Problem Corner

Solutions are invited to the following problems. They should be addressed to Chris Starr c/o The Mathematical Association, Charnwood Building, Holywell Park, Loughborough University Science and Enterprise Park, Leicestershire, LE11 3AQ (e-mail: czqstarr@gmail.com) and should arrive not later than 10th July 2026.

Proposals for problems are equally welcome. They should also be sent to Chris Starr at the above address or e-mail and should be accompanied by solutions and any relevant background information.

109.I (Michael Fox)

The opposite sides of the convex plane hexagon $ABCDEF$ are parallel and equal in length. The circle passing through points A, B and C, and the circle passing through points and C, D and E meet at N.

(a) Prove that the circle passing through E, F and A also passes through N;

(b) Find a fourth circle that passes through three other vertices of the hexagon that also passes through N.

109.J (Stan Dolan)

For any positive integer n, let $S(n)$ be the set of non-negative integers of the form $n - x^2$ for some integer x. Let $a(n)$ be the least positive integer of $S(n)$.

As an example, $S(9) = \{0, 5, 8, 9\}$ and $a(9) = 5$. Note that $a(9)$ divides one of the other elements of $S(9)$.

Find all n such that $a(9)$ divides **no** other element of $S(9)$.

109.K (Albert Natian)

An ant finds itself at a vertex of an n-dimensional hypercube. Every time the ant is at a vertex of the hypercube, it randomly chooses, with probability $\frac{1}{n}$, one of the edges emanating from that vertex. It then walks along that edge and takes one second to reach the next vertex.

Starting from a vertex, how long, on average, will it take the ant to return to that vertex?

109.L (George Stoica)

Let $a, b, \alpha, \beta > 0$, $a \neq b$. Find all continuous functions $f : (0, \infty) \to (0, \infty)$ such that $f(x) = \alpha f(ax) = \beta f(bx)$ for all $x > 0$.

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Solutions and comments on **109.A**, **109.B**, **109.C**, **109.D** (March 2025)

109.A (Stan Dolan)

Show that for $k \geqslant 1$, the Fibonacci sequence modulo 2^k is cyclic with period 3.2^{k-1}. For example, the first 12 Fibonacci numbers modulo 4 are 1, 1, 2, 3, 1, 0, 1, 1, 2, 3, 1, 0… .

Solution:

The most common method of solution to this attractive problem was to use a Fibonacci identity and then establish the result by induction. The following is based on the solution offered by Z. Retkes.

Using the "addition formula" $F_{m+n} = F_{m-1}F_n + F_mF_{n+1}$, where F_n is the n-th Fibonacci number, we may derive the following two formulas:

$$F_{2n} = F_{n+1}^2 - F_{n-1}^2, \tag{1}$$

$$F_{2n+1} = F_n^2 + F_{n+1}^2. \tag{2}$$

We may use these to establish that

$$F_{3.2^{k-1}} \equiv 0 \left(\bmod \left(2^k\right)\right) \text{ and } F_{3.2^{k-1}-1} \equiv 1 \left(\bmod \left(2^k\right)\right)$$

as follows:

(a) $F_{3.2^{k-1}} \equiv 0 \left(\bmod \left(2^k\right)\right)$:

If $k = 2$, then $F_6 = 8 \equiv 0 \left(\bmod \left(2^2\right)\right)$. Assuming the result is true for $n = k$, we find, using (1):

$$F_{3.2^k} = F_{2.(3.2^{k-1})} = F_{3.2^{k-1}+1}^2 - F_{3.2^{k-1}-1}^2 = \left(F_{3.2^{k-1}} + F_{3.2^{k-1}-1}\right)^2 - F_{3.2^{k-1}-1}^2$$

$$= F_{3.2^{k-1}}^2 + 2F_{3.2^{k-1}}F_{3.2^{k-1}-1} \equiv 0\left(\bmod\left(2^{k+1}\right)\right).$$

The statement (a) is therefore true by induction

(b) $F_{3.2^{k-1}} \equiv 1 \left(\bmod \left(2^k\right)\right)$:

