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FARTHEST POINTS AND THE FARTHEST DISTANCE MAP

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In this paper, we consider farthest points and the farthest distance map of a closed bounded set in a Banach space. We show, *inter alia*, that a strictly convex Banach space has the Mazur intersection property for weakly compact sets if and only if every such set is the closed convex hull of its farthest points, and recapture a classical result of Lau in a broader set-up. We obtain an expression for the subdifferential of the farthest distance map in the spirit of Preiss' Theorem which in turn extends a result of Westphal and Schwartz, showing that the subdifferential of the farthest distance map is the unique maximal monotone extension of a densely defined monotone operator involving the duality map and the farthest point map.

1. INTRODUCTION

We work with real scalars. The closed unit ball and the unit sphere of a Banach space X will be denoted by $B(X)$ and $S(X)$ respectively. Our notations are otherwise standard. Any unexplained terminology can be found in [**3**].

For a closed and bounded set K in a Banach space X, the farthest distance map r_K is defined as

$$r_K(x) = \sup\{\|z - x\| : z \in K\},$$

$x \in X$. For $x \in X$, we define the farthest point map as

$$Q_K(x) = \{z \in K : \|z - x\| = r_K(x)\},$$

that is, the set of points of K farthest from x. Note that this set may be empty. Let

$$D(K) = \{x \in X : Q_K(x) \neq \emptyset\}.$$

The set of farthest points of K will be denoted by $\text{far}(K)$, that is,

$$\text{far}(K) = \cup\{Q_K(x) : x \in D(K)\}.$$

Call a closed and bounded set K densely remotal if $D(K)$ is norm dense in X.

We say that a Banach space X has the Mazur Intersection Property if every closed bounded convex set in X is the intersection of closed balls containing it. The Mazur

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Intersection Property is a well studied notion in geometry of Banach space and several authors have studied Mazur-like intersection properties for different families of closed bounded convex sets. See [**1, 2**] and references thereof for a survey and unified treatments. However, no complete characterisation is available, in particular, for every weakly compact convex set in X to be intersection of balls.

Lau [**6**, Theorem 3.3] had shown that a reflexive Banach space X has the Mazur Intersection Property if and only if every closed bounded convex set in X is the closed convex hull of its farthest points. In Section 2, we show that in a strictly convex Banach space X, every weakly compact convex set is intersection of balls if and only if every such set is the closed convex hull of its farthest points. Similar conclusions hold for compact convex sets, compact convex sets of finite affine dimension. And if X has the Radon-Nikodým Property, then similar result holds for $w*$-compact convex sets in X^*.

Recall that the subdifferential of a convex function $\phi : X \to \mathbb{R}$ at $x \in X$ is

$$\partial\phi(x) = \left\{x^* \in X^* : x^*(y - x) \leqslant \phi(y) - \phi(x) \text{ for all } y \in X\right\}.$$

The subdifferential of the function $\phi(x) = \|x\|^2/2$ is referred to as the duality map on X and is denoted by $\mathcal{D}$.

Since r_K is a continuous convex function, ∂r_K is a maximal monotone operator defined on X. In [**8**, Proposition 4.3], the authors showed that if X is a reflexive Banach space with X^* Fréchet smooth, then for a closed bounded set K, ∂r_K is the unique maximal monotone extension of $\mathcal{D}(I - Q_K)/r_K$ and for each $x \in X$,

$$\partial r_K(x) = \bigcap_{\delta>0} \overline{\mathrm{co}}\left\{\mathcal{D}\frac{I - Q_K}{r_K}(y) : \|y - x\| < \delta\right\}. \tag{1}$$

Note that this is actually the Preiss' Theorem (see [**7**]) for ∂r_K.

For a nonreflexive space, such a statement needs qualification as $\mathcal{D}(I - Q_K/r_K)(y)$ may be empty for some y. Nonetheless, even in nonreflexive spaces, for a densely remotal set K, $\mathcal{D}(I - Q_K)/r_K$ is a well-defined monotone operator with dense domain. We show that if X is locally uniformly convex/rotund, then ∂r_K remains the unique maximal monotone extension of $\mathcal{D}(I - Q_K)/r_K$ and an analogue of (1) is available where we need to take the $w*$-closure and choose y from the set $D_1(K)$ defined below. We believe that this is the only version of the Preiss' Theorem for ∂r_K available in the nonreflexive case.

