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A prime ideal and its quotient ring

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Given arbitrary rings A, B such that $A \subset B$ and an arbitrary prime ideal P of A, we show how to construct a quotient ring A_P such that $A \subset A_P \subset B$. The ring A_P contains a prime ideal $\bar{P}$ lying over P and the prime ring A/P may be embedded in $A_P/\bar{P}$.

The ring A_P is also defined in [1]. However, many of the proofs given there are valid only for rings with identity. The present paper generalizes these proofs to arbitrary rings. Furthermore the definition of the "lying over" prime ideal $\bar{P}$ of A_P (that is $\bar{P} \cap A = P$), although equivalent to that given in [1], has been slightly simplified. Some further observations have been made, such as the fact that $A_P/\bar{P}$ may be considered an extension ring of A/P.

Let A, B be arbitrary rings such that $A \subset B$. Let P be any prime ideal of A. Then A_P is defined to be the set of elements $b \in B$ such that there exists $s \in A - P$ (the complement of P in A) for which $sAb \subset A$ and $bAs \subset A$.

PROPOSITION 1. *A_P is a ring containing A.*

Proof. If $b \in A$ choose any $s \in A - P$. Hence $A_P \supset A$.

Let $b_1, b_2 \in A_P$ and suppose that $s_1, s_2 \in A - P$ are such that $s_1Ab_1 \subset A$, $b_1As_1 \subset A$, $s_2Ab_2 \subset A$ and $b_2As_2 \subset A$. Then since P is prime there exists $a \in A$ such that $s = s_1as_2 \in A - P$. Then

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$$\begin{aligned} sA(b_1+b_2)' &= s_1as_2A(b_1+b_2) \\ &\subset s_1(as_2A)b_1 + s_1a(s_2Ab_2) \\ &\subset A \ . \end{aligned}$$

Similarly $(b_1+b_2)As \subset A$ and hence $b_1 + b_2 \in A_P$.

Also

$$sAb_2b_1 = s_1a(s_2Ab_2)b_1 \subset s_1Ab_1 \subset A$$

and

$$b_2b_1As = b_2(b_1As_1)as_2 \subset b_2As_2 \subset A \ .$$

Hence $b_2b_1 \in A_P$ and therefore A_P is a ring.

Note. If S is any m-system of A (see [2], p. 195) then we can define more generally a ring A_S containing A . (See [1].)

Now let P denote the set of elements $b \in A_P$ such that there exists an $s \in A - P$ for which $sAbAs \subset P$.

PROPOSITION 2. *P is an ideal of A_P .*

Proof. Let $b_1, b_2 \in P$. Then there exist $s_1, s_2 \in A - P$ such that $s_1Ab_1As_1 \subset P$ and $s_2Ab_2As_2 \subset P$. Since P is prime there exists $a \in A$ such that $s = s_1as_2 \in A - P$. Then

$$\begin{aligned} sA(b_1-b_2)As &\subset s_1as_2Ab_1As_1as_2 + s_1as_2Ab_2As_1as_2 \\ &\subset (s_1Ab_1As_1)as_2 + s_1a(s_2Ab_2As_2) \\ &\subset P \ . \end{aligned}$$

Hence $b_1 - b_2 \in P$. Similarly for any $b \in A_P$, we can show $bb_1, b_1b \in P$. This proves that P is an ideal of A_P .

PROPOSITION 3. $P \cap A = P$.

Proof. Let $b \in P \cap A$. Then there exists $s \in A - P$ such that $sAbAs \subset P$. Hence, since P is prime, $sAb \subset P$ and therefore $b \in P$. This shows that $P \cap A \subset P$.

Now let $p \in P$. Then for any $s \in A - P$, $sApAs \subset P$. Hence $p \in P$ and $P \cap A = P$.

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THEOREM 1. *P is a prime ideal of A_P lying over P.*

Proof. From the previous propositions it remains only to prove that P is prime. Let $b_1 A_P b_2 \subset P$ where $b_1, b_2 \in A_P$. Suppose $s_1, s_2 \in A - P$ are such that $s_1 A b_1 \subset A$, $b_1 A s_1 \subset A$, $s_2 A b_2 \subset A$ and $b_2 A s_2 \subset A$. Since $b_1 A b_2 \subset P$ we have $s_1 A b_1 A b_2 A s_2 \subset P$. Hence by Proposition 3, $s_1 A b_1 A b_2 A s_2 \subset P$. Since $s_1 A b_1 \subset A$, $b_2 A s_2 \subset A$ and P is prime this implies $s_1 A b_1 \subset P$ or $b_2 A s_2 \subset P$. If $s_1 A b_1 \subset P$ then $s_1 A b_1 A s_1 \subset P$ and hence $b_1 \in P$. Similarly, if $b_2 A s_2 \subset P$ then $b_2 \in P$. Hence either b_1 or $b_2 \in P$, that is P is prime.

A ring is called a *prime ring* if 0 is a prime ideal. Thus when P is prime the rings A/P and A_P/P are examples of prime rings. It will be shown that A_P/P may be considered as an extension ring of A/P.

THEOREM 2. *If P is a prime ideal of A then A/P may be embedded in A_P/P.*

Proof. The subring of A_P/P consisting of all elements of A_P/P of the form $a + P$ where $a \in A$ will be shown to be isomorphic to A/P. In fact the mapping f defined by $f(a+P) = a + P$ is a monomorphism from A/P into A_P/P. Clearly f is a homomorphism, and to verify that f is one-to-one, suppose $f(a+P) = f(a'+P)$ where $a, a' \in A$. Then $a + P = a' + P$, that is, $a - a' \in P$ and hence, by Theorem 1, $a - a' \in P$, that is, $a + P = a' + P$.

References

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