\mathscr{F}$ is again in $\mathscr{F}$ and hence bounded from zero on L and then one applies the fact that an appropriate multiple of $\log |f|$ lies in $\mathscr{F}$.

Proof. By enlarging L we may suppose that L^0 is non-empty. We argue by contradiction and suppose that no such α exists. Then for $n = 1, 2, \ldots$ there exist f_n holomorphic on Ω such that

(i) $\|f_n\|_\Omega = 1$, and

(ii) $\|f_n\|_L > \|f_n\|_K{}^{1/n}$.

Choose $c_n > 0$ so that

$$\max_L c_n \log |f_n| = -1,$$

and set $\varphi_n = c_n \log |f_n|$. Then φ_n is plurisubharmonic on Ω, $\varphi_n < 0$, and

(iii) $\max_L \varphi_n = -1$.

Put $\varphi = \limsup \varphi_n$ on Ω. By (ii) and (iii), $\varphi_n < -n$ on K and so $\varphi \equiv -\infty$ on K. We claim that φ is not $\equiv -\infty$ on L. Otherwise, by the Hartogs Lemma 3.1, φ_n would converge uniformly to $-\infty$ on L, contradicting (iii). Hence we can choose $z_0 \in L$ and $q \neq -\infty$ and

$n_j \geqq j$ for $j = 1, 2, \ldots$ such that $\varphi_{n_j}(z_0) > q$ for each j. Now set

$$\psi = \sum_1^\infty \frac{1}{j^2} \varphi_{n_j};$$

ψ is plurisubharmonic (as a decreasing sequence of psh functions). We have $\psi(z_0) \not\equiv -\infty$ and $\psi \equiv -\infty$ on K, contradicting the assumption that K is not pluripolar. This proves the lemma.

THEOREM 3.3. *Let Σ be an irreducible local subvariety of $\mathbf{P}^{n-1}$ which is not contained in a hyperplane and let E be a compact subset of Σ which is not a locally pluripolar subset of Σ. Then $\mathscr{C}(E) > 0$.*

Remarks. Molzon, Shiffman and Sibony proved this in the case when Σ is a global (closed) subvariety of $\mathbf{P}^{n-1}$ (hence algebraic by Chow's theorem). Theorem 3.3 also contains another of their results; namely, if γ is a non-degenerate real analytic arc imbedded in $\mathbf{P}^{n-1}$, then $\mathscr{C}(\gamma) > 0$. In fact, γ is then a non-locally polar set in the holomorphic curve Σ obtained by extending the imbedding map for γ from the real axis to a domain in the complex plane.

We shall apply the following fact which will be proved below.

LEMMA 3.4. *With Σ as in Theorem 3.3, let $F \subseteq \Sigma$ be a closed ball in some local coordinates. Then $\mathscr{C}(F) > 0$.*

Proof of Theorem 3.3. We may assume that Σ is a local submanifold of $\mathbf{P}^{n-1}$ (since E could not be contained in the singular set of Σ). With $\Pi : \mathbf{C}^n \backslash 0 \rightarrow \mathbf{P}^{n-1}$ the natural projection, let Ω be the complex manifold

$$\Pi^{-1}(\Sigma) \cap \{z \in \mathbf{C}^n : 1/2 < \|z\| < 2\}$$

and let

$$K = \Pi^{-1}(E) \cap \partial B \subseteq \Omega.$$

It is straightforward to deduce that K is not locally pluripolar in Ω from the fact that E is not locally pluripolar in Σ (cf. [**8**], Lemma 2.5).

Choose a compact set $F \subseteq \Sigma$ which is a ball in local coordinates and let $L = \Pi^{-1}(F) \cap \partial B \subseteq \Omega$. By Lemma 3.4, $\mathscr{C}(L) > 0$. By Lemma 3.2, there exists α, $0 < \alpha < 1$, such that

$$(*) \qquad \|f\|_L \leqq \|f\|_K{}^\alpha \|f\|_\Omega{}^{1-\alpha}$$

for all f holomorphic on Ω.