If $k = 2$, then $F_5 = 5 \equiv 1 \left(\bmod \left(2^2\right)\right)$. Assuming the result is true for $n = k$, we find, using (2), and the result from part (a):

$$F_{3.2^k-1} = F_{2.(3.2^{k-1}-1)+1} = F_{3.2^{k-1}-1}^2 + F_{3.2^{k-1}}^2 \equiv 1 \left(\bmod \left(2^{k+1}\right)\right).$$

If we now let $m = 3.2^{k-1}$ in the addition formula and apply both results, we have:

$$F_{n+3.2^{k-1}} = F_{3.2^{k-1}-1}F_n + F_{3.2^{k-1}}F_{n+1} \equiv 1.F_n + 0.F_{n+1} \equiv F_n\left(\bmod\left(2^k\right)\right).$$

Therefore, the sequence modulo 2^k is cyclic with period 3.2^{k-1}.

G. Howlett investigated further and came up with the conjecture that, for every $m > 1$, there exists a $k_{\min} > 0$ and an integer c such that the period of the Fibonacci sequence modulo m^k is given by $c.m^{k-1}$ for all $k > k_{\min}$.

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Moreover, when m is odd or a power of 2, then $k_{\min} = 1$. As an example, if we write the Fibonacci sequence modulo 3, the terms are 1, 1, 2, 0, 2, 2, 1, 0, 1, 1, 2, 0, 2, 2, 1, 0, … which has a period of 8, so we have $k_{\min} = 1$, $m = 3^1$, period = $8 \times 3^0 = 8$, so $c = 8$. Other examples of values of $(m, k_{\min}, c)$ are (5, 1, 20), (7, 1 16), (9, 1 24), (10, 3, 15). It would be interesting to see if anyone can develop this further.

Correct solutions were received from: D. Buckland, M. G. Elliott. M. Golushka, M. Hennings, G. Howlett, A. Jha, J. A. Mundie, Z. Retkes and the proposer, S. Dolan.

109.B (Seán Stewart)

Prove that

$$\sum_{n=1}^{\infty} \frac{1}{2^{2n}} \operatorname{sech}^2\left(\frac{\pi}{2^{n+1}}\right) = \frac{4}{\pi^2} - \operatorname{cosech}^2\left(\frac{\pi}{2}\right).$$

Solution:

The following solution, based on that by a 6th former D. Buckland, caught my eye:

It is known that $\dfrac{\sin x}{x} = \displaystyle\prod_{n=1}^{\infty} \cos \frac{x}{2^n}$. If we take logs of each side we get:

$$\ln(\sin x) - \ln x = \sum_{n=1}^{\infty} \ln\left(\cos \frac{x}{2^n}\right).$$

If we then differentiate both sides twice we obtain:

$$-\operatorname{cosec}^2 x + \frac{1}{x^2} = -\sum_{n=1}^{\infty} \frac{1}{2^n} \sec^2\left(\frac{x}{2^n}\right).$$

Of course, care must be taken to justify each of these steps, but J. Santmeyer found this formula in [1].

Replacing x with ix gives

$$\sum_{n=1}^{\infty} \frac{1}{2^{2n}} \operatorname{sech}^2\left(\frac{x}{2^n}\right) = \frac{1}{x^2} - \operatorname{cosech}^2 x.$$

Finally, replacing x with $\frac{1}{2}\pi$ achieves the desired result.

Reference

1. M. R. Spiegel, *Mathematical Handbook* p. 582, Schaum's Outline Series, McGraw-Hill Book Company (1968).

Correct solutions were received from D. Buckland, N. Curwen, M. G. Elliott. M. Hennings, G. Howlett, R. Mortini and R. Rupp, J. A. Mundie, Z. Retkes, J. Santmyer and the proposer, S. Stewart.

