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ARTICLE

Random embeddings of bounded-degree trees with optimal spread

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Abstract
A seminal result of Komlós, Sárközy, and Szemerédi states that any n-vertex graph G with minimum degree at least $(1/2+\alpha)n$ contains every n-vertex tree T of bounded degree. Recently, Pham, Sah, Sawhney, and Simkin extended this result to show that such graphs G in fact support an *optimally spread distribution* on copies of a given T, which implies, using the recent breakthroughs on the Kahn-Kalai conjecture, the *robustness* result that T is a subgraph of sparse random subgraphs of G as well. Pham, Sah, Sawhney, and Simkin construct their optimally spread distribution by following closely the original proof of the Komlós-Sárközy-Szemerédi theorem which uses the blow-up lemma and the Szemerédi regularity lemma. We give an alternative, regularity-free construction that instead uses the Komlós-Sárközy-Szemerédi theorem (which has a regularity-free proof due to Kathapurkar and Montgomery) as a black box. Our proof is based on the simple and general insight that, if G has linear minimum degree, almost all constant-sized subgraphs of G inherit the same minimum degree condition that G has.

Keywords: Tree embedding; Dirac's theorem; spread distributions
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1. Introduction

There is a large body of results in extremal graph theory focusing on determining the minimum degree threshold which forces the containment of a target subgraph. For example, a classical result of Dirac [6] states that any n-vertex graph with minimum degree at least $n/2$ contains a Hamilton cycle. Although this result is tight, graphs with minimum degree $n/2$ are quite dense, so it is natural to suspect that they are Hamiltonian in a rich sense. In this direction, Sárközy, Selkow, and Szemerédi [25] showed that n-vertex graphs with minimum degree $n/2$ contain $\Omega(n)^n$ distinct Hamilton cycles (we refer to such results as *enumeration* results; see also [5]). Moreover, randomly sparsifying the edge set of an n-vertex graph with minimum degree $n/2$ yields, with high probability, another Hamiltonian graph, as long as each edge is kept with probability $\Omega(\log n/n)$. This follows from an influential result of Krivelevich, Lee, and Sudakov [17] (such results are referred to as *robustness* results; see [26]), which generalises Pósa's celebrated result stating that the random graph $G(n, C\log n/n)$ is Hamiltonian with high probability.

The study of random graphs was recently revolutionised by Frankston, Kahn, Narayanan, and Park's [7] proof of the *fractional expectation threshold vs. threshold* conjecture of Talagrand [27] (see also [21] for a proof of the even stronger Kahn–Kalai conjecture [11]). In our context, these breakthroughs imply that the enumeration and robustness results stated in the first paragraph, which themselves are fairly general, admit a a further common generalisation. To state this generalisation, we need the language of *spread distributions* which we will define momentarily. In a nutshell, the key idea is to show that a graph G, with minimum degree large enough to necessarily contain a copy of a target graph H, actually supports a *random embedding* of H that (roughly speaking) resembles a uniformly random function from $V(H)$ to $V(G)$. The formal definition we use is below.

Definition 1 [24]. *Let X and Y be finite sets, and let μ be a probability distribution over injections $\varphi{:}X \to Y$. For $q \in [0,1]$, we say that μ is q-spread if for every two sequences of distinct vertices $x_1, \ldots, x_s \in X$ and $y_1, \ldots, y_s \in Y$,*

$$\mu\left(\left\{\varphi{:}\varphi(x_i) = y_i \text{ for all } i \in [s]\right\}\right) \leqslant q^s.$$

In our context, $X = V(H)$, $Y = V(G)$, $|X| = |Y| = n$, and μ is a probability distribution over embeddings of H into G. The gold standard for us is constructing distributions μ that are $O(1/n)$-spread. Such distributions have the optimal spread (up to the value of the implied constant factor) that is also attained by a uniformly random injection from $V(H)$ to $V(G)$. We remark that Definition 1, originally introduced in [24], is different to the usual definition of spreadness phrased in terms of edges instead of vertices. However, for embedding spanning subgraphs, the above definition turns out to be more convenient (see [14, 24] for more details).

The breakthroughs on the Kahn–Kalai conjecture have created a lot of incentive to show "spread versions" of Dirac-type results in graphs and hypergraphs, as such results directly imply enumeration and robustness results, thereby coalescing two streams of research which have, until now, been investigated independently. We refer the reader to the recent papers [2, 9, 12, 14, 24] that obtain several results in this direction (see also [1]). Most of the aforementioned work focuses on constructing spread distributions for target graphs with rather simple structures, such as perfect matchings or Hamilton cycles. One notable exception is the result from [24] for bounded-degree trees. To introduce this result, we first cite the following classical result in extremal graph theory.

Theorem 2 (Komlós–Sárközy–Szemerédi [15]). *For every $\Delta \in \mathbb{N}$ and $\alpha > 0$, there exists $n_0 \in \mathbb{N}$ such that the following holds for all $n \geqslant n_0$. If G is an n-vertex graph with $\delta(G) \geqslant (1+\alpha)\frac{n}{2}$, then G contains a copy of every n-vertex tree with maximum degree bounded by Δ.*

Theorem 2 admits a spread version, as demonstrated in [24].

Theorem 3 (Pham, Sah, Sahwney, Simkin [24]). *For every $\Delta \in \mathbb{N}$ and $\alpha > 0$, there exists $n_0, C \in \mathbb{N}$ such that the following holds for all $n \geqslant n_0$. If G is an n-vertex graph with $\delta(G) \geqslant (1+\alpha)\frac{n}{2}$, and T is a n-vertex tree with $\Delta(T) \leqslant \Delta$, there exists a (C/n)-spread distribution on embeddings of T onto G.*

Using the $s = n$ case of Definition 1, Theorem 3 allows us to deduce that in the context of Theorem 2, G contains $\Omega(n)^n$ copies of a given bounded-degree tree (see [10] for a more precise result). Furthermore, Theorem 3 implies that the random subgraph $G' \subseteq G$ obtained by keeping each edge of G with probability $\Omega(\log n/n)$ also contains a given bounded-degree tree (see [24] for a precise statement).