For a closed and bounded set $K \subseteq X$, $x \in X$ and $\alpha > 0$, a crescent of K determined by x and α is the set

$$C(K, x, \alpha) = \left\{z \in K : \|z - x\| > r_K(x) - \alpha\right\}.$$

Let K be a closed and bounded set in X. Let $x \in X$ and $k \in Q_K(x)$. We say $x \in D_1(K)$ if k is contained in crescents of K determined by x of arbitrarily small diameter. It is easy to note that if $x \in D_1(K)$, then $Q_K(x)$ is necessarily singleton.

2. Intersection of Balls and Farthest Points

Here is the main theorem of this section. As mentioned in the introduction, this, in particular, gives the only known characterisation of when every weakly compact convex set in X is intersection of balls.

Theorem 2.1.

(a) *If X is a strictly convex Banach space and $\mathcal{C}$ is one of the following families of sets,*

(i) $\mathcal{K} = \{$*all compact convex sets in* $X\}$.

(ii) $\mathcal{F} = \{$*all compact convex sets in* X *with finite affine dimension*$\}$.

(iii) $\mathcal{W} = \{$*all weakly compact convex set in* $X\}$.

then every $K \in \mathcal{C}$ is intersection of balls if and only if every $K \in \mathcal{C}$ is the closed convex hull of its farthest points.

(b) *If X has the Radon-Nikodým Property, then X^* has the $w*$-Mazur Intersection Property if and only if every $w*$-compact convex set in X^* is the $w*$-closed convex hull of its farthest points.*

(c) [**6**] *If X is reflexive, then X has the Mazur Intersection Property if and only if every closed bounded convex set in X is the closed convex hull of its farthest points.*

Proof: (a) We give the proof for the family $\mathcal{W}$ of weakly compact sets. The same proof works in the other cases too.

Necessity. Let $K \in \mathcal{W}$ and thus by [**6**, Theorem 2.3], K is densely remotal. We claim every crescent of K contains a farthest point of K.

To see this, let $C(K, x, \alpha)$ be any crescent of K. Choose ε and β such that $0 < \varepsilon < \alpha/2$ and $0 < \beta < \alpha - 2\varepsilon$. Since K is densely remotal, there exists $y \in D(K)$ such that $\|x - y\| < \varepsilon$. Then clearly

$$Q_K(y) \subseteq C(K, y, \beta) \subseteq C(K, x, \alpha).$$

Now let $L = \overline{\text{co}}(\text{far}(K))$. Then $L \in \mathcal{W}$ is also intersection of balls. So if $K \setminus L \neq \emptyset$, there exists a crescent C of K disjoint from L. By the above claim, $C \cap \text{far}(K) \neq \emptyset$. But, of course, $\text{far}(K) \subseteq L$. This proves the necessity.

Sufficiency. Suppose there exists $K \in \mathcal{W}$ that is not intersection of balls. Let $\widetilde{K} = \cap\{B : B \text{ is a closed ball and } K \subseteq B\}$. Let $x_0 \in \widetilde{K} \setminus K$. Choose $y_0 \in K$ and $0 < \lambda < 1$ such that $z_0 = \lambda x_0 + (1 - \lambda) y_0 \notin K$.

Let $K_1 = \text{co}(K \cup \{z_0\})$. Then $K_1 \in \mathcal{W}$. We shall show that $\text{far}(K_1) \subseteq K$, and hence, $K_1 \neq \overline{\text{co}}(\text{far}(K_1))$.

Let $x \in X$. Then

$$\widetilde{K} \subseteq \{u \in X : \|u - x\| \leqslant r_K(x)\}.$$

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Note that

$$r_K(x) \leqslant r_{K_1}(x) \leqslant r_{\widetilde{K}}(x) = r_K(x).$$

Clearly, z_0 as well as any point of the form

$$v = \alpha z_0 + (1-\alpha)z, \quad \alpha \in (0,1], \quad z \in K, \tag{2}$$

are not extreme points of $\widetilde{K}$, and since X is strictly convex, they are not farthest points as well. Therefore, $\|v - x\| < r_K(x)$. Thus, $Q_{K_1}(x) \subseteq K$. Since $x \in X$ was arbitrary, $\mathrm{far}(K_1) \subseteq K$.

(b) If X has the Radon-Nikodým Property, by [**4**, Proposition 3], each $w*$-compact set $K \subseteq X^*$ is densely remotal. Thus, necessity can be proved as in (a).

