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TERM BY TERM DYADIC DIFFERENTIATION

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1. Introduction. Let $\psi_0, \psi_1, \ldots$ denote the Walsh-Paley functions and let $\dot{+}$ denote the group operation which Fine [**5**] defined on the interval $[0, 1)$. Thus, if $k \geqq 0$ is an integer and if u, t are points in the interval $[0, 1)$ then

$$\psi_k(u \dot{+} t) = \psi_k(u)\psi_k(t), \quad t \dot{+} 2^{-k} = t + (-1)^{\alpha_k} 2^{-k}$$

(where $\alpha_k = 0$ or 1 represents the kth coefficient of the binary expansion of t), and

$$\psi_{k \cdot 2^n}(t)\psi_j(t) = \psi_{k \cdot 2^n + j}(t) \quad \text{for } n = 1, 2, \ldots \text{ and } 0 \leqq j < 2^n.$$

A real-valued function f, is said to be *dyadically differentiable* at a point $x \in [0, 1)$ if f is defined at x and at $x \dot{+} 2^{-n-1}$, $n = 0, 1, \ldots$, and if the sequence

$$(1) \qquad d_N(f, x) = \sum_{n=0}^{N-1} 2^{n-1}(f(x) - f(x \dot{+} 2^{-n-1}))$$

converges as $N \to \infty$. In this case, we shall denote the limit of (1) by $\dot{d}f(x)$ and call it the dyadic derivative of f at x. This definition was introduced by Butzer and Wagner [**1**], who proved that every Walsh function is dyadically differentiable on $[0, 1)$ with $\dot{d}\psi_k = k\psi_k$, $k = 0, 1, \ldots$, and that if N and k are any non-negative integers and if k_0 satisfies $0 \leqq k_0 < 2^N$ and $k \equiv k_0 (\text{mod } 2^N)$ then

$$(2) \qquad \sum_{n=0}^{N-1} 2^{n-1}[1 - \psi_k(2^{-n-1})] = k_0.$$

In a later paper, Butzer and Wagner [**2**] began to study the problem of determining which Walsh series were term by term dyadically differentiable, that is to say, under what conditions would a function

$$(3) \qquad f(x) \equiv \sum_{k=0}^{\infty} a_k \psi_k(x)$$

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have a dyadic derivative which satisfies

$$(4) \qquad \dot{d}f(x) = \sum_{k=1}^{\infty} k a_k \psi_k(x)$$

at a certain point x?

They proved that (4) holds a.e. if both $\{a_k\}$ and $\{ka_k\}$ are quasi-convex and $ka_k \to 0$ as $k \to \infty$, that (4) holds everywhere if $\sum_{k=1}^{\infty} k|a_k| < \infty$, and they conjectured that (4) would hold a.e. if $ka_k \downarrow 0$ as $k \to \infty$. This conjecture was verified by Schipp [7], who showed that (4) holds, in this case, for all but countably many $x \in [0, 1)$.

In Section 2 we shall derive a condition sufficient to conclude that (4) holds at a particular point x. In Section 3 we shall use this condition to study dyadic derivatives, growth of Walsh-Fourier coefficients, and conditions sufficient to conclude that a continuous function is constant. In the process, we shall show that if $k^\alpha a_k \downarrow 0$, as $k \to \infty$, for some $\alpha > 1$ then (4) holds everywhere in $(0, 1)$. Hence, a tightening of the hypothesis in Schipp's theorem leads to a stronger conclusion.

2. The main theorem. In this section we shall outline a proof of the following result.

THEOREM. *Let x be a point in the interval $[0, 1)$, let $a_0, a_1, \ldots$ be a sequence of real numbers and suppose that $\alpha > 1$. If the series*

$$(5) \qquad \sum_{k=0}^{\infty} k^\alpha a_k \psi_k(x)$$

converges then the function

$$f(t) \equiv \sum_{k=0}^{\infty} a_k \psi_k(t)$$

is dyadically differentiable at x and (4) is satisfied.