The original proof [15] of Theorem 2 constitutes one of the early applications of the Szemerédi regularity lemma (used in conjunction with the blow-up lemma of Komlós, Sárközy, and Szemerédi). The proof of the more general Theorem 3 in [24] can be interpreted as a randomised version of the proof in [15]. Indeed, readers familiar with applications of the regularity/blow-up lemma would know that whilst embedding a target subgraph with this method, there is actually a lot of flexibility for where each vertex can go. Thus, a choice can be made randomly from the available options as a reasonable strategy towards proving Theorem 3.

The main contribution of the current paper is a proof of Theorem 3 that uses Theorem 2 as a black box. The most obvious advantage of such a proof is that, as Theorem 2 has a more modern

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proof due to Montgomery and Kathapurkar [13] that circumvents the use of Szemerédi regularity lemma, our proof yields a regularity-free proof of Theorem 3 which naturally has better dependencies between the constants (see Remark 17). Our proof is presented in Section 3, and Section 3.1 contains an overview explaining the key ideas.

1.1 Future directions

Our proof can also be interpreted as modest progress towards a more ambitious research agenda, hinted to in [14], which asserts that *all* Dirac-type results admit a spread version, regardless of the target structure being embedded. One reason why such a general result could hold is that in Dirac-type results, host graphs have linear minimum degree. Thus, Chernoff's bound can be used to show that almost all $O(1)$-sized induced subgraphs of such dense host graphs maintain the same (relative) minimum degree condition. If the target graph itself has some recursive structure, we may use this to our advantage whilst constructing a random embedding with optimal spread. The strategy would be to first break up the target graph into pieces of size $O(1)$, for example, in the case of Hamilton cycles, we would simply break up the cycle into several subpaths. For each $O(1)$-sized subpath, almost all $O(1)$-sized subsets of the host graph have large enough minimum degree to necessarily contain a copy of the subpath (simply by invoking Dirac's theorem), so we may choose one such host subset randomly while constructing a random embedding with good spread.

A variant of the above strategy was successfully implemented in [14] in the context of hypergraph Hamilton cycles. In this paper, we devise a novel strategy that works for bounded-degree spanning trees, which, like Hamilton cycles, have a recursive structure, albeit a lot more complex than that of Hamilton cycles. In particular, a Hamilton cycle can be thought of as a union of subpaths of essentially the same length, whereas any partition of a tree into further subtrees needs to use subtrees of a wide range of possible sizes (see Section 2.1), which makes it difficult to extend the techniques of [14] to bounded-degree trees.

Our proof is fairly short; however, we explain the key idea in Section 3.1. We believe our methods are fairly general, and they could translate to construct spread distributions in the context of directed trees [13], hypertrees [22, 23], or other related structures such as spanning grids.

It remains an interesting open problem to find an even larger class of target graphs for which the Dirac-type theorem admits a spread generalisation. For example, it would be natural to investigate graph families with sublinear bandwidth, and we believe our methods could be applicable here. Note that this would entail more than simply randomising the blow-up lemma-based proof of the Bandwidth Theorem [3], as this theorem does not always give the optimal minimum degree condition for the containment of every graph family with sublinear bandwidth. Though, of course, obtaining a spread version of the bandwidth theorem would be of independent interest.

Komlós, Sárközy, and Szemerédi [16] actually proved a stronger result than Theorem 2, where the maximum degree hypothesis is relaxed as $\Delta(T) = o(n/\log n)$. It would be interesting to similarly strengthen Theorem 3 by weakening the assumption on $\Delta(T)$.

2. Preliminaries

We use the standard notation for hierarchies of constants, writing $x \ll y$ to mean that there exists a non-decreasing function $f : (0, 1] \to (0, 1]$ such that the subsequent statements hold for $x \leqslant f(y)$. Hierarchies with multiple constants are defined similarly. We use standard graph theory notations, given a graph G, $V(G)$ denote the vertex set of G, $E(G)$ the edges set, and $|G| = |V(G)|$ denotes the size of the vertex set.

We will use the following theorem to embed our subtrees of bounded size. It generalises Theorem 2 and was proved in [20], using tools from [13]. In particular, the proof of the following theorem does not rely on the Szemerédi regularity lemma.

Theorem 4 (Theorem 4.4 [20]). *Let $1/n \ll 1/\Delta, \alpha$. Let G be an n-vertex graph with $\delta(G) \geqslant (1/2 + \alpha)n$. Let T be an n-vertex tree with $\Delta(T) \leqslant \Delta$. Let $t \in V(T)$ and $v \in V(G)$. Then, G contains a copy of T with t copied to v.*

2.1 Tree-splittings

Definition 5. *Let T be an n-vertex tree. A tree-splitting of size ℓ is a family of edge-disjoint subtrees $(T_i)_{i\in[\ell]}$ of T such that $\bigcup_{i\in[\ell]} E(T_i) = E(T)$. Note that for any $i \neq j$, the subtrees T_i and T_j intersect on at most one vertex. Given a tree-splitting $(T_i)_{i\in[\ell]}$ of an n-vertex tree T, we define the bag-graph of the tree-splitting to be the graph whose nodes are indexed by $[\ell]$ and in which the nodes i and j are adjacent if $V(T_i) \cap V(T_j) \neq \emptyset$. A bag-tree of a tree-splitting is simply a spanning tree of the bag-graph.*

We will use the following simple proposition to divide a tree into subtrees (see, for example, [19, Proposition 3.22]).

Proposition 6. *Let $n, m \in \mathbb{N}$ such that $1 \leqslant m \leqslant n/3$. Given any n-vertex tree T containing a vertex $t \in V(T)$, there are two edge-disjoint trees $T_1, T_2 \subset T$ such that $E(T_1) \cup E(T_2) = E(T)$, $t \in V(T_1)$ and $m \leqslant |T_2| \leqslant 3m$.*

This implies that a tree can be divided into many pieces of roughly equal size, as follows (see [20] for the simple proof).

Corollary 7. *Let $n, m \in \mathbb{N}$ satisfy $m \leqslant n$. Given any n-vertex tree T, there exists a tree-splitting $(T_i)_{i\in[\ell]}$ of T such that for each $i \in [\ell]$, we have $m \leqslant |T_i| \leqslant 4m$.*

2.2 Probabilistic results

Below we give a lemma that encapsulates a simple argument often used when computing the spreadness of a random permutation.