For sufficiency, note that if there exists a $w*$-compact set K that is not intersection of balls, since $K = \bigcap_{\lambda>0} [K + \lambda B(X^*)]$, passing to some $K + \lambda B(X^*)$ if necessary, we may assume that K has nonempty interior. Now if we choose $y_0 \in \mathrm{int}(K)$, then z_0 and any point of the form (2) are interior points of $\widetilde{K}$, and hence the result follows as before. ∎

The following observation is immediate from the above arguments.

Proposition 2.2. *Every closed bounded convex set in X is the closed convex hull of its farthest points if and only if*

(a) *X has the Mazur Intersection Property ; and*

(b) *for every closed bounded convex set $K \subseteq X$, every crescent of K contains a farthest point of K.*

Note that the proof of Theorem 2.1 shows that if K is densely remotal, then K satisfies (b) above. This also captures the essential argument in the proof of [**6**, Theorem 3.2]. Following example shows that we cannot dispense with (b).

Example 2.3. The space c_0 has a strictly convex Fréchet differentiable renorming [**3**, Theorem 7.1(ii)] which, thus, has the Mazur Intersection Property . However, since the unit ball of the usual norm on c_0 lacks extreme points, it must lack farthest points in the new norm.

Remark 2.4. This also shows that even if X^* has the Radon-Nikodým Property, there may exist a closed bounded convex set in X with $\mathrm{far}(K) = \emptyset$ (Compare this with [**4**, Proposition 3]).

Observe that since the bidual of a space with the Mazur Intersection Property has the $w*$- Mazur Intersection Property , Theorem 2.1 (b) shows that every $w*$-compact convex set in ℓ^∞, with the bidual of the above norm, is the $w*$-closed convex hull of its farthest points. Thus, there is a closed bounded convex set $K \subseteq c_0$, such that no farthest point of the $w*$-closure of K in X^{**} comes from K.

3. The Farthest Distance Map

We begin by collecting some simple properties of the set $D_1(K)$. Recall that a sequence $\{z_n\} \subseteq K$ is called a maximising sequence for x if $\|x - z_n\| \longrightarrow r_K(x)$.

Proposition 3.1. *Let K be closed bounded set in a Banach space X.*

(a) *$x \in D_1(K)$ if and only if any maximising sequence for x converges.*

(b) *If $x \in D_1(K)$, then Q_K is single valued and continuous at x and $Q_K(x)$ is a strongly exposed point of K.*

(c) *$D_1(K)$ is a G_δ in X.*

The following proposition shows that any discussion on $D_1(K)$ naturally require some convexity conditions on the norm.

Proposition 3.2.

(a) *A Banach space X is strictly convex if and only if for every compact set K and $k \in \operatorname{far}(K)$, there exists $x \in D_1(K)$, such that $Q_K(x) = \{k\}$.*

(b) *A Banach space X is locally uniformly convex/rotund if and only if for every closed bounded set K and $k \in \operatorname{far}(K)$, there exists $x \in D_1(K)$, such that $Q_K(x) = \{k\}$.*

Proof: (a) Let K be a compact set in a strictly convex Banach space X. Let $k \in \operatorname{far}(K)$. Then, $k \in Q_K(x)$ for some $x \in D(K)$. Let $t > 1$. Strict convexity of the norm shows that for $y = k + t(x - k)$, $Q_K(y) = \{k\}$. Now compactness shows that $y \in D_1(K)$.

Conversely, if X is not strictly convex, there exists $x, y \in S(X)$ such that the line segment $[x, y] \subseteq S(X)$. Clearly, $K = [x, y]$ is compact and $K \subseteq Q_K(0)$. But any point of the open segment (x, y) cannot be strongly exposed and therefore, cannot be in the set $Q_K\big(D_1(K)\big)$.

(b) Observe that $S(X) \subseteq Q_{B(X)}(0)$. For any $x \in S(X)$ and any sequence $\{x_n\} \subseteq B(X)$ that is maximising for $-x$, we have $\|x + x_n\| \to 2$. So if X is locally uniformly convex/rotund, $x_n \to x$. Thus $-x \in D_1\big(B(X)\big)$ and $x \in Q_{B(X)}(-x)$. Now for a closed bounded set $K \subseteq X$ and $k \in \operatorname{far}(K)$, get $x \in D(K)$ such that $k \in Q_K(x)$, and apply this argument with suitable translation and scaling to the ball $B\big[x, r_K(x)\big]$.

To prove the converse, let $K = B(X)$. Then, $S(X) = \operatorname{far}\big(B(X)\big)$. So by the hypothesis, it follows that every point in $S(X)$ is a strongly exposed point of $B(X)$, and therefore, X is strictly convex.

