*Math. Proc. Camb. Phil. Soc.* (2025), **178**, 321–330
doi:10.1017/S0305004125000155
First published online 28 April 2025


# Hypergraph removal with polynomial bounds

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(*Received* 17 *February* 2022*; revised* 16 *October* 2024*; accepted* 17 *December* 2024)

*Abstract*

Given a fixed $k$-uniform hypergraph $F$, the $F$-removal lemma states that every hypergraph with few copies of $F$ can be made $F$-free by the removal of few edges. Unfortunately, for general $F$, the constants involved are given by incredibly fast-growing Ackermann-type functions. It is thus natural to ask for which $F$ one can prove removal lemmas with polynomial bounds. One trivial case where such bounds can be obtained is when $F$ is $k$-partite. Alon proved that when $k = 2$ (i.e. when dealing with graphs), only bipartite graphs have a polynomial removal lemma. Kohayakawa, Nagle and Rödl conjectured in 2002 that Alon's result can be extended to all $k > 2$, namely, that the only $k$-graphs $F$ for which the hypergraph removal lemma has polynomial bounds are the trivial cases when $F$ is $k$-partite. In this paper we prove this conjecture.

2020 Mathematics Subject Classification: 05C35 (Primary)

## 1. *Introduction*

The hypergraph removal lemma is one of the most important results of extremal combinatorics. It states that for every fixed integer $k$, $k$-uniform hypergraph ($k$-graph for short) $F$ and positive $\varepsilon$, there is $\delta = \delta(F, \varepsilon) > 0$ so that if $G$ is an $n$-vertex $k$-graph with at least $\varepsilon n^k$ edge-disjoint[1] copies of $F$, then $G$ contains $\delta n^{v(F)}$ copies of $F$. This lemma was first conjectured by Erdős, Frankl and Rödl [**5**] as an alternative approach for proving Szemerédi's theorem [**15**]. The quest to proving this lemma, which involved the development of the hypergraph extension of Szemerédi's regularity lemma [**16**], took more than two decades, culminating in several proofs, first by Gowers [**8**] and Rödl–Skokan–Nagle–Schacht [**11, 13**] and later

† During this work, LG was supported by SNSF grant 200021_196965.

‡ Supported in part by ISF Grant 1028/16, ERC Consolidator Grant 863438 and NSF-BSF Grant 20196.

1 The lemma's assumption is sometimes stated as $G$ being $\varepsilon$-far from $F$-freeness, meaning that one should remove at least $\varepsilon n^k$ edges to turn $G$ into an $F$-free hypergraph. It is easy to see that up to constant factors, this notion is equivalent to having $\varepsilon n^k$ edge-disjoint copies of $F$.

by Tao [17]. For the sake of brevity, we refer the reader to [12] for more background and references on the subject.

While the hypergraph removal lemma has far-reaching qualitative applications, its main drawback is that it supplies very weak quantitative bounds. Specifically, for a general $k$-graph $F$, the function $1/\delta(F,\varepsilon)$ grows like the $k$th Ackermann function. It is thus natural to ask for which $k$-graphs $F$ one can obtain more sensible bounds. Further motivation for studying such questions comes from the area of graph property testing [7], where graph and hypergraph removal lemmas are used to design fast randomised algorithms.

Suppose first that $k=2$. In this case it is easy to see that if $F$ is bipartite then $\delta(F,\varepsilon)$ grows polynomially with $\varepsilon$. Indeed, if $G$ has $\varepsilon n^2$ edge-disjoint copies of $F$ then it must have at least $\varepsilon n^2$ edges, which implies by the well-known Kővári–Sós–Turán theorem [10], that $G$ has at least $\text{poly}(\varepsilon)n^{v(F)}$ copies of $F$. In the seminal paper of Ruzsa and Szemerédi [14] in which they proved the first version of the graph removal lemma, they also proved that when $F$ is the triangle $K_3$, the removal lemma has a super-polynomial dependence on $\varepsilon$. A highly influential result of Alon [1] completed the picture by extending the result of [14] to all non-bipartite graphs F.