Lemma 8. *Let $n \in \mathbb{N}$, $s \leqslant n$, and $L_1, \ldots, L_s \subseteq [n]$. For any distinct integers $1 \leqslant x_1, \ldots, x_s \leqslant n$, a uniformly sampled permutation π of $[n]$ satisfies $\mathbb{P}\left[\bigwedge_{i=1}^s \pi(x_i) \in L_i\right] \leqslant \prod_{i=1}^s \frac{e|L_i|}{n}$.*

Proof of Lemma 8. We have the following,

$$\mathbb{P}\left[\bigwedge_{i=1}^{s} \pi(x_i) \in L_i\right] = \prod_{i=1}^{s} \mathbb{P}\left[\pi(x_i) \in L_i \,\middle|\, \bigwedge_{j=1}^{i-1} \pi(x_j) \in L_j\right] \leqslant \prod_{i=1}^{s} \frac{|L_i|}{n-i+1} \leqslant \prod_{i=1}^{s} \frac{e|L_i|}{n},$$

where in the last step we used the fact that $\prod_{i=1}^s n-i+1 \geqslant \left(\frac{n}{e}\right)^s$, which is a well-known application of Stirling's approximation. □

Next, we present a lemma that records several properties we need from a random vertex partition of a dense graph.

Lemma 9. *Let $1/n \ll \eta, 1/C, 1/K, \delta, \alpha$, and suppose $1/C \ll 1/K \ll \alpha$. Fix two sequences of integers $(a_c)_{C\leqslant c\leqslant 4C} \geqslant 1$ and $(b_c)_{C\leqslant c\leqslant 4C} \geqslant 1$ such that $\sum_{C\leqslant c\leqslant 4C} b_c a_c < n$, $b_c \geqslant \eta n$ and $C - K \leqslant a_c \leqslant 4C - K$ for each $C \leqslant c \leqslant 4C$. Set $\varepsilon := e^{-\alpha^2 C/12}$. Let G be a n-vertex graph with $\delta(G) \geqslant (\delta + \alpha)n$ and let $v \in V(G)$. Then, there exists a random labelled partition $\mathcal{R} = (R_c^j)_{C\leqslant c\leqslant 4C, j\in[b_c]}$ of a subset of $V(G) \setminus \{v\}$ into $\sum_{C\leqslant c\leqslant 4C} b_c$ parts, with the following properties,*

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A1 $\forall c \in [C, 4C], \forall j \in [b_c], |R_c^j| = a_c$, *meaning there are exactly* b_c *parts of size* a_c;

A2 $\forall R_c^j \in \mathcal{R}$, $|\{u \in V(G) \mid \deg(u, R_c^j) \geqslant (\delta + \frac{\alpha}{2})|R_c^j + u|\}| \geqslant (1 - 3e^{-\frac{\alpha^2 C}{10}})|V(G)|$;

A3 $\forall u \in V(G)$, $|\{R_c^j \in \mathcal{R} \mid \delta(G[R_c^j + u]) \geqslant (\delta + \frac{\alpha}{2})|R_c^j|\}| \geqslant (1 - 3e^{-\frac{\alpha^2 C}{10}})|\mathcal{R}|$.

Moreover, call $R_c^j \in \mathcal{R}$ *good if* $\delta(G[R_c^j]) \geqslant (\delta + \alpha/2)|R_c^j|$. *Call* $R_c^j, R_d^k \in \mathcal{R}$ *a good pair if* $\delta(G[R_c^j + v]) \geqslant (\delta + \alpha/2)|R_c^j + v|$ *for each* $v \in R_d^k$, *and the same statement holds with* j *and* c *interchanged with* k *and* d. *Let* A *be the auxiliary graph with vertex set* $\mathcal{R}$ *where* $R_c^j \sim_A R_d^k$ *if and only if* R_c^j, R_d^k *is a good pair. Then, there exists a subgraph* A' *of* A *such that*

B1 $\forall c \in [C, 4C], A'_c := \{R_c^j \in A'\}$ *has size at least* $(1 - \varepsilon)b_c$;

B2 $\forall R_c^j \in A', \forall d \in [C, 4C], \deg_{A'}(R_c^j, A'_d) \geqslant (1 - \varepsilon)|A'_d|$;

B3 $\forall R_c^j \in A', R_c^j$ *is good.*

Furthermore, the following spreadness property holds. For any function $f \colon \{u_1, \ldots, u_s\} \to \mathcal{R}$ *(where* $\{u_1, \ldots, u_s\} \subseteq V(G)$*),* $\mathbb{P}[u_i \in f(u_i)] \leqslant (12C/n)^s$. □

The proof of Lemma 9 consists of standard applications of well-known concentration inequalities, that we state before proving Lemma 9.

Lemma 10 (Chernoff bound [4]). *Let* $X := \sum_{i=1}^m X_i$ *where* $(X_i)_{i \in [m]}$ *is a sequence of independent indicator random variables, and let* $\mathbb{E}[X] = \mu$. *For every* $\gamma \in (0, 1)$, *we have* $\mathbb{P}[|X - \mu| \geqslant \gamma\mu] \leqslant 2e^{-\mu\gamma^2/3}$.

Lemma 11 ([8], Lemma 3.5). *Let* $\ell \in \mathbb{N}$, $0 < \delta' < \delta < 1$ *and* $1/n, 1/\ell \ll \delta - \delta'$. *Let* G *be a* n*-vertex graph with* $\delta(G) \geqslant \delta n$. *If* $A \subseteq V(G)$ *is a vertex set of size* ℓ *chosen uniformly at random, then for every* $v \in V(G)$ *we have that* $\mathbb{P}\left[\deg(v, A) < \delta'\ell\right] \leqslant 2\exp(-\ell(\delta - \delta')^2/2)$.

Finally, we need the following result due to McDiarmid, which appears in the textbook of Molloy and Reed [18, Chapter 16.2]. Here, a *choice* is the position that a particular element gets mapped to in a permutation.

Lemma 12 (McDiarmid's inequality for random permutations). *Let* X *be a non-negative random variable determined by a uniformly sampled random permutation* π *of* $[n]$ *such that the following holds for some* $c, r > 0$:

1. *Interchanging two elements of* π *can affect the value of* X *by at most* c
2. *For any* s, *if* $X \geqslant s$ *then there is a set of at most* rs *choices whose outcomes certify that* $X \geqslant s$.