109.C (Narendra Bhandari)

Prove:

$$\int_0^1 \int_0^1 \frac{(1 + xy) \log x (\log y)^2}{(2x - 2)(1 - y)(1 - xy)^2}\, dx\, dy = \zeta(2) + \zeta(3).$$

Solution:

The proposer, N. Bhandari established this result by using double integration, but the other solvers used infinite series approaches, such as this neat solution offered by M. Hennings.

Note that
$$\frac{1}{(1 - xy)^2} = \sum_{n=0}^{\infty} (n + 1)(xy)^n$$
$$\frac{1 + xy}{(1 - xy)^2} = \sum_{n=0}^{\infty} (2n + 1)(xy)^n$$
$$\frac{1 + xy}{(1 - x)(1 - y)(1 - xy)^2} = \sum_{r,s,n=0}^{\infty} (2n + 1) x^{r+n} y^{s+n}$$

with all these series being valid for $0 < x, y < 1$. Since

$$\int_0^1 x^k \ln x\, dx = -\frac{1}{(k + 1)^2} \qquad \text{and} \qquad \int_0^1 x^k (\ln x)^2\, dx = \frac{2}{(k + 1)^3}$$

for any $k > 0$, we deduce that

$$\int_0^1 \int_0^1 \frac{(1 + xy) \ln x (\ln y)^2}{2(x - 1)(1 - y)(1 - xy)^2} dx\, dy$$
$$= -\frac{1}{2} \int_0^1 \int_0^1 \sum_{r,s,n=0}^{\infty} (2n + 1) x^{r+n} y^{s+n} \ln x (\ln y)^2 dx\, dy$$
$$= \sum_{r,s,n=0}^{\infty} \frac{(2n + 1)}{(r + n + 1)^2 (s + n + 1)^3}.$$

Since $x^{r+n} y^{s+n} \ln x (\ln y)^2 \leqslant 0$ for all $0 < x, y < 1$ and all $r, s, n \geqslant 0$, the Monotone Convergence Theorem justifies this identity, provided that the infinitesum converges. Thus

$$\int_0^1 \int_0^1 \frac{(1 + xy) \ln x (\ln y)^2}{2(x - 1)(1 - y)(1 - xy)^2} dx\, dy$$
$$= \sum_{R,S=1}^{\infty} \sum_{n=0}^{\min(R,S) - 1} \frac{2n + 1}{R^2 S^3} = \sum_{R,S=1}^{\infty} \frac{(\min(R, S))^2}{R^2 S^3}$$
$$= \sum_{S=1}^{\infty} \left(\sum_{R=1}^{S} \frac{R^2}{R^2 S^3} + \sum_{R=S+1}^{\infty} \frac{S^2}{R^2 S^3} \right)$$

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$$= \sum_{S=1}^{\infty} \frac{1}{S^2} + \sum_{S=1}^{\infty} \sum_{R=S+1}^{\infty} \frac{1}{R^2 S} = \zeta(2) + \sum_{R=2}^{\infty} \sum_{S=1}^{R-1} \frac{1}{R^2 S}$$

$$= \zeta(2) + \sum_{R=2}^{\infty} \frac{H_{R-1}}{R^2} = \zeta(2) + \sum_{R=1}^{\infty} \frac{H_R}{(R+1)^2}. \qquad (1)$$

Now

$$\sum_{n=1}^{\infty} H_n t^n = -\frac{\ln(1-t)}{1-t} \qquad |t| < 1$$

and hence

$$\sum_{n=1}^{\infty} \frac{H_n}{(n+1)^2} = \int_0^1 \frac{\ln t \ln(1-t)}{1-t} dt = \int_0^1 \frac{\ln t \ln(1-t)}{t} dt = \frac{1}{2} \int_0^1 \frac{(\ln t)^2}{1-t} dt$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} \int_0^1 t^n (\ln t)^2 dt = \sum_{n=0}^{\infty} \frac{1}{(n+1)^3} = \zeta(3)$$

and upon substituting into (1) we achieve the desired result.