Moving now to general $k>2$, it is natural to ask for which $k$-graphs the function $\delta(F,\varepsilon)$ depends polynomially on $\varepsilon$. Let us say that in this case the *$F$-removal lemma is polynomial*. It is easy to see that like in the case of graphs, the $F$-removal lemma is polynomial whenever $F$ is $k$-partite. This follows from Erdős's [4] well-known hypergraph extension of the Kővári–Sós–Turán theorem. Motivated by Alon's result [1] mentioned above, Kohayakawa, Nagle and Rödl [9] conjectured in 2002 that the $F$-removal lemma is polynomial if and only if $F$ is $k$-partite. They further proved that the $F$-removal lemma is not polynomial when $F$ is the complete $k$-graph on $k+1$ vertices. Alon and the second author [2] proved that a more general condition guarantees that the $F$-removal lemma is not polynomial, but fell short of covering all non-$k$-partite $k$-graphs. In this paper we complete the picture, by fully resolving the problem of Kohayakawa, Nagle and Rödl [9].

THEOREM 1. *For every $k$-graph $F$, the $F$-removal lemma is polynomial if and only if $F$ is $k$-partite.*

As a related remark, we note that for $k\geq 3$, the analogous problem for the *induced* $F$-removal lemma (that is, a characterisation of $k$-graphs for which the induced $F$-removal lemma has polynomial bounds) was recently settled in [6], following a nearly-complete characterisation given in [2].

Before proceeding, let us recall the notion of a *core*, which plays an important role in the proof of Theorem 1. Recall that for a pair of $k$-graphs $F_1, F_2$, a homomorphism from $F_1$ to $F_2$ is a map $\varphi : V(F_1)\rightarrow V(F_2)$ such that for every $e\in E(F_1)$ it holds that $\{\varphi(x) : x\in e\}\in E(F_2)$. The *core* of a $k$-graph $F$ is the smallest (with respect to the number of vertices) subgraph of $F$ to which there is a homomorphism from $F$. It is not hard to show that the core of $F$ is unique up to isomorphism[2]. Also, note that the core of a $k$-graph $F$ is a single edge if and

[2] Indeed, suppose that $F_1, F_2$ are both cores of $F$. Then $F_1$ is homomorphic to $F_2$ (by taking a homomorphism from $F$ to $F_2$ and restrincting it to $V(F_1)$) and similarly $F_2$ is homomorphic to $F_1$. Also, by the minimality of a core, both homomorphisms $\varphi : F_1\rightarrow F_2$ and $\psi : F_2\rightarrow F_1$ must be surjective. Indeed, if e.g. $\varphi$ is not surjective, then by composing $\varphi$ with a homomorphism from $F$ to $F_1$, we get a homomorphism from $F$ to a proper subgraph of $F_2$, a contradiction. So $|V(F_1)|=|V(F_2)|$ and $\varphi,\psi$ are in fact bijections. It follows that $F_1, F_2$ are isomorphic.

only if $F$ is $k$-partite. In particular, if a $k$-graph is not $k$-partite, then neither is its core. We say that $F$ is a core if it is the core of itself.

Alon's [1] approach relies on the fact that the core of every non-bipartite graph has a cycle. It is then natural to try and prove Theorem 1 by finding analogous sub-structures in the core of every non-$k$-partite $k$-graphs. Indeed, this was the approach taken in [2, 9]. The main novelty in this paper, and what allows us to handle all cases of Theorem 1, is that instead of directly inspecting the $k$-graph $F$, we study the properties of a certain graph associated with $F$. More precisely, given a $k$-graph $F=(V,E)$, we consider its *2-shadow*, which is the graph on the same vertex set $V$ in which $\{u,v\}$ is an edge if and only if $u$,$v$ belong to some $e\in E$. The proof of Theorem 1 relies on the two lemmas described below.

LEMMA 1·1. *Suppose a $k$-graph $F$ is a core and its 2-shadow contains an induced cycle of length at least* 4. *Then the $F$-removal lemma is not polynomial.*[3]

Note that this is a generalisation of Alon's result mentioned above since the 2-shadow of every non-bipartite graph $F$ (which is of course $F$ itself in this case) must contain a cycle. Our second lemma is the following.

LEMMA 1·2. *Suppose a $k$-graph $F$ is a core and its 2-shadow contains a clique of size* $k+1$. *Then the $F$-removal lemma is not polynomial.*

Note that this is a generalisation of the result of Kohayakawa, Nagle and Rödl [9] mentioned above since the 2-shadow of the complete $k$-graph on $k+1$ vertices is a clique of size $k+1$.