Arguing by contradiction we suppose that $\mathscr{C}(K) = 0$. Then, by Theorem 2.1, $\mathscr{S}$-cap$(K) = 0$ and so there exists a sequence $\{f_k\}$ with $f_k \in \mathscr{S}_k$ such that $\|f_k\|_K^{1/k} \rightarrow 0$. Since $\|z\| < 2$ for $z \in \Omega$ we have

$$\|f_k\|_\Omega \leqq 2^k \|f_k\|_B \leqq 2^k.$$

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Thus, taking a kth root in (*) gives

$$\|f_k\|_L^{1/k} \leqq (\|f_k\|_K^{1/k})^\alpha 2^{1-\alpha}.$$

This implies that $\|f_k\|_L^{1/k} \to 0$; i.e., $\mathscr{S}$-cap$(L) = 0$. Hence, by Theorem 2.1, $\mathscr{C}(L) = 0$, a contradiction.

Proof of Lemma 3.4. This can be deduced from the work of Molzon, Shiffman and Sibony [**8**] but we shall give a direct proof based on the following elementary lemma.

LEMMA 3.5. *Let $\varphi_1, \varphi_2, \ldots, \varphi_n$ be holomorphic functions on the closed unit ball B in $\mathbf{C}^s$. Suppose that $\{\varphi_k\}_1^n$ are linearly independent over $\mathbf{C}$. Then there exists a real constant C such that*

$$\int_B \log \left|\sum_{k=1}^n z_k\varphi_k(\zeta)\right| d\lambda(\zeta) \geqq \log \|z\| - C$$

where $d\lambda$ is unit volume on $B \subseteq \mathbf{C}^s$.

Proof. Without loss of generality we may assume that $\sum\|\varphi_k\|_B^2 \leqq 1$ and hence that

$$|\sum z_k\varphi_k(\zeta)| \leqq \|z\| \text{ for } \zeta \in B.$$

Then there exists $A > 1$ such that

$$(*) \quad \int_B \log\left(\frac{\left|\sum_{k=1}^n z_k\varphi_k(\zeta)\right|}{\|z\|}\right) d\lambda(\zeta) \geqq A \cdot \sup_{\|\zeta\|\leqq 1/2} \log\left(\frac{\left|\sum_{k=1}^n z_k\varphi_k(\zeta)\right|}{\|z\|}\right)$$

for $z \neq 0$. In fact Jensen's inequality is the case with $\zeta = 0$ in the quotient on the right hand side, with the sup deleted and with $A = 1$; (*) follows from this by applying automorphisms of the ball which move the origin, using the negativity of the integrand. We get

$$(**) \quad \int_B \log \left|\sum z_k\varphi_k(\zeta)\right| d\lambda \geqq A \log |||z||| - (A - 1) \log \|z\|$$

where

$$|||z||| \equiv \sup \{|\sum z_k\varphi_k(\zeta)| : \|\zeta\| \leqq \tfrac{1}{2}\}.$$

The assumption of linear independence implies that $|||z||| \neq 0$ for $z \neq 0$. It follows easily that $|||\cdot|||$ is a norm for $\mathbf{C}^n$ and since these are all equivalent, we have $D > 0$ such that $|||z||| \geqq D\|z\|$. Thus the right hand side of (**) dominates

$$\log \|z\| + A \log D.$$

This gives Lemma 3.5.

Now Lemma 3.4 follows easily. Let μ be the probability measure on F which is induced from the unit volume measure λ on the ball B in $\mathbf{C}^s$ by the local coordinates on Σ. With $\zeta_1, \zeta_2, \ldots, \zeta_n$ the homogeneous global projective coordinates on $\mathbf{P}^{n-1}$ (we may assume that $\zeta_n \neq 0$ on F) let ψ_k, $1 \leqq k \leqq n$, be the holomorphic function on B which correspond to ζ_k/ζ_n on F. From the fact that Σ does not lie in a hyperplane in $\mathbf{P}^{n-1}$ it follows that $\{\varphi_k\}_1^n$ are linearly independent in B. Now Lemma 3.5 transplanted back to F yields

$$\int \log(|z \cdot \zeta|/\|\zeta\|)d\mu(\zeta) \geqq \log \|z\| - C$$

for some C; i.e., $\mathscr{C}(F) > 0$.