Then, for any $0 \leqslant t \leqslant \mathbb{E}[X]$, *we have that* $\mathbb{P}(|X - \mathbb{E}[X]| \geqslant t + 60c\sqrt{r\mathbb{E}[X]}) \leqslant 4\exp(-t^2/(8c^2 r\mathbb{E}[X]))$.

Proof of Lemma 9. Let us consider a uniformly sampled partition $\mathcal{R}$ of a subset of $V(G) \setminus v$ with exactly b_c parts of size a_c. We show that $\mathcal{R}$, conditional on the events **A1**, **A2**, **A3**, **B1**, **B2** and **B3**, has the desired spreadness property. First, we show the intersection of the events **A1**, **A2**, **A3**, **B1**, **B2** and **B3** has probability $1 - o(1)$.

We denote by $\mathcal{R}_c := \bigcup_{j=1}^{b_c} R_c^j$ the set of all vertices contained in a set of size a_c. For $u \in V(G)$ and $R_c^j \in \mathcal{R}$, we say that (u, R_c^j) is a *good pair* if $\delta(G[R_c^j + u]) \geqslant (\delta + 2\alpha/3)|R_c^j + u|$, and we say it is a *bad pair* if it is not a good pair. By Lemma 11, for any u and R_c^j, we have that

$$\mathbb{P}[(u, R_c^j) \text{ is a good pair}] \geqslant 1 - e^{-\alpha^2|R_c^j|/10} \geqslant 1 - e^{-\alpha^2 C/10} \qquad (*)$$

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Consider some $R_c^j \in \mathcal{R}$ and some $d \in [C, 4C]$, and let $X_{R_c^j,d}$ be the random variable counting the number of good pairs (u, R_c^j) where $u \in \mathcal{R}_d \setminus R_c^j$. We can naturally view the sets in $\mathcal{R} \setminus \{R_c^j\}$ as being obtained by looking at consecutive intervals of given lengths in a random permutation π of $V(G) \setminus R_c^j$. Then, interchanging two elements from π affects $X_{R_c^j,d}$ by at most 2, and for each s if $X_{R_c^j,d} \geqslant s$, we can certify this with specifying $4Cs$ coordinates of π (as each $a_d \leqslant 4C$). Thus Lemma 12 gives us the following inequality (using that $\mathbb{E}[X_{R_c^j,d}] \geqslant \left(1 - e^{-\alpha^2 C/10}\right) a_d b_d$, which follows from $(*)$),

$$\mathbb{P}\left[X_{R_c^j,d} \leqslant \left(1 - 3e^{-\alpha^2 C/10}\right) a_d b_d\right] \leqslant 4e^{-\frac{e^{-\alpha^2 C/10}}{100}\eta n} = o(n^{-1}).$$

By taking a union bound over all sets R_c^j in the partition $\mathcal{R}$, we obtain, with probability $1 - o(1)$, that for every set $R_c^j \in \mathcal{R}$ and for every $d \in [C, 4C]$, the number of bad pairs (u, R_c^j), with $u \in \mathcal{R}_d$, is at most $3e^{-\alpha^2 C/10} b_d$. Summing over all d, we see that A2 is satisfied with probability $1 - o(1)$. A similar reasoning over the random variable that counts the number of good pairs (u, R_c^j) for a fixed u instead of a fixed R_c^j, implies that A3 is satisfied with probability $1 - o(1)$.

For given sets $R_d^k, R_c^j \in \mathcal{R}$, a union bound and $(*)$ shows that

$$\mathbb{P}[\forall u \in R_d^k, (R_c^j, u) \text{ is a good pair}] \geqslant 1 - e^{-\alpha^2 C/20}. \tag{$**$}$$

Define A' to consist of the good sets $R_c^j \in \mathcal{R}$. The events **B1** and **B2** hold with high probability. This follows from $(**)$ and by computing the corresponding expectation and invoking 12 as before.

Let E denote the conjunctions of all of **A1**, **A2**, **A3**, **B1**, **B2** and **B3**, noting E has probability $1 - o(1)$. Consider s distinct vertices $u_1, \ldots, u_s \in V(G)$ and a function $f{:}\{u_1 \ldots, u_s\} \mapsto \mathcal{R}_c^j$,

$$\mathbb{P}\left[\bigwedge_{i=1}^{s} u_i \in f(u_i) \,\middle|\, E\right] \leqslant \mathbb{P}(E)^{-1} \cdot \mathbb{P}\left[\bigwedge_{i=1}^{s} u_i \in f(u_i)\right] \leqslant (1 - o(1)) \left(\frac{4eC}{n}\right)^s \leqslant \left(\frac{12C}{n}\right)^s,$$

where the second inequality follows from Lemma 8 for $x_i = u_i$ and $L_i = f(u_i)$. □

2.3 Good spread with high minimum degree

If the minimum degree of the host graph is larger than the size of the target graph we wish to embed, a simple random greedy algorithm can find a distribution with good spread. The following lemma records a version of this where vertices of the target and host graphs are coloured, and the embedding we produce respects the colour classes. This is used in the proof whilst (randomly) embedding the bag-tree of a tree splitting into an auxiliary graph where vertices represent random subsets of a host graph, edges represent good pairs (as in Lemma 9), and the colours represent the size of the random set.

Lemma 13. *Let $1/n \ll \gamma \ll \eta \leqslant 1$. Let G be a graph and $v \in V(G)$. Let $V_1 \cup \ldots \cup V_k$ be a partition of $V(G) \setminus \{v\}$ where $|V_i| \geqslant \eta n$ for all $i \in [k]$ and so that for all $i \in [k]$, for all $u \in V(G)$, $\deg(u, V_i) \geqslant (1-\gamma)|V_i|$. Let T be a tree and let t be a vertex of T. Let c be a k-colouring of $T - t$ such that the number of vertices coloured i is at most $(1-\eta)|V_i|$. Then, there exists a random embedding $\phi\colon T \to G$ such that the following all hold.*

1. *With probability 1, $\phi(t) = v$.*
2. *With probability 1, any i-coloured vertex of T is embedded in V_i.*
3. *The random embedding induced via ϕ by restricting to the forest $T - t$ is a $\left(\frac{2}{\eta^2 n}\right)$-spread embedding.*