The proofs of Lemmas 1·1 and 1·2 appear in Section 2, but let us first see why they together allow us to handle all non-$k$-partite $k$-graphs, thus proving Theorem 1.

*Proof of Theorem* 1. The "if" part was discussed above. As for the "only if" part, suppose $F$ is a $k$-graph which is not $k$-partite and assume first that $F$ is a core. Let $G$ denote the 2-shadow of $F$. If $G$ contains an induced cycle of length at least 4, then the result follows from Lemma 1·1. Suppose then that $G$ contains no such cycle, implying that $G$ is chordal. Since $F$ is not $k$-partite, $G$ is not $k$-colourable. Since $G$ is assumed to be chordal, and chordal graphs are well-known to be perfect, this means that $G$ has a clique of size $k+1$. Hence, the result follows from Lemma 1·2.

To prove the result when $F$ is not necessarily a core, one just needs to observe that if $F'$ is the core of $F$, then (i) as noted earlier, $F'$ is not $k$-partite, and (ii) since the $F'$ removal lemma is not polynomial (by the previous paragraph), then neither is the $F$-removal lemma (see Claim 2·1 for the short proof of this fact).

## 2. *Proofs of Lemmas 1.1 and 1.2*

We start by introducing some recurring notions. Recall that the *b-blowup* of a $k$-graph $H=(V,E)$ is the $k$-graph obtained by replacing every vertex $v\in V$ with a $b$-tuple of vertices $S_v$, and then replacing every edge $e=\{v_1,\ldots,v_k\}\in E$ with all possible $b^k$ edges $S_{v_1}\times S_{v_2}\times\cdots\times S_{v_k}$. Note that if $H'$ is the $b$-blowup of $H$, then the map sending $S_v$ to $v$

[3] The proof of this lemma also works if the 2-shadow of $F$ contains a triangle $x$, $y$, $z$ and $|e\cap\{x,y,z\}|\leq 2$ for every $e\in E(F)$, but we will not require this; in fact, this case follows from Lemma 2·9.

is a homomorphism from $H'$ to $H$. We will frequently refer to this as the *natural* homomorphism from $H'$ to $H$. We say that a $k$-graph $H$ is *homomorphic* to a $k$-graph $F$ if there is a homomorphism from $H$ to $F$. We first prove the following assertion, which was used in the proof of Theorem 1.

CLAIM 2·1. *Let $F$ be a $k$-graph and let $C$ be a subgraph of $F$ so that $F$ is homomorphic to $C$. Then, if the $C$-removal lemma is not polynomial, then neither is the $F$-removal lemma.*

*Proof.* Since the $C$-removal lemma is not polynomial, there is a function $\delta : (0, 1) \to (0, 1)$ such that $1/\delta(\varepsilon)$ grows faster than any polynomial in $1/\varepsilon$, and such that for every $\varepsilon > 0$ and large enough $n$ there is an $n$-vertex $k$-graph $H_1$ which contains a collection $\mathcal{C}$ of $\varepsilon n^k$ edge-disjoint copies of $C$ but only $\delta n^{v(C)}$ copies of $C$ altogether. Let $H$ be the $v(F)$-blowup of $H_1$. Note that the $v(F)$-blowup of $C$ contains a copy of $F$. Also, copies of $F$ corresponding to different copies of $C$ from $\mathcal{C}$ are edge-disjoint. Hence, $H$ has a collection of $\varepsilon n^k = \varepsilon(v(H)/v(F))^k = \Omega(\varepsilon \cdot v(H)^k) = \varepsilon' v(H)^k$ edge-disjoint copies of $F$, for a suitable $\varepsilon' = \Omega(\varepsilon)$. Let us bound the total number of copies of $F$ in $H$. Since $C$ is a subgraph of $F$, each copy of $F$ must contain a copy of $C$. Let $\varphi : V(H) \to V(H_1)$ be the natural homomorphism from $H$ to $H_1$ (as defined above). For each copy $C'$ of $C$ in $H$, consider the subgraph $\varphi(C')$ of $H_1$. The number of copies $C'$ of $C$ with $v(\varphi(C')) < v(C)$ is at most $v(F)^{v(C)} \cdot O(n^{v(C)-1}) \leq \delta n^{v(C)}$, provided that $n$ is large enough. The number of copies $C'$ of $C$ with $\varphi(C') \cong C$ is at most $v(F)^{v(C)} \cdot \delta n^{v(C)} = O(\delta n^{v(C)})$, because $H_1$ contains at most $\delta n^{v(C)}$ copies of $C$. So in total, $H$ contains at most $O(\delta n^{v(C)})$ copies of $C$. This means that $H$ contains at most $O(\delta n^{v(C)}) \cdot v(H)^{v(F)-v(C)} = O(\delta \cdot v(H)^{v(F)}) = \delta' v(H)^{v(F)}$ copies of $F$, for a suitable $\delta' = O(\delta)$. Note that $1/\delta'$ is super-polynomial in $1/\varepsilon'$. This shows that the $F$-removal lemma is not polynomial.