We begin by observing that convergence of (5) implies that $k^\alpha a_k \to 0$ as $k \to \infty$. It follows that

$$\sum_{k=0}^{\infty} |a_k| < \infty.$$

Hence, $f(t)$ is absolutely convergent for all $t \in [0, 1]$.

Let $N \geqq 1$ be an integer, and observe that

$$d_N(f, x) = \sum_{n=0}^{N-1} 2^{n-1} \sum_{k=1}^{\infty} a_k[\psi_k(x) - \psi_k(x \dotplus 2^{-n-1})].$$

If we apply the identity $\psi_k(x \dotplus 2^{-n-1}) = \psi_k(x)\psi_k(2^{-n-1})$, we can rewrite the expression displayed above in the following form:

$$(6) \qquad d_N(f, x) = \sum_{k=1}^{\infty} \left(\sum_{n=0}^{N-1} 2^{n-1}[1 - \psi_k(2^{-n-1})] \right) a_k \psi_k(x).$$

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Multiplying the k^{th} term of (6) by $1 = k^{\alpha} \cdot k^{-\alpha}$, we have that

$$(7) \qquad d_N(f, x) = \sum_{k=1}^{\infty} k^{-\alpha} \left(\sum_{n=0}^{N-1} 2^{n-1} [1 - \psi_k(2^{-n-1})] \right) k^{\alpha} a_k \psi_k(x).$$

In § 4 we shall verify that if (5) converges then the sequence (7) has a limit, as $N \to \infty$, and that this limit can be obtained by replacing N by ∞ on the right-hand-side of (7) (see Lemma 5). In view of (2), this means that

$$\lim_{N \to \infty} d_N(f, x) = \sum_{k=1}^{\infty} k a_k \psi_k(x),$$

which completes the proof of our theorem.

3. Applications. Throughout this section, let $f(t)$ represent the Walsh series $\sum_{k=0}^{\infty} a_k \psi_k(t)$. Since any Walsh series whose coefficients are bounded variation converges everywhere on the interval (0, 1), our main theorem contains the following result.

COROLLARY 1. *If the sequence* $\{k^{\alpha} a_k\}$ *is of bounded variation for some* $\alpha > 1$, *then* $f(x)$ *has a dyadic derivative which satisfies* (4) *everywhere on* (0, 1).

The hypothesis of Corollary 2 is surely satisfied if $k^{\alpha} a_k \downarrow 0$ as $k \to \infty$. Thus Coury's example [**3**] $g(x) = \sum_{k=0}^{\infty} 2^{-k} \psi_k(x)$ is both classically differentiable a.e. and dyadically differentiable everywhere on (0, 1), with

$$\dot{d}g(x) = \sum_{k=1}^{\infty} k 2^{-k} \psi_k(x).$$

Our main theorem, together with the convergence theorems of [**6**], [**9**], and [**8**], yield the following sufficient conditions for global dyadic differentiability.

COROLLARY 2. *Suppose that* $\{n_j\}$ *is a lacunary sequence of integers and that* $\{a_k\}$ *is a sequence of real numbers satisfying* $a_k = 0$ *unless* $k = n_j$ *for some* j. *If for every point* x *in some non-degenerate interval* I *there exists an* $\alpha > 1$ *such that*

$$\limsup_{n \to \infty} \left| \sum_{k=1}^{n} k^{\alpha} a_k \psi_k(x) \right| < \infty,$$

then $f(x)$ *has a dyadic derivative which satisfies* (4) *everywhere on* [0, 1).