Proof. Let us consider an ordering $t, t_1, t_2, \ldots, t_m$ of the vertices of T rooted in t, where $T[\{t, t_1, \ldots, t_i\}]$ is a subtree for each $i \in [m]$. We define $\phi: T \to G$ greedily vertex by vertex following the ordering of $V(T)$. Let $\phi(t) := v$. We denote by p_i the parent of t_i in T for all $i \geqslant 1$, and define $\phi(t_i)$ conditioned on some value of $\phi(t), \phi(t_1), \ldots \phi(t_{i-1})$ to be the random variable following the uniform distribution over $\left(V_{c(t_i)} \cap N(\phi(p_i))\right) \setminus \bigcup_{j=1}^{i-1} \phi(t_j)$. Note that,

$$\left|\left(V_{c(t_i)} \cap N(\phi(p_i))\right) \setminus \bigcup_{j=1}^{i-1} \{\phi(t_j)\}\right| \geqslant \deg(p_i, V_{c(t_i)}) - |c^{-1}(c(t_i))|$$
$$\geqslant ((1-\gamma) - (1-\eta))\, |V_{c(t_i)}| \geqslant (\eta - \gamma)\eta n.$$

We now discuss the spreadness of ϕ. Fix an integer $s \leqslant k$ and two sequences $t'_1, \ldots, t'_s \in V(T) \setminus \{t\}$ and $v_1, \ldots, v_s \in V(G) \setminus \{v\}$ of distinct elements. Moreover, let us suppose that $t'_1, t'_2, \ldots, t'_s$ appear in this order in the ordering chosen above. Observe that

$$\mathbb{P}\left[\bigwedge_{i=1}^{s} \phi(t'_i) = v_i\right] = \prod_{i=1}^{s} \mathbb{P}\left[\phi(t'_i) = v_i \,\middle|\, \bigwedge_{j=1}^{i-1} \phi(t'_j) = v_j\right] \leqslant \left(\frac{1}{(\eta-\gamma)\eta n}\right)^s.$$

The last inequality follows as for any $u \in V(G)$ and $i \in [k]$ the probability that $\phi(t_i) = u$ conditioned on any value of $\phi(t), \phi(t_1), \ldots, \phi(t_{i-1})$ is at most $\frac{1}{(\eta-\gamma)\eta n}$. In particular, the lemma follows as we have $\gamma \leqslant \eta/2$. □

2.4 Spread distributions on star matchings

At some point during our analysis, we will have to adjust precisely the size of some random partition of G, and we need to do this without damaging the randomness properties or the minimum degree conditions of the partition (coming from Lemma 9). The following lemma gives us a way to achieve this in the special case where each random set is meant to receive exactly one new element. In this context, the lemma below should be interpreted as being applied to an auxiliary bipartite graph, where one side of the bipartition represents the leftover vertices to be reassigned, and the other side represents the parts of the partition. There is an edge between a vertex v and a part R if and only if adding v to R preserves the randomness properties stated in Lemma 9.

Lemma 14 (Lemma 3.1 in [14]). *There exists an absolute constant C_{14} with the following property. If G is a balanced bipartite graph on $2n$ vertices with $\delta(G) \geqslant 3n/4$, then there exists a random perfect matching M of G such that for any collection of edges $e_1, \ldots, e_s \in E(G)$, $\mathbb{P}[\bigwedge_{i\in[s]} e_i \in M] \leqslant (C_{14}/n)^s$.*

We will need a slightly more general version, where a single part of the random partition will be assigned multiple vertices. The next result generalises the previous lemma into this context.

Corollary 15. *There exists an absolute constant C_{15} with the following property. Let $1/n \ll 1/k$. Let G be a bipartite graph with partition (A, B), with $|A| = n$, $|B| = kn$. Suppose for each $a \in A$, $d(a, B) \geqslant (99/100)|B|$ and for each $b \in B$, $d(b, A) \geqslant (99/100)|A|$. Then, there is a random $K_{1,k}$-perfect-matching M of G (where the centres of the $K_{1,k}$ are embedded in A) such that for any collection of edges $e_1, \ldots, e_s \in E(G)$, $\mathbb{P}[\bigwedge_{i\in[s]} e_i \in M] \leqslant (C_{15}/n)^s$.*

Proof. There exists an equipartition of B as $B_1, \ldots, B_k$ such that each $G[A \cup B_i]$ ($i \in [k]$) is a graph with minimum degree at least $(98/100)n$. Indeed, a random partition of B would have this property with high probability (as $n/k \to \infty$, see, for example, Lemma 3.5 from [8]). Now, Lemma 14 gives us a random perfect matching M_i in each $G[A \cup B_i]$ ($i \in [k]$), and $\bigcup_{i\in[k]} M_i =: M$ is a random $K_{1,k}$-perfect-matching of G. M clearly has the desired spread with $C_{15} = C_{14}$. □

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3. Proof of Theorem 3

3.1 Overview

As briefly discussed in Section 1.1, our proof capitalises on several desirable properties (as collected in Lemma 9) satisfied by a random partition of the vertex-set of the host graph G. In this way, our proof bears resemblance to the proof in [14]; however, we emphasise that the specific constructions of the distributions are otherwise quite different. In particular, the key idea in the current work could be used to give a distinct and more concise proof of the results in [14].

To start, we split T into $O(1)$-sized edge-disjoint subtrees via Corollary 7 and take a random partition $\mathcal{R}$ of the host graph given by Lemma 9 (we will comment on the choice of parameters momentarily). Almost all of the random subsets R in $\mathcal{R}$ have good enough minimum degree to contain all bounded-degree trees of size $|R|$ by just applying Theorem 2 as a black box. Now, we need to decide (randomly), which subtrees of T will embed into which random subsets of $V(G)$. This corresponds to randomly embedding a bag-tree of the tree-splitting into A', the auxiliary graph given in Lemma 9 that encodes the pairs of random sets with good minimum degree. Thus, we reduce Theorem 3 to a weaker version of itself where the host graph G is nearly complete (thanks to B2). Unfortunately, we do not have a way of directly producing the necessary random embedding even in this simpler context where the minimum degree of the host graph is extremely large.[1]

To circumvent this problem, we introduce the following trick, which we hope might have further applications. While applying Lemma 9, we make the sizes of the random sets an ε-fraction smaller than the sizes of the subtrees they are meant to contain. This gives us extra space as we then have more random sets than subtrees we need to embed. Afterwards, using a simple random greedy strategy (see Lemma 13), we can produce the necessary random embedding ψ of the bag-tree into A'. Two problems remain: some random sets are unused by ψ and the random sets that are used by ψ are too small to contain the subtrees we wish to embed. We fix both of these issues by randomly reallocating all vertices of the unused random sets into the used random sets using Corollary 15.