Since the core of $F$ satisfies the properties of $C$ in the above claim, it indeed establishes the assertion which we used when proving Theorem 1, namely that it suffices to prove the theorem when $F$ is a core.

It thus remains to prove Lemmas 1·1 and 1·2. We begin preparing these proofs with some auxiliary lemmas. The following is a key property of cores that we will use in this section.

CLAIM 2·2. *Let $F$ be a core $k$-graph, let $H$ be a $k$-graph, and let $\varphi : H \to F$ be a homomorphism. Then for every copy $F'$ of $F$ in $H$, the map $\varphi_{|V(F')}$ is an isomorphism.*

*Proof.* We first observe that every homomorphism from a core $F$ to itself is an isomorphism. Indeed, by definition, $F$ is the core of itself, meaning that there is no homomorphism from $F$ to a subgraph $F_0$ of $F$ with $V(F_0) \subsetneq V(F)$. Hence, every homomorphism from $F$ to itself is a bijection, and hence an isomorphism. The assertion of the claim now follows from the fact that $\varphi_{|V(F')}$ is a homomorphism from $F'$ (which is a copy of $F$) to $F$.

The following definition will play an important role in our proofs. Let $F$ be a $k$-graph on vertex-set $[f]$ and let $G$ be an $f$-partite $k$-graph with sides $V_1, \ldots, V_f$. A *canonical copy* of $F$ in $G$ is a copy consisting of vertices $v_1 \in V_1, \ldots, v_f \in V_f$ in which $v_i$ plays the role of $i \in V(F)$ for each $i = 1, \ldots, f$. Note that if $G$ is homomorphic to $F$ via the homomorphism mapping $V_i$ to $i$ (for each $i = 1, \ldots, f$), and if furthermore $F$ is a core, then every copy of $F$ in $G$ is canonical; this follows from Claim 2·2.

We now describe our approach for proving Lemma 1·1 (the approach for Lemma 1·2 is similar). Let $I \subseteq V(F)$ be a set of vertices so that the 2-shadow of $F$ induced on $I$ is a cycle $C_t$, $t \geq 4$. Then $|I \cap e| \leq 2$ for every $e \in E(F)$. We first use a construction from [1], giving a $t$-partite graph which consists of many edge-disjoint canonical copies of $C_t$, yet contains only few canonical copies of $C_t$ altogether. The second step is then to extend the graph thus constructed into a $k$-graph containing many edge-disjoint copies of $F$ yet few copies of $F$. The following lemma will help us in performing this extension. For $\ell \geq 1$, two sets are called *$\ell$-disjoint* if their intersection has size at most $\ell - 1$. Two subgraphs of a hypergraph are called $\ell$-disjoint if their vertex-sets are $\ell$-disjoint. In what follows, when considering an $s$-partite hypergraph with parts $V_1, \ldots, V_s$, we will refer to the edges as sets or $s$-tuples, interchangeably. Moreover, we will use both set notation and $s$-tuple notation. For example, for $F \in V_1 \times \ldots \times V_s$, we write $F(i)$ for the $i$'th coordinate of $F$; and for $F_1, F_2 \in V_1 \times \ldots \times V_s$, we write $F_1 \cap F_2$ for the intersection of $F_1, F_2$ as sets.