COROLLARY 3. *If for every point* x *in some set* E *of positive measure there*

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exists an $\alpha > 1$ *such that*

$$\sum_{k=1}^{\infty} 2^{\alpha k} a_k \psi_{2^k}(x)$$

converges, then $g(x) \equiv \sum_{k=0}^{\infty} a_k \psi_{2^k}(x)$ *is dyadically differentiable a.e. on* $[0, 1)$ *and*

$$\dot{d}g(x) = \sum_{k=0}^{\infty} a_k 2^k \psi_{2^k}(x) \quad a.e.\ x \in [0, 1).$$

COROLLARY 4. *Suppose that* g *is a function which belongs to* $L \log^+ L \log^+ \log^+ L$ *and that* $a_k = k^{-\alpha}\hat{g}(k)$ *for some* $\alpha > 1$, *where* $\hat{g}(k)$ *represents the Walsh-Fourier coefficients of* g, $k = 0, 1, \ldots$. *Then* $f(x)$ *has a dyadic derivative which satisfies* (4) *a.e.*

In particular, if the function $\sum_{k=1}^{\infty} k^{\epsilon} a_k \psi_k(x)$, $\epsilon > 0$, has a strong L^p dyadic derivative for some $p > 1$ (see [**1**]), then $f(x)$ has a dyadic derivative which satisfies (4) a.e.

For any integer $N \geqq 1$ and any point $x \in [0, 1)$, consider the series

$$(8) \qquad R_N(x) \equiv \sum_{j=1}^{\infty} \sum_{k=j2^N}^{(j+1)2^N - 1} j 2^N a_k \psi_k(x).$$

There is a strong connection between the convergence of (8) and the formal dyadic derivative of $f(x)$.

PROPOSITION. *Suppose that* $f(t)$ *exists for* $t = x$ *and* $t = x \dotplus 2^{-n-1}$, $n = 0, 1, \ldots$, *and suppose that* N *is any positive integer. Then* $R_N(x)$ *exists and is finite if and only if*

$$g(x) \equiv \lim_{\tau \to \infty} \sum_{k=1}^{\tau 2^N - 1} k a_k \psi_k(x) \quad \textit{exists},$$

in which case,

$$(9) \qquad d_N(f, x) = g(x) - R_N(x).$$

In particular, if $\sum_{k=1}^{\infty} k a_k \psi_k(x)$ *converges then* $\dot{d}f(x)$ *exists if and only if*

$$R(x) = \lim_{N \to \infty} R_N(x) \ \textit{exists},$$

in which case,

$$(10) \qquad \dot{d}f(x) = \sum_{k=1}^{\infty} k a_k \psi_k(x) - R(x).$$

In other words, (4) holds if and only if $R(x) = 0$.

To establish this proposition, we need only verify (9). To accomplish this, return to (6) and apply (2) to obtain

$$d_N(f, x) = \sum_{k=0}^{2^N-1} ka_k\psi_k(x) + \sum_{j=1}^{\infty} \sum_{k=j2^N}^{(j+1)2^N-1} (k - j2^N)a_k\psi_k(x).$$

Replace "$\sum_{j=1}^{\infty}$" by "$\lim_{\tau\to\infty} \sum_{j=1}^{\tau-1}$" and conclude that

$$d_N(f, x) = \lim_{\tau\to\infty} \left\{ \sum_{k=0}^{\tau 2^N-1} ka_k\psi_k(x) + \sum_{j=1}^{\tau-1} \sum_{k=j2^N}^{(j+1)2^N-1} (-j2^N)a_k\psi_k(x) \right\}.$$

Since this limit exists, we have proved that $g(x)$ exists if and only if $R_N(x)$ exists. Taking the limit, as $\tau \to \infty$, we have also established (9).

COROLLARY 5. *If $a_k \downarrow 0$ as $k \to \infty$ and if $\sum_{k=1}^{\infty} |a_k| < \infty$, then f has a dyadic derivative which satisfies* (4) *everywhere on* (0, 1).