To finish the embedding, we need to convert ψ into a random embedding $\phi{:}V(T) \to V(G)$. We may do this by ordering the subtrees so that each subtree intersects the previous ones in at most one vertex and using Theorem 4, which is a slight strengthening of Theorem 2 that allows us to prescribe the location of a root vertex in advance. To illustrate, suppose $T_1, \ldots, T_{i-1}$ are already embedded, and suppose that $\psi(T_i)$ is an empty random set large enough to contain T_i. Say there exists some $t \in V(T_i) \cap V(T_j)$ for some $j < i$, then $\phi(t)$ is already determined, as T_j is already embedded. The properties of ψ, coming from Lemma 9 and Lemma 13, guarantee that $\phi(t) \cup \psi(T_i)$ has large minimum degree, so we may invoke Theorem 4 to extend ϕ to embed vertices of T_i in $\psi(T_i)$, respecting the previous choice of $\phi(t)$, as desired.

3.2 Proof of Theorem 3

We actually prove a stronger version of Theorem 3 where the location of a root vertex is specified in advance (similar to Theorem 4) as we believe this stronger result could have further applications. The unrooted version, i.e. Theorem 3, follows simply by choosing $t \in V(T)$ arbitrarily, and $v \in V(G)$ uniformly at random, setting $\phi(t) = v$, and using Theorem 16 to complete this to a full $O(1/n)$-spread embedding of T.

Theorem 16. *Let $1/n \ll 1/C_* \ll \alpha, 1/\Delta$. Let G be a n-vertex graph with $\delta(G) \geqslant (1/2 + \alpha)n$. Let T be a tree on n vertices, with $\Delta(T) \leqslant \Delta$. Let $t \in V(T)$ and let $v \in V(G)$. Then, there exists a random embedding $\phi\colon T \to G$ such that $\phi(t) = v$ with probability 1 and ϕ restricted to $T - t$ is $\left(\frac{C_*}{n}\right)$-spread.*

[1] In contrast, a similar method is employed in [14] to embed hypergraph Hamilton cycles, but here the "bag-tree" of a hypergraph Hamilton cycle is simply a 2-uniform Hamilton cycle, which is simpler to embed randomly with good spread using elementary methods.

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Remark 17. *All of the dependencies between the constants that arise from our proof are polynomial. However, we also need that C_* is at least a polynomial function in $f(\alpha, \Delta)$, where f is the function from Theorem 4 that ultimately relies on [13]. Unfortunately, [13] does not cite an explicit bound (though their proof does not use the Szemerédi regularity lemma).*

Proof of Theorem 16. Let C be a new constant such that $1/n \ll 1/C_* \ll 1/C \ll \alpha, 1/\Delta$. Let $(T_i)_{i\in[\ell]}$ be a tree-splitting of T obtained by Corollary 7 applied with $m \leftarrow C$. Notice that adding $T_* := \{t\}$ to this tree-splitting produces another tree-splitting. Let T' be a bag-tree of $(T_1, \dots T_\ell, T_*)$ rooted in T_*. Since T has maximum degree at most Δ, each vertex of T belongs to at most Δ subtrees of any tree-splitting, moreover $|T_i| \leqslant 4C$ for all $i \in [l] \cup \{*\}$, so T' has maximum degree at most $4C\Delta$.

Step 1: Randomly partition $V(G) \setminus \{v\}$. We assign to each subtree T_i a colour that corresponds to its size. Formally, define the colouring $f: V(T') \setminus \{T_*\} \to [C, 4C]$ via $f(T_i) := |T_i|$ for each subtree T_i. In particular, $|f^{-1}(c)|$ counts the number of subtrees of size c in the tree-splitting. Fix an integer K to be the amount by which the random clusters are shrunk compared to the subtrees that we embed into them, such that $1/C \ll 1/K \ll \alpha$. For each colour $c \in [C, 4C]$, let $a_c := c - 1 - K$ and $b_c := |f^{-1}(c)| + \left\lfloor \frac{K}{32C^3} n \right\rfloor$.

We use Lemma 9 on G with the following parameters, $(a_c)_{C\leqslant c\leqslant 4C} \leftarrow (a_c)_{C\leqslant c\leqslant 4C}$, $(b_c)_{C\leqslant c\leqslant 4C} \leftarrow (b_c)_{C\leqslant c\leqslant 4C}$, $\delta \leftarrow \frac{1}{2}$, $K \leftarrow K+1$, and $C, \alpha, v \leftarrow C, \alpha, v$. To do so, we only need to check that $\sum_c a_c b_c < n$, as the other conditions follow directly from our choice of constants. Observe

$$\sum_{c=C}^{4C} a_c b_c = \sum_{i=1}^{\ell} (|T_i| - 1 - K) + \left\lfloor \frac{K}{32C^3} n \right\rfloor \sum_{c=C}^{4C} (c - 1 - K) < |T| - \ell K + \left\lfloor \frac{K}{32C^3} n \right\rfloor \frac{5C(3C+1)}{2}$$
$$\leqslant n - \frac{K}{4C} n + \frac{K}{4C} n.$$

We can thus obtain a random partition $\mathcal{R} = (R_c^j)_{C\leqslant c\leqslant 4C, j\in[b_c]}$ of a subset of $V(G) \setminus \{v\}$ and an auxiliary graph A' whose vertex set $V(A')$ is a subset of $\mathcal{R}$, satisfying the conditions listed in Lemma 9. Add to A' a vertex $R_* = \{v\}$ adjacent to all $R_c^j \in \mathcal{R}$ such that $\delta(G[R_c^j + v]) \geqslant \left(\frac{1}{2} + \frac{\alpha}{2}\right) |R_c^j + v|$. For each colour $c \in [C, 4C]$, let us denote by $V_c := \{R_c^j \mid j \in [b_c]\} \cap V(A')$. We define the colouring $g: V(A') \to [C, 4C]$ that associates the colour c to all parts in V_c. Formally, $\forall c \in [C, 4C], \forall R_c^j \in V_c, g(R_c^j) = c$.