LEMMA 2·3. *Let $r, s, k, \ell \geq 0$ satisfy $k \geq \ell$ and $r \geq k - \ell$. Let $V_1, \ldots, V_s, V_{s+1}, \ldots, V_{s+r}$ be pairwise-disjoint sets of size $n$ each. Let $\mathcal{S} \subseteq V_1 \times \ldots \times V_s$ be a family of $\ell$-disjoint sets. Then there is a family $\mathcal{F} \subseteq V_1 \times \ldots \times V_{s+r}$ with the following properties*:

*(i) for every $F \in \mathcal{F}$ it holds that $F|_{V_1 \times \ldots \times V_s} \in \mathcal{S}$;*

*(ii) $|\mathcal{F}| = \Omega_{r,s,k}(|\mathcal{S}| n^{k-\ell})$;*

*(iii) for every pair of distinct $F_1, F_2 \in \mathcal{F}$, if $|F_1 \cap F_2| \geq k$ then*

$$\#\{s + 1 \leq i \leq s + r : F_1(i) = F_2(i)\} \leq k - \ell - 1.$$

*Proof.* We construct the family $\mathcal{F}$ as follows. For each $S \in \mathcal{S}$ and each $r$-tuple $A \in V_{s+1} \times \ldots \times V_{s+r}$, add $S \cup A$ to $\mathcal{F}$ with probability $1/(Cn^{r-k+\ell})$ and independently, where $C$ is a large constant to be chosen later. (i) is satisfied by definition. Let us estimate the number of pairs $F_1, F_2 \in \mathcal{F}$ violating (iii); denote this number by $B$. We claim that

$$\mathbb{E}[B] = O_{s,r,k}\left(\frac{1}{C^2}\right) \cdot |\mathcal{S}| \cdot n^{k-\ell}. \tag{2.1}$$

To this end, suppose that $F_1, F_2 \in \mathcal{F}$ violate (iii), and write $F_1 = S_1 \cup A_1$ and $F_2 = S_2 \cup A_2$, where $S_1, S_2 \in \mathcal{F}$ and $A_1, A_2 \in V_{s+1} \times \ldots \times V_{s+r}$. Suppose first that $S_1 = S_2$. Then there are $|\mathcal{S}|$ choices for $S_1, S_2$. Also, to violate (iii), it must hold that $|A_1 \cap A_2| \geq k - \ell$. The number of choices of $A_1, A_2 \in V_{s+1} \times \ldots \times V_{s+r}$ with $|A_1 \cap A_2| \geq k - \ell$ is at most $n^r \cdot \binom{r}{k-\ell} \cdot n^{r-k+\ell}$. Finally, the probability that $F_1, F_2 \in \mathcal{F}$ is $1/(Cn^{r-k+\ell})^2$. Hence, the expected number of violations of this type (i.e., with $S_1 = S_2$) is at most $|\mathcal{S}| \cdot n^r \cdot \binom{r}{k-\ell} \cdot n^{r-k+\ell} \cdot 1/(Cn^{r-k+\ell})^2 = O_{s,r,k}\left(1/C^2\right) \cdot |\mathcal{S}| \cdot n^{k-\ell}$.

Now consider the case that $S_1 \neq S_2$, and put $t := |S_1 \cap S_2|$. As the sets in $\mathcal{S}$ are pairwise $\ell$-disjoint, we have $t \leq \ell - 1$. Also, the number of choices for $S_1, S_2 \in \mathcal{S}$ with $|S_1 \cap S_2| = t$ is at most $|\mathcal{S}| \cdot \binom{s}{t} \cdot n^{\ell-t}$, again using that the sets in $\mathcal{S}$ are pairwise $\ell$-disjoint. In order for $F_1, F_2$ to violate (iii), we must have $|A_1 \cap A_2| \geq k - t$. The number of choices for $A_1, A_2 \in V_{s+1} \times \ldots \times V_{s+r}$ with $|A_1 \cap A_2| \geq k - t$ is at most $n^r \cdot \binom{r}{k-t} \cdot n^{r-k+t}$. Finally, as before, the probability that $F_1, F_2 \in \mathcal{F}$ is $1/(Cn^{r-k+\ell})^2$. Hence, the expected number of violations of this type (i.e., with $S_1 \neq S_2$) is at most

$$\sum_{t=0}^{\ell-1}\left[|\mathcal{S}|\cdot\binom{s}{t}\cdot n^{\ell-t}\cdot n^{r}\cdot\binom{r}{k-t}\cdot n^{r-k+t}\cdot\left(\frac{1}{Cn^{r-k+\ell}}\right)^{2}\right]=O_{s,r,k}\left(\frac{1}{C^{2}}\right)\cdot|\mathcal{S}|\cdot n^{k-\ell}.$$