To establish this corollary, we begin by proving that the sequence $\{ka_k\}$ is of bounded variation. Indeed, since the sequence $\{a_k\}$ is monotone decreasing, it must be the case that

$$|ka_k - (k + 1)a_{k+1}| \leqq k(a_k - a_{k+1}) + a_{k+1},$$

for any integer $k \geqq 1$. Consequently,

$$\sum_{k=1}^{N} |ka_k - (k + 1)a_{k+1}| \leqq 2 \cdot \sum_{k=1}^{\infty} |a_k| + Na_N.$$

But absolute convergence of the series $\sum_{k=1}^{\infty} a_k$ implies that $Na_N \to 0$ as $N \to \infty$. Thus the sequence $\{ka_k\}$ is of bounded variation.

It follows, therefore, that the Walsh series $\sum_{k=0}^{\infty} ka_k\psi_k(x)$ converges for all $x \in (0, 1)$. A similar argument establishes the fact that $R_N(x) \to 0$ as $N \to \infty$ for all $x \in (0, 1)$. By the proposition above, then, $\check{d}f(x)$ exists for all $x \in (0, 1)$ and satisfies (4).

We conjecture that Corollary 5 holds if the condition "$a_k \downarrow 0$ as $k \to \infty$" is replaced by "$|a_k| \leqq b_k$ and $b_k \downarrow 0$ as $k \to \infty$".

When the proposition above is applied, in conjunction with Corollaries 1–4, we obtain some rather delicate growth conditions for certain types of Walsh series. For example,

COROLLARY 6. *If for every x (respectively, for a.e. x) in some interval I there exists an $\alpha > 1$ such that* (5) *converges then $R_N(x) \to 0$, as $N \to \infty$, for all x (respectively, for a.e. x) in I.*

Coury [**4**, Theorem 6] has shown that if f is continuous and if

$$(11) \quad \sum_{k=1}^{\infty} k|a_k| < \infty,$$

then f is constant. Our last corollary generalizes this result.

COROLLARY 7. *Suppose that f is continuous on* $(0, 1)$, *and that for each* $x \in (0, 1)$ *there exists an* $\alpha > 1$ *such that* (5) *converges, or that* $\sum_{k=1}^{\infty} ka_k\psi_k$ *converges on* $(0, 1)$ *and that* $\lim_{N\to\infty} R_N(t)$ *exists for each* $t \in (0, 1)$. *Then f is constant.*

To prove this result, apply Corollary 1 or the proposition above to conclude that $\dot{d}f(x)$ exists for each $x \in (0, 1)$. Thus the sequence (1) converges. It follows that the terms of the series on the right-hand-side of (1) must tend to zero:

(12) $2^{n-1}(f(x) - f(x \dot{+} 2^{-n-1})) \rightarrow 0$ as $n \rightarrow \infty$.

Recall that $f(x \dot{+} 2^{-n-1}) = f(x \pm 2^{-n-1})$. Thus (12) implies that the upper Dini derivate of f is non-negative and that the lower Dini derivate of f is non-positive. Since f is continuous, it now follows that f is constant.

It is clear that this technique can be combined with other corollaries above to obtain sufficient conditions that a continuous function f be constant on $(0, 1)$, or on some non-degenerate interval I.

4. Unavoidable technicalities. Throughout this section $b_{n,k}$, b_k, and x_k will denote real numbers for $k = 0, 1, \ldots, n = 0, 1, \ldots$, which satisfy

(13) $\sum_{k=0}^{\infty} |b_{n,k} - b_{n,k+1}| \leqq M \quad n = 0, 1, \ldots,$

(14) $\lim_{n\to\infty} b_{n,k} = b_k \quad k = 0, 1, \ldots$

and

(15) $\sum_{k=0}^{\infty} x_k$ converges.

We begin with two elementary observations. First, since

$$b_{n,j} = \sum_{k=0}^{j-1} (b_{n,k} - b_{n,k+1}) + b_{n,0},$$

conditions (13) and (14) prove that the sequences $\{b_{n,k}\}$ and $\{b_k\}$ are bounded:

LEMMA 1. *There exists an* $A < \infty$ *such that* $|b_{n,k}| \leqq A$ *and* $|b_k| \leqq A$ *for all integers* $n \geqq 0$ *and* $k \geqq 0$.