Now let $x_0 \in S(X)$. By hypothesis, there exists $x \in D_1\big(B(X)\big)$ such that $Q_K(x) = \{x_0\}$. Then $\|x - x_0\| = r_{B(X)}(x) = 1 + \|x\|$. By strict convexity, it follows that $x = \alpha x_0$ for some $\alpha \in \mathbf{R}$ and $|\alpha - 1| = 1 + |\alpha|$. Therefore, $\alpha < 0$.

To show X is locally uniformly convex/rotund, let $\{x_n\} \subseteq B(X)$ be such that $\|x_n + x_0\| \to 2$. For each n consider the function on $(0,1)$,

$$f_n(\lambda) = 1 - \left\|\lambda x_n + (1-\lambda)x_0\right\|$$

Then for all $\lambda \in (0,1)$, $f_n(\lambda) \geqslant 0$. And by triangle inequality,

$$2f_n(1/2) \geqslant f_n(\lambda) + f_n(1-\lambda) \geqslant f_n(\lambda) \geqslant 0$$

By assumption, $f_n(1/2) \to 0$. Thus, for any $\lambda \in (0,1)$, $f_n(\lambda) \to 0$. In particular, putting $\lambda = 1/(1-\alpha)$, we get $\|x_n - \alpha x_0\| \to (1-\alpha)$, that is $\{x_n\}$ is a maximising sequence for $x = \alpha x_0$. Hence, $x_n \to x_0$. ∎

The following two lemmas are crucial in proving our main theorem of this section.

Lemma 3.3. *Suppose X is locally uniformly convex/rotund and $K \subseteq X$ is densely remotal. Then $D_1(K)$ is a dense G_δ in X.*

Proof: By Proposition 3.1 (d), it suffices to show that $D_1(K)$ is dense in X.

Let $x \in D(K)$. Get $k \in Q_K(x)$. $0 < \varepsilon < 1$. Let $y = k + (1+\varepsilon)(x-k)$. Then, $\|x - y\| = \varepsilon r_K(x)$. It is easy to see that $r_K(y) = (1+\varepsilon)r_K(x)$ and by strict convexity, k is a unique farthest point from y.

We now claim $y \in D_1(K)$. Let $\{z_n\} \subseteq K$ be a maximising sequence for y. That is, $\|z_n - y\| \to (1+\varepsilon)r_K(x)$. Then,

$$\left\|\frac{(z_n - x) + \varepsilon(k-x)}{(1+\varepsilon)}\right\| \to r_K(x)$$

Then

$$y_n = (z_n - x)/r_K(x) \in B(X), \quad y_0 = (k-x)/r_K(x) \in S(X),$$

and for $\lambda = 1/(1+\varepsilon)$, we have $\left\|\lambda y_n + (1-\lambda)y_0\right\| \to 1$. Notice that since $\varepsilon < 1$, $1/2 < \lambda < 1$. As in the proof of Proposition 3.2 (b), let

$$f_n(\lambda) = 1 - \left\|\lambda y_n + (1-\lambda)y_0\right\|.$$

By convexity of the norm,

$$f_n(\lambda) \geqslant (2-2\lambda)f_n(1/2) \geqslant 0$$

Since $f_n(\lambda) \to 0$, we have that $f_n(1/2) \to 0$, that is, $\|y_n + y_0\| \to 2$. Since X is locally uniformly convex/rotund, $y_n \to y_0$ and hence, $z_n \to k$. ∎

Remark 3.4. It follows that for any weakly compact set K in a locally uniformly convex/rotund Banach space, $D_1(K)$ is a dense G_δ in X. So our result is more general than [**5**, Corollary 2.8], where it is proved that if the norm on X^* is Fréchet differentiable, then for any closed and bounded subset $K \subseteq X$, $D_1(K)$ is residual.

Lemma 3.5. *Let $x \in D_1(K)$. Then, $\partial r_K(x) = \mathcal{D}(I - Q_K/r_K)(x)$.*

Moreover, r_K is Gâteaux (respectively Fréchet) differentiable at x if and only if the norm is Gâteaux (respectively Fréchet) differentiable at $x - Q_K(x)$.

Proof: Let $Q_K(x) = \{k\}$ and $x^* \in \mathcal{D}\big(x - k/r_K(x)\big)$. Then $x^*(x-k) = r_K(x)$. For

$$z \in X, \quad x^*(z - x) = x^*(z) - x^*(k) - r_K(x) \leqslant r_K(z) - r_K(x).$$

Thus $x^* \in \partial r_K(x)$.