This proves (2·1). Now note that the expected size of $\mathcal{F}$ is $|\mathcal{S}|\cdot n^r\cdot 1/Cn^{r-k+\ell}=1/C\cdot|\mathcal{S}|\cdot n^{k-\ell}$. So by choosing $C$ to be large enough (as a function of $s$,$r$,$k$), we can guarantee that $\mathbb{E}[|\mathcal{F}|-B]\geq 1/2C\cdot|\mathcal{S}|\cdot n^{k-\ell}$. By fixing such a choice of $\mathcal{F}$ and deleting one set $F\in\mathcal{F}$ from each violation, we get the required conclusion.

The following well-known fact is an easy corollary of Lemma 2·3.

LEMMA 2·4. *Let $1\leq k\leq r$, and let $V_1,\ldots,V_r$ be pairwise-disjoint sets of size n each. Then there is $\mathcal{F}\subseteq V_1\times\ldots\times V_r$, $|\mathcal{F}|\geq\Omega(n^k)$, such that the r-sets in $\mathcal{F}$ are k-disjoint.*

*Proof.* Apply Lemma 2·3 with $s=\ell=0$ and $\mathcal{S}=\{\emptyset\}$.

The next lemma shows why constructing a $k$-graph with many edge-disjoint copies of $F$ but at most $n^{v(F)-1}$ copies of $F$ in total can be boosted to prove Lemmas 1·1 and 1·2. The lemma makes crucial use of the fact that $F$ is a core.

LEMMA 2·5. *Let F be a core k-graph, and suppose that for every $\delta>0$ and large enough n, there is an n-vertex k-graph H which is homomorphic to F, has a collection of at least $n^{k-\delta}$ edge-disjoint copies of F, but has at most $n^{v(F)-1}$ copies of F altogether. Then the F-removal lemma is not polynomial.*

*Proof.* Let $\varepsilon>0$ and let $n$ be large enough. Let $m$ be the largest integer satisfying $m^\delta\leq 1/\varepsilon$, so that $m\geq(1/\varepsilon)^{1/(2\delta)}$, say. Let $H$ be the $k$-graph guaranteed to exist by the assumption of the lemma, but with $m$ in place of $n$. So $H$ has $m$ vertices, is homomorphic to $F$, contains a collection $\mathcal{F}$ of $m^{k-\delta}\geq\varepsilon m^k$ edge-disjoint copies of $F$, but has at most $m^{v(F)-1}$ copies of $F$ altogether.

Let $G$ be the $n/m$-blowup of $H$. Each $F'\in\mathcal{F}$ gives rise to $\Omega((n/m)^k)$ $k$-disjoint (and hence also edge-disjoint) copies of $F$ in $G$, by Lemma 2·4 applied with $r=v(F)$ and with $n/m$ in place of $n$. Copies arising from different $F'_1,F'_2\in\mathcal{F}$ are edge-disjoint, because the copies in $\mathcal{F}$ are edge-disjoint. Altogether, this gives a collection of $\varepsilon m^k\cdot\Omega((n/m)^k)=\Omega(\varepsilon n^k)$ edge-disjoint copies of $F$ in $G$.

Let us upper-bound the total number of copies of $F$ in $G$. By assumption, there is a homomorphism $\varphi$ from $H$ to $F$. Let $\psi$ be the "natural" homomorphism from $G$ to $H$ (as described in the beginning of this section). Then $\varphi\circ\psi$ is a homomorphism from $G$ to $F$. By Claim 2·2, for every copy $F'$ of $F$ in $G$ the map $(\varphi\circ\psi)|_{V(F')}$ is an isomorphism from $F'$ to $F$. We claim that this means that $\psi$ maps every copy $F'$ of $F$ in $G$ onto a copy of $F$ in $H$. Indeed, $\psi|_{V(F')}$ must be injective (otherwise $(\varphi\circ\psi)|_{V(F')}$ would not be an isomorphism), and since $\psi|_{V(F')}$ must map edges to edges (on account of being a homomorphism) its image must contain a copy of $F$. We thus see that every copy of $F$ in $G$ must come from the blown-up copies of $F$ in $H$. But each copy of $F$ in $H$ gives rise to $(n/m)^{v(F)}$ copies of $F$ in $G$. Hence, the total number of copies of $F$ in $G$ is at most

$$m^{v(F)-1}\cdot(n/m)^{v(F)}=n^{v(F)}/m\leq\varepsilon^{1/(2\delta)}\cdot n^{v(F)}.$$

Since $\delta>0$ is arbitrary, this shows that the $F$-removal lemma is not polynomial.