Secondly, since $b_k - b_{k+1}$ can be written in the form

$$(-b_{n,k} + b_k) - (-b_{n,k} + b_{n,k+1}) - (b_{k+1} - b_{n,k+1})$$

we can apply (13) and (14) to show that $\{b_k\}$ is also of bounded variation:

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LEMMA 2. *For each integer* $N > 0$,

$$\sum_{k=0}^{N} |b_k - b_{k+1}| \leqq M.$$

This leads to the following convergence result.

LEMMA 3. *Both* $\sum_{k=0}^{\infty} b_{n,k}\, x_k$ *and*

$$\sum_{k=0}^{\infty} b_k x_k \tag{16}$$

are convergent series.

Comparing Lemma 2 with (13), it suffices to show that (16) converges. By Abel's transformation, however,

$$\sum_{k=n}^{m} b_k x_k = b_n \cdot \sum_{j=n}^{m} x_j - b_m \cdot \sum_{j=n}^{m-1} x_j + \sum_{k=n}^{m-1} \sum_{j=n}^{k} x_j (b_k - b_{k+1}).$$

Hence by Lemmas 1 and 2, we conclude that

$$\left| \sum_{k=n}^{m} b_k x_k \right| \leqq (2A + M) \sup_{k \geqq n} \left| \sum_{j=n}^{k} x_j \right| .$$

This inequality, together with (15) establishes the convergence of (16).

LEMMA 4. *If conditions* (13), (14) *and* (15) *are satisfied, then*

$$\lim_{n \to \infty} \sum_{k=0}^{\infty} b_{n,k} x_k = \sum_{k=0}^{\infty} b_k x_k.$$

To prove this result let $\epsilon > 0$ and fix an integer $N \geqq 1$ so that

$$\left| \sum_{l=n}^{m} x_l \right| < \epsilon \quad \text{when } n, m > N. \tag{17}$$

Next, observe by Lemma 3 that

$$\sum_{k=0}^{\infty} b_{n,k} x_k - \sum_{k=0}^{\infty} b_k x_k = \sum_{k=0}^{N} (b_{n,k} - b_k) x_k + \sum_{k=N+1}^{\infty} (b_{n,k} - b_k) x_k.$$

Since N is fixed, (14) implies that the first term above is negligible as $n \to \infty$. It suffices, therefore, to show that

$$\Sigma_n \equiv \sum_{k=N+1}^{\infty} (b_{n,k} - b_k) x_k$$

converges to zero, as $n \to \infty$.

Toward this, apply Abel's transformation to Σ_n, obtaining the following identity:

$$\Sigma_n = \sum_{k=N+1}^{\infty} (b_{n,k} - b_{n,k+1} - b_k + b_{k+1}) \cdot \sum_{l=k}^{\infty} x_l + (b_{n,N} - b_N) \cdot \sum_{l=N+1}^{\infty} x_l.$$

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In particular, it follows from (15), Lemma 2, (17), and Lemma 1 that

$$|\Sigma_n| \leqq \epsilon(2M + 2A).$$

Since $\epsilon > 0$ was arbitrary, this inequality completes the proof of Lemma 4.

This lemma can be used to justify the interchange of limit and sum sign used in § 2. Indeed, if we let $k^\alpha a_k \psi_k(x)$ play the role of x_k, and observe by Lemma 3 that if (5) converges for $\alpha = \alpha_0$ then (5) converges for all $\alpha \leqq \alpha_0$, then we need verify only the following result.