J. Santmeyer went further to establish the following:

$$I_a = \int_{y=0}^{1} \int_{x=0}^{1} \frac{(1+axy)\ln(x)\ln^2(y)}{(2x-2)(1-y)(1-xy)^2} = \frac{1+a}{2}\left[\zeta(2) + \zeta(3)\right] + (1-a)\zeta(4).$$

I leave the details for the interested reader.

Correct solutions were received from: D, Buckland, M. G. Elliott, M. Hennings, J. A. Mundie, J. Santmeyer, S. Stewart and the proposer, N. Bhandari.

109.D (Dorin Marghidanu)

Prove that if $x_1, x_2, x_3, \dots, x_n > 0$, $p_1, p_2, p_3, \dots, p_n > 0$, with $p_1 + p_2 + p_3 + \dots + p_n = 1$, and $n, r \in \mathbb{N}$, then

$$\frac{x_1 + x_2 + \dots + x_n}{n}$$

$$\leqslant \sqrt[r]{\frac{(p_1x_1 + p_2x_2 + \dots + p_nx_n)^r + (p_2x_1 + p_3x_2 + \dots + p_1x_n)^r + \dots + (p_nx_1 + p_1x_2 + \dots + p_{n-1}x_n)^r}{n}}$$

$$\leqslant \sqrt[r]{\frac{x_1^r + x_2^r + \dots + x_n^r}{n}}.$$

Solution:

This problem was dealt with very neatly by A. Plaza as follows:

Since $p_1 + p_2 + p_3 + \dots + p_n = 1$, the arithmetic mean of $p_1x_1 + p_2x_2 + \dots + p_nx_n$, $p_2x_1 + p_3x_2 + \dots p_1x_n$, $\dots$, $p_nx_1 + p_1x_2 + \dots + p_{n-1}x_n$ is $\frac{1}{n}(x_1 + x_2 + \dots + x_n)$, so the first inequality is established by the power mean inequality.

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For the second inequality, note that the function $f : x \to x^r$, $r \in \mathbb{N}$ is convex, so using Jensen's inequality:

$$(p_1x_1 + p_2x_2 + \dots + p_nx_n)^r \leqslant p_1x_1^r + p_2x_2^r + \dots + p_nx_n^r$$

$$(p_2x_1 + p_3x_2 + \dots + p_1x_n)^r \leqslant p_2x_1^r + p_3x_2^r + \dots + p_1x_n^r$$

$$\vdots$$

$$\vdots$$

$$(p_nx_1 + p_1x_2 + \dots + p_{n-1}x_n)^r \leqslant p_nx_1^r + p_1x_2^r + \dots + p_{n-1}x_n^r.$$

Summing these inequalities and using $p_1 + p_2 + p_3 + \dots + p_n = 1$, the second inequality is established.

Correct solutions were received from: P. De, N. Curwen, S. Dolan, M. G. Elliott, M. Hennings, G. Howlett, A. Plaza, S. Riccarelli, and the proposer, D. Marghidanu.

The following solutions were received after the publication date: 108.I, 108.J, 108.K, 108.L (M. G. Elliott), 108.I (C. Jones), 108.K, 108.L (S. Riccarelli).

On a final note, I would like to thank Gerry Leversha for giving me the opportunity to be the Editor of Problem Corner and also for his support in getting me started in the role. I would also thank Bill Richardson for his advice and expert typesetting. I wish them all the best in their retirement.

10.1017/mag.2025.10153 C. STARR
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A Correction

The Author of Note 109.14 'Extensions of a Geometric Inequality', which appeared in the March 2025 *Gazette*, has pointed out two errors which were overlooked.

The corrected versions are as follows:

- On page 157, the final equation should be

$$IA_1 \times IA_2 \times IA_3 \times IA_4 = 2r^3\left(\sqrt{4R^2 + r^2} - r\right)$$

- The 11th line on page 158, should be

$$\frac{4r^4}{\sin A_1 \sin A_2} = 4r^4 \times \frac{2R^2}{r^2 + r\sqrt{4R^2 + r^2}} = 2r^3\sqrt{4R^2 + r^2}$$

We apologise to Dr Yun for these errors.

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