4. Zero capacity and locally pluripolar sets. We shall end with short proofs of Theorems 6.4 and 6.7 of [**1**]. The original proof of the latter involved a complicated application of a proposition of Josefson [**7**]. It is much simpler to apply the basic theorem of Josefson that locally polar in $\mathbf{C}^n$ implies globally polar; a nice proof of this, based on their theory of the Monge–Ampere operator, has recently been given by Bedford and Taylor [**2**].

THEOREM 4.1. *Let $K \subseteq \mathbf{P}^{n-1}$ be a compact locally pluripolar set. Then* $\operatorname{cap}(K) = 0$.

Proof. Let $\Pi : \mathbf{C}^n \backslash \{0\} \to \mathbf{P}^{n-1}$ be the natural projection. The fact that K is locally pluripolar in $\mathbf{P}^{n-1}$ implies that $\Pi^{-1}(K)$ is locally pluripolar in $\mathbf{C}^n$ and hence globally polar in $\mathbf{C}^n$. Say $\Pi^{-1}(K) \subseteq \{\varphi = -\infty\}$ with φ psh on $\mathbf{C}^n$, $\varphi \not\equiv -\infty$. Let $E = \Pi^{-1}(K) \cap \partial B$. We want to show that $\operatorname{cap}(E) = 0$. It suffices by [**1**] to show that $\hat{E}$ does not contain a neighborhood of the origin. Suppose otherwise, that $\hat{E} \supseteq B_\delta$. Then, as the polynomial hull agrees with the psh hull (see [**6**], p. 91), we have

$$\sup_{B_\delta} \varphi \leqq \sup_E \varphi = -\infty.$$

Therefore $\varphi = -\infty$ on B_δ and hence on $\mathbf{C}^n$, a contradiction.

THEOREM 4.2. *Let L be a compact non-locally pluripolar subset of $\mathbf{P}^{n-1}$ or, more generally, let K be a non-locally pluripolar subset of an irreducible subvariety Σ of $\mathbf{P}^{n-1}$. (In the first case take Σ to be $\mathbf{P}^{n-1}$.) Then $\hat{K}$ contains a neighborhood of* 0 *in* $\Pi^{-1}(\Sigma) \cup \{0\} \subseteq \mathbf{C}^n$.

Remark. The general case was obtained in [**8**] as a consequence of the case $\Sigma = \mathbf{P}^{n-1}$ which is Theorem 6.4 of [**1**]. The following proof shows that it is a direct consequence of Lemma 3.2 and the method of [**1**].

Proof. We view K as a circled subset of ∂B which (as noted in the proof of Theorem 3.3) is non-locally polar in

$$\Omega \equiv (\Pi^{-1}(\Sigma) \cup \{0\}) \cap B_2.$$

Take $L = \Omega \cap B_1$ in Lemma 3.2 to get

$$\| f \|_{B_1 \cap \Omega} \leqq \| f \|_K{}^{\alpha} \| f \|_{\Omega}{}^{1-\alpha}$$

for f holomorphic on Ω. Apply this to a homogeneous polynomial f of degree k to get

$$\| f \|_{B_1 \cap \Omega} \leqq \| f \|_K{}^{\alpha} (2^k \| f \|_{B_1 \cap \Omega})^{1-\alpha}.$$

Hence

$$(*) \qquad \|f\|_{B_1 \cap \Omega} \leqq \frac{1}{\rho^k} \|f\|_K$$

for $\rho = 1/2^{(1/\alpha - 1)}$. Now, by the argument of [**1**] Section 4, (*) implies $\hat{K} \supseteq B_\rho \cap \Omega$, as claimed.

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