LEMMA 5. *If for some* α, $1 < \alpha < 3/2$,

$$b_{N,k} = k^{-\alpha}\left(\sum_{n=0}^{N-1} 2^{n-1}[1 - \psi_k(2^{-n-1})]\right),$$

$N = 1, 2, \ldots$ *and* $k = 0, 1, \ldots$ *and* $b_{0,k} \equiv 1$ *for* $k = 0, 1, \ldots,$ *then* (15) *is satisfied.*

In order to prove this lemma, we begin by establishing that if $j > 0$ and $0 \leqq k < 2^{N+1} - 1$, then

$$\text{(18)} \quad \frac{k}{(j2^{N+1} + k)^\alpha} \leqq \frac{k+1}{(j2^{N+1} + k + 1)^\alpha}.$$

Indeed, for any positive real number B, it is the case that

$$\left(1 + \frac{1}{B}\right)^\alpha \leqq \left(1 + \frac{1}{B}\right)^{3/2}.$$

Furthermore, if $B \geqq 1$ then $1/B^3 \leqq 1/B^2$, so the following inequality holds:

$$\left(1 + \frac{1}{B}\right)^3 \leqq \left(1 + \frac{2}{B}\right)^2.$$

It follows, therefore, that if $B \geqq 1$ and $C \geqq 0$ satisfies $2C \leqq B$, then

$$\left(1 + \frac{1}{B}\right)^\alpha \leqq 1 + \frac{1}{C}.$$

Hence (18) is obtained by setting $B = j2^{N+1} + k$ and $C = k$.

We shall establish Lemma 5 by showing that the sequence

$$S_N \equiv \sum_{j=0}^{\infty} \sum_{k=0}^{2^{N+1}-1} \left| (j2^{N+1} + k)^{-\alpha} \cdot \sum_{n-0}^{N-1} 2^{n-1}[1 - \psi_{j2^{N+1}+k}(2^{-n-1})] - (j2^{N+1} + k + 1)^{-\alpha} \cdot \sum_{n=0}^{N-1} 2^{n-1}[1 - \psi_{j2^{N+1}+k+1}(2^{-n-1})]\right|$$

is uniformly bounded in N. To evaluate the terms inside the square

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brackets, recall that for such j, k, and n,

$$\psi_{j2^{N+1}+k}(2^{-n-1}) = \psi_{j2^{N+1}}(2^{-n-1})\cdot\psi_k(2^{-n-1}) \equiv 1\cdot\psi_k(2^{-n-1}).$$

Hence, by equation (2),

$$\sum_{n=0}^{N-1} 2^{-n-1}[1 - \psi_{j2^{N+1}+k}(2^{-n-1})] = k$$

for $0 \leqq k < 2^{N+1}$ and $j = 0, 1, \ldots$. If $k = 2^{N+1}$, then

$$\psi_{j2^{N+1}+k}(2^{-n-1}) \equiv 1$$

so in this case the term inside the square brackets is exactly zero. It follows that S_N is dominated by T_N, where

$$T_N \equiv \sum_{j=0}^{\infty}\left\{\sum_{k=0}^{2^{N+1}-2} |(j2^{N+1}+k)^{-\alpha}k - (j2^{N+1}+k+1)^{-\alpha}(k+1)| - |j2^{N+1}+2^{N+1}-1|^{-\alpha}(2^{N+1}-1)\right\}.$$

The $j = 0$ term of T_N has the form

$$(2^{N+1}-1)^{1-\alpha} + \sum_{k=0}^{2^{N+1}-2} |k^{1-\alpha} - (k+1)^{1-\alpha}|$$

and thus telescopes to a sequence dominated by 3. According to inequality (18), each term of T_N corresponding to $j > 0$ also telescopes, leaving us with

$$\{2(j2^{N+1}+2^{N+1}-1)^{-\alpha}(2^{N+1}-1)\}.$$

In particular,

$$S_N \leqq 3 + 2\sum_{j=1}^{\infty}(j2^{N+1}+2^{N+1}-1)^{-\alpha}2^{N+1} \leqq 3 + 2\cdot\sum_{j=1}^{\infty} j^{-\alpha}.$$

Since $\alpha > 1$, we have obtained the uniform bound for S_N and have completed the proof of Lemma 5.

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