Conversely, let $x^* \in \partial r_K(x)$. Since $\mathcal{D}(I - Q_K/r_K)(x)$ is a $w*$-closed convex subset of $S(X^*)$, it is enough to show that for any $z \in S(X)$, there is an $x_0^* \in \mathcal{D}(I - Q_K/r_K)(x)$ such that $x^*(z) \leqslant x_0^*(z)$.

Let $\{k_n\} \subseteq K$ be such that

$$\|x + z/n - k_n\| > r_K(x + z/n) - 1/n^2.$$

Then $\{k_n\}$ is a maximising sequence for x, and hence, $k_n \to k$. Now

$$x^*\left(\frac{z}{n}\right) = x^*\left(x + \frac{z}{n}\right) - x^*(x) \leqslant r_K\left(x + \frac{z}{n}\right) - r_K(x) < \left\|x + \frac{z}{n} - k_n\right\| - r_K(x) + \frac{1}{n^2}.$$

Choose $x_n^* \in \mathcal{D}(x + z/n - k_n)$. Then

$$x_n^*\left(\frac{z}{n}\right) = x_n^*\left(x + \frac{z}{n} - k_n\right) - x_n^*(x - k_n) \geqslant \left\|x + \frac{z}{n} - k_n\right\| - r_K(x).$$

Combining the two, we have $x^*(z) \leqslant x_n^*(z) + 1/n$. Let x_0^* be a $w*$-cluster point of $\{x_n^*\}$. Since $x + z/n - k_n$ converges to $x - k$ in norm, we have $x_0^* \in \mathcal{D}(I - Q_K/r_K)(x)$ and $x^*(z) \leqslant x_0^*(z)$, as desired.

Thus, the norm is Gâteaux differentiable is at $x - k \Leftrightarrow \mathcal{D}\big(x - k/r_K(x)\big)$ is singleton $\Leftrightarrow$ so is $\partial r_K(x) \Leftrightarrow r_K$ is Gâteaux differentiable at x.

Now, let $\{x^*\} = \partial r_K(x) = \mathcal{D}\big(x - k/r_K(x)\big)$. For any $\lambda \in \mathbb{R}$ and

$$z \in B(X), \quad x^*(\lambda z) \leqslant \|x + \lambda z - k\| - \|x - k\| \leqslant r_K(x + \lambda z) - r_K(x).$$

Therefore,

$$\left|\frac{\|x + \lambda z - k\| - \|x - k\|}{\lambda} - x^*(z)\right| \leqslant \left|\frac{r_K(x + \lambda z) - r_K(x)}{\lambda} - x^*(z)\right|.$$

Thus, Fréchet differentiability of r_K at x implies that of the norm at $x - k$.

Conversely, let the norm be Fréchet differentiable at $x - k$. Let $x_n \to x$, $x_n^* \in \partial r_K(x_n)$ and $x^* \in \partial r_K(x)$, then $\{x_n^*\} \subseteq B(X^*)$ and since r_K is Gâteaux differentiable at x, $x_n^* \to x^*$ in the $w*$-topology. Since $x^* \in \mathcal{D}(I - Q_K/r_K)(x)$, x^* is a $w*$-norm point of continuity of $B(X^*)$, and therefore, $x_n^* \to x^*$ in norm. It follows that r_K is Fréchet differentiable at x. ∎

REMARK 3.6. [5, Theorem 3.2(a)] proves only the "necessity" part of this result. Our proof is also simpler.

Combining Lemma 3.5 with Lemma 3.3, it follows that in a Banach space with smooth locally uniformly convex/rotund norm, the farthest distance map r_K of a densely remotal set K is Gâteaux differentiable on a dense G_δ.

We now state the main theorem of this section. This gives an expression for ∂r_K in the spirit of Preiss' Theorem [7]. Note that our result does not need smoothness of the norm and with (Fréchet) smoothness, by Theorem 3.5, we get back Preiss' Theorem for ∂r_K.