Step 2: Construct $\psi_{\mathcal{R}}: T' \to A'$. Conditional on a fixed outcome of $\mathcal{R}$ (and thus, A'), we describe $\psi_{\mathcal{R}}$, which is a random embedding of T' into A'. We apply Lemma 13 to A' with partition $V_C \cup \dots \cup V_{4C}$ and T' coloured by f, with parameters $t \leftarrow T_*$, $v \leftarrow R_*$, $\gamma \leftarrow e^{-\alpha^2 C/12}$ and $\eta \leftarrow \left\lfloor \frac{K}{32C^3} \right\rfloor$ to obtain a random embedding $\psi_{\mathcal{R}}$. To apply the lemma, we need the following conditions to be satisfied:

- $1/n \ll \gamma = e^{-\alpha^2 C/12} \ll \eta = \Theta\left(\frac{K}{C^3}\right)$,
- for all $i \in [C, 4C]$, for all $u \in V(A')$, $\deg(u, V_i) \geqslant (1 - \gamma)|V_i|$,
- for all c, the number of subtrees coloured c is at most $(1 - \eta)|V_c|$.

The first condition is satisfied by our constant hierarchy. Our choice of γ and condition **B1** of Lemma 9 is tailored so that $V(A')$ satisfies the second condition. The third condition is less direct. Let n_c be the number of subtrees coloured c in T', what we aim to show is $n_c \leqslant (1 - \eta)|V_c|$.

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By Condition **B1** of Lemma 9 and the definition of b_c, we have that $|V_c| \geqslant \left(1 - e^{-\frac{\alpha^2 C}{12}}\right)(n_c + \eta n)$. Hence,

$$(1-\eta)|V_c| \geqslant (1-\eta)\left(1 - e^{-\frac{\alpha^2 C}{12}}\right)(n_c + \eta n) \geqslant \left(1 - \eta - e^{-\frac{\alpha^2 C}{12}}\right)(n_c + \eta n) \geqslant (1-2\eta)\left(1 + \frac{\eta n}{n_c}\right) n_c$$
$$\geqslant (1-2\eta)(1+C\eta)n_c \geqslant n_c$$

where we used that $e^{-\frac{\alpha^2 C}{12}} \ll \eta = \Theta\left(\frac{K}{C^3}\right)$, and for the last step that $n_c \leqslant \frac{n}{C}$ and $C \geqslant 4$.
Recall that $\psi_{\mathcal{R}}: V(T') \to V(A')$ denotes the random embedding obtained from Lemma 13. By construction, $\psi_{\mathcal{R}}$ restricted to $T' \setminus \{T_*\}$ is $(\frac{2^{15}C^6}{n})$-spread, and note that this spreadness condition holds independently of the values of $\mathcal{R}$ and A' that we condition on. Lemma 13 also ensures that with probability 1, $\psi_{\mathcal{R}}$ preserves the colouring given by g, i.e. $\forall i \in [\ell], f(T_i) = g(\psi_{\mathcal{R}}(T_i))$.

Step 3: Adjust the size of the bags. In this step, we describe how to obtain a randomised partition $\mathcal{M}$ of $V(G) \setminus \{v\}$. We define $\mathcal{M}$ conditional on fixed values of $\mathcal{R}$ and $\psi_{\mathcal{R}}$. Informally, our goal is to build, for all $R_c^j \in \mathrm{Im}(\psi_{\mathcal{R}})$, a set M_c^j satisfying $R_c^j \subseteq M_c^j$ and $|M_c^j| = c - 1$ while preserving the minimum degree condition given by Lemma 9 for the set $\mathrm{Im}(\psi_{\mathcal{R}})$ and the edges induced by $\psi_{\mathcal{R}}$. Formally, defining $N(R_c^j) = \{R \in \mathrm{Im}(\psi_{\mathcal{R}}) | \{\psi_{\mathcal{R}}^{-1}(R_c^j), \psi_{\mathcal{R}}^{-1}(R)\} \in E(T')\}$, we want $\mathcal{M}$ to satisfy the following three properties:

C1 $\forall R_c^j \in \mathrm{Im}(\psi_{\mathcal{R}}), |M_c^j| = c - 1$;

C2 $\forall R_c^j \in \mathrm{Im}(\psi_{\mathcal{R}}), \delta(G[M_c^j]) \geqslant \left(\frac{1}{2} + \frac{\alpha}{3}\right)|M_c^j|$;

C3 $\forall R_c^j \in \mathrm{Im}(\psi_{\mathcal{R}}), \forall R_{c'}^{j'} \in N(R_c^j), \forall v \in M_c^j, \delta\left(G\left[M_{c'}^{j'} + v\right]\right) \geqslant \left(\frac{1}{2} + \frac{\alpha}{3}\right)|M_{c'}^{j'} + v|$.

Consider the following bipartite graph H with bipartition (A, B) where $A = V(G) \setminus \left(\{v\} \cup \bigcup_{R \in \mathcal{R}} R\right)$ and $B = \mathcal{R}$. Put an edge between $u \in A$ and $R \in B$ if $\forall R' \in \{R\} \cup N(R), \delta(G[R' + u]) \geqslant \left(\frac{1}{2} + \frac{\alpha}{2}\right)|R' + u|$. Note that, for all $(a, b) \in A \times B$, **A2** and **A3** imply that $d(a, B) \geqslant (1 - (4\Delta C + 1)3e^{-\frac{\alpha^2 C}{10}})|B| \geqslant \frac{99}{100}|B|$ and that $d(b, A) \geqslant (1 - (4\Delta C + 1)3e^{-\frac{\alpha^2 C}{10}}) \geqslant \frac{99}{100}|B|$. Therefore, we may use Corollary 15 on H with $k \leftarrow K$, to associate to each R_c^j a disjoint random set of K elements of B, satisfying the spreadness property stated in the lemma (regardless of the value of $\mathcal{R}$ and $\psi_{\mathcal{R}}$ that is being conditioned upon). Consider $L_c^j \subseteq A$, the set of random vertices that are matched by the $K_{1,K}$-perfect-matching to R_c^j, and define $M_c^j := R_c^j \cup L_c^j$. Note C1 is then directly satisfied. The definition of an edge in H and the fact that $K/C \ll \alpha$ imply that C2 and C3 are also both satisfied. We define $\mathcal{M} := \bigcup_{c,j}\{M_c^j\}$.