THEOREM 3.7. *Let K be such that $D_1(K)$ is dense in X and $x \in X$. Then*

$$\partial r_K(x) = \bigcap_{\delta>0} \overline{\mathrm{co}}^*\{\partial r_k(y) : y \in D_1(K) \text{ and } \|y-x\| < \delta\}.$$

PROOF: Let $x^* \in$ right hand side and $\varepsilon > 0$. Choose $\delta < \varepsilon/3$. For $z \in X$, choose $y \in D_1(K)$ and $y^* \in \partial r_K(y)$ such that $\|y-x\| < \delta$ and $x^*(z-x) < y^*(z-x) + \delta$. Thus,

$$\begin{aligned} x^*(z-x) &< y^*(z-x) + \delta = y^*(z-y) + y^*(y-x) + \delta \leqslant r_K(z) - r_K(y) + 2\delta \\ &\leqslant r_K(z) - r_K(x) + 3\delta \leqslant r_K(z) - r_K(x) + \varepsilon. \end{aligned}$$

Since ε is arbitrary, we have $x^* \in \partial r_K(x)$.

Conversely, let $x^* \in \partial r_K(x)$. As in Lemma 3.5, we shall show given any $z \in S(X)$ there is an $x_0^* \in$ right hand side such that $x^*(z) \leqslant x_0^*(z)$.

For each n, get $y_n \in D_1(K)$ such that $\|x + z/n - y_n\| < 1/n^2$. Then

$$x^*\left(\frac{z}{n}\right) \leqslant r_K\left(x + \frac{z}{n}\right) - r_K(x) \leqslant r_K(y_n) - r_K\left(y_n - \frac{z}{n}\right) + \frac{2}{n^2}.$$

Let $x_n^* \in \partial r_K(y_n)$. Let $k_n \in Q_K(y_n)$. Then

$$x_n^*\left(\frac{z}{n}\right) = x_n^*(y_n - k_n) - x_n^*\left(y_n - \frac{z}{n} - k_n\right) \geqslant r_K(y_n) - r_K\left(y_n - \frac{z}{n}\right).$$

Thus $x^*(z) \leqslant x_n^*(z) + 2/n$. Let x_0^* be a $w*$-cluster point of x_n^*. Then $x_0^* \in$ right hand side and $x^*(z) \leqslant x_0^*(z)$. ∎

Combining the Lemma 3.3, Lemma 3.5 and Theorem 3.7 we obtain the following:

COROLLARY 3.8. *Suppose K is a densely remotal set in a locally uniformly convex/rotund Banach space X. Then ∂r_K is the unique maximal monotone extension of the densely defined monotone operator $\mathcal{D}(I - Q_K)/r_K$ and for each $x \in X$, we have,*

$$\partial r_K(x) = \bigcap_{\delta>0} \overline{\mathrm{co}}^*\left\{\mathcal{D}\frac{I - Q_K}{r_K}(y) : y \in D_1(K) \text{ and } \|y-x\| < \delta\right\}.$$

REMARK 3.9. In [8, Proposition 4.3] obtained the similar conclusion for reflexive Banach spaces with X^* Fréchet smooth.

We end this section with a result on range of ∂r_K. Compare this with [8, Theorem 4.2].

THEOREM 3.10. *Let X be a smooth (respectively Fréchet smooth) Banach space. Let $K \subseteq X$ be a closed and bounded set such that $D_1(K)$ is dense in X, then the image of $D_1(K)$ under ∂r_K is $w*$-dense (respectively norm dense) in $S(X^*)$.*

PROOF: Let $NA(X)$ denote the set of norm attaining functionals in $S(X^*)$. By Bishop–Phelps Theorem, $NA(X)$ is norm dense in $S(X^*)$. Let $x_0^* \in NA(X)$ and $x_0 \in S(X)$ such that $x_0^*(x_0) = 1$. By density of $D_1(K)$, choose $x_n \in D_1(K)$ such that $\|x_n - nx_0\| < 1/n$ and let $x_n^* \in \partial r_K(x_n)$. Then $\|x_n\| \to \infty$. Therefore, by [8, Lemma 4.1], $\lim x_n^*\big(x_n/\|x_n\|\big) = \lim \|x_n^*\| = 1$. But since $x_n/\|x_n\| \to x_0$ in norm, $x_n^*(x_0) \to 1$ as well. Thus, any $w*$-cluster point of $\{x_n^*\}$ is in $\mathcal{D}(x_0)$. Since the norm is smooth, this set is singleton. Hence, $x_n^* \to x_0^*$ in $w*$-topology.

Now, if the norm on X is Fréchet smooth, then x_0^* chosen above is a $w*$-norm point of continuity of $B(X^*)$. Thus $\partial r_K\big(D_1(K)\big)$ is norm dense in $S(X^*)$. ∎

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