Having defined the random variable $\mathcal{M}$, we now show the following spreadness property. To clarify, $\psi_{\mathcal{R}}$ and $\mathcal{R}$ are not considered to be fixed anymore.

Claim 17.1. *For any $s \in \mathbb{N}$ and any function $h: \{v_1, \ldots, v_s\} \to \mathcal{M}$ where $\{v_1, \ldots, v_s\} \subseteq V(G)$, we have $\mathbb{P}\left[\wedge_{i=1}^s v_i \in h(v_i)\right] \leqslant (\frac{12C \cdot C_{15}}{n})^s$.* □

Proof. Note that if $v \in \{v_1, \ldots, v_s\}$ then $\mathbb{P}[v \in h(v)] = 0$ because $\mathcal{M}$ is a random partition of $V(G) \setminus \{v\}$. Suppose this is not the case, and let us partition $\{v_1, \ldots, v_s\}$ into $\{x_1, \ldots, x_{s_1}\} \subseteq A$ and $\{y_1, \ldots, y_{s_2}\} \subseteq V(G) \setminus (A \cup \{v\})$. Observe that

$$\mathbb{P}\left[\bigwedge_{i=1}^{s} v_i \in h(v_i)\right] = \mathbb{P}\left[\bigwedge_{i=1}^{s_2} y_i \in h(y_i)\right] \mathbb{P}\left[\bigwedge_{i=1}^{s_1} x_i \in h(x_i) \,\middle|\, \bigwedge_{i=1}^{s_2} y_i \in h(y_i)\right] \leqslant \left(\frac{12C}{n}\right)^{s_2}\left(\frac{C_{15}}{n}\right)^{s_1},$$

where we used the spreadness property from Lemma 9 and Corollary 15 in the last step. □

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Step 4: Embed the subtrees. From now on, we redefine g and $\psi_{\mathcal{R}}$ as being maps to $\mathcal{M} \cup \{R_*\}$ (by composing g and $\psi_{\mathcal{R}}$ with the natural bijection $\mathcal{R} \to \mathcal{M} \cup \{R_*\}$ that associates M_c^j to R_c^j, and R_* to itself).

We fix $\phi(t) := v$, by doing so we embed T_* into R_*. The goal is now to embed each T_i in $\psi_{\mathcal{R}}(T_i) \in \mathcal{M}$. Note that $|\psi_{\mathcal{R}}(T_i)| = |T_i| - 1$ for all $i \in [\ell]$ (by 1). We define ϕ as follows: While there exists a subtree T_i that is not fully embedded into G, pick a subtree T_i that has exactly one vertex t_i already embedded say $\phi(t_i) = v_i$ and apply Theorem 4 to embed the rest of T_i in $\psi_{\mathcal{R}}(T_i)$. We can use Theorem 4, due to C1, C2 and C3. This procedure is well defined because T' is a tree. Let us define the *native atom* of a vertex $y \in V(T)$, denoted by $T(y)$, to be the first T_i that contains y.

Checking spreadness. We now prove that the random embedding ϕ constructed this way is $\left(\frac{C_*}{n}\right)$-spread. The spreadness of this embedding comes from two different randomness sources: the partition $\mathcal{M}$ via Claim 17.1, and the random embedding $\psi_{\mathcal{R}}$ via Lemma 13.

Let us fix two sequences of distinct elements $y_1, \dots, y_s \in V(G - v)$ and $x_1, \dots x_s \in V(T - t)$. Let $b := |\{T_{x_i} \mid i \in [s]\}|$, where T_{x_i} is the first subtree of the tree-splitting that is embedded, among those containing x_i. We may suppose, up to reordering, that $x_1, \dots, x_b$ each have distinct native atoms, this way we have $\{T_{x_1}, \dots, T_{x_b}\} = \{T_{x_1}, \dots, T_{x_s}\}$. Let us split our probability on the two sources of spreadness as follows. Set $C_0 := 12C \cdot C_{15}$.

$$\begin{aligned}
\mathbb{P}\left[\bigwedge_{i=1}^{s} \phi(x_i) = y_i\right] &= \sum_{h:[b]\to\mathcal{M}} \mathbb{P}\left[\bigwedge_{i=1}^{s} \phi(x_i) = y_i \,\middle|\, \bigwedge_{i=1}^{s} y_i \in h(i)\right] \cdot \mathbb{P}\left[\bigwedge_{i=1}^{s} y_i \in h(i)\right] \\
&\leqslant \sum_{h:[b]\to\mathcal{M}} \mathbb{P}\left[\bigwedge_{i=1}^{s} \phi(x_i) = y_i \,\middle|\, \bigwedge_{i=1}^{s} y_i \in h(i)\right] \cdot \left(\frac{C_0}{n}\right)^s && \text{by Claim 17.1} \\
&\leqslant \sum_{h:[b]\to\mathcal{M}} \mathbb{P}\left[\bigwedge_{i=1}^{b} \psi_{\mathcal{R}}(T(x_i)) = h(i)\right] \cdot \left(\frac{C_0}{n}\right)^s \\
&\leqslant \sum_{h:[b]\to\mathcal{M}} \left(\frac{2C^2}{\eta^2 n}\right)^b \left(\frac{C_0}{n}\right)^s \leqslant |\mathcal{M}|^b \left(\frac{2C^2}{\eta^2 n}\right)^b \left(\frac{C_0}{n}\right)^s \\
&\leqslant n^b \left(\frac{2C^2}{\eta^2 n}\right)^b \left(\frac{C_0}{n}\right)^s && \text{by Lemma 9} \\
&\leqslant \left(\frac{2C^2 C_0}{\eta^2 n}\right)^s \leqslant \left(\frac{C_*}{n}\right)^s && b \leqslant s
\end{aligned}$$

To justify the second inequality, it is sufficient to observe that $\phi(x_i) = y_i$ only if $\psi_{\mathcal{R}}(T(x_i)) = h(i)$. Moreover by the remark made above, $T(x_1), \dots, T(x_b)$ are all distinct, so we can indeed invoke the spreadness property of $\psi_{\mathcal{R}}$ coming from Lemma 13. □

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