The following result is implicit in [1]. For the sake of completeness, we include a proof.

LEMMA 2·6. *Let $t \geq 3$. Then for every large enough $n$, there is a $t$-partite graph $G$ with sides $V_1, \ldots, V_t$, each of size $n$, such that $G$ has a collection of $n^2/e^{O(\sqrt{\log n})} = n^{2-o(1)}$ 2-disjoint canonical copies of $C_t$, but at most $n^{t-1}$ canonical copies of $C_t$ altogether.*

*Proof.* Suppose that the vertices of $C_t$ are $1, 2, \ldots, t$ (appearing in this order along the cycle). Take a set $B \subseteq [n/t]$, $|B| \geq n/e^{O\sqrt{\log n}}$, with no non-trivial solution to the linear equation $y_1 + \ldots + y_{t-1} = (t-1)y_t$ with $y_1, \ldots, y_t \in B$ (where a solution is trivial if $y_1 = y_2 = \cdots = y_t$). The existence of such a set $B$ is by a simple generalisation of Behrend's construction [3] of sets avoiding 3-term arithmetic progressions, see [1, lemma 3·1]. Take pairwise-disjoint sets $V_1, \ldots, V_t$ of size $n$ each, and identify each $V_i$ with $[n]$. For each $x \in [n/t]$ and $y \in B$, add to $G$ a canonical copy $S_{x,y}$ of $C_t$ on the vertices $v_i = x + (i-1)y \in V_i$, $i = 1, \ldots, t$. Note that $x + (i-1)y \leq x + (t-1)y \leq n$, so $v_i$ indeed "fits" into $V_i = [n]$. The copies $S_{x,y}$ (where $x \in [n/t], y \in B$) are 2-disjoint. Indeed, if $S_{x_1,y_1}, S_{x_2,y_2}$ intersect in $V_i$ and in $V_j$, then $x_1 + (i-1)y_1 = x_2 + (i-1)y_2$ and $x_1 + (j-1)y_1 = x_2 + (j-1)y_2$, and solving this system of equations gives $x_1 = x_2, y_1 = y_2$. The number of copies $S_{x,y}$ is $n/t \cdot |B| \geq n^2/e^{O\sqrt{\log n}}$.

Let us bound the total number of canonical copies of $C_t$ in $G$. Fix a canonical copy with vertices $v_1, \ldots, v_t, v_i \in V_i$. For $1 \leq j \leq t-1$, let $x_j \in [n/t], y_j \in B$ be such that $v_j, v_{j+1} \in S_{x_j,y_j}$. Similarly, let $x_t \in [n/t], y_t \in B$ such that $v_1, v_t \in S_{x_t,y_t}$. Then we have $v_{j+1} - v_j = y_j$ for every $1 \leq j \leq t-1$, and $v_t - v_1 = (t-1)y_t$. So $y_1 + \ldots + y_{t-1} = (t-1)y_t$. By our choice of $B$, we have $y_1 = \ldots = y_t =: y$. Now, for each $1 \leq j \leq t-1$ we have $x_j = v_{j+1} - j \cdot y = x_{j+1}$, so $x_1 = \ldots = x_t =: x$. So we see that the only canonical copies of $C_t$ in $G$ are the copies $S_{x,y}$. Their number is at most $n^2 \leq n^{t-1}$, as required.

Recall that $K_s^{(s-1)}$ is the $(s-1)$-graph with vertices $1, \ldots, s$ and all $s$ possible edges. The following construction appears implicitly in [9] (see also [2]). Again, for completeness, we include a proof.

LEMMA 2·7. *Let $s \geq 3$. For every large enough $n$, there is an $s$-partite $(s-1)$-graph $G$ with sides $V_1, \ldots, V_s$, each of size $n$, such that $G$ has a collection of $n^{s-1}/e^{O(\sqrt{\log n})} = n^{s-1-o(1)}$ $(s-1)$-disjoint canonical copies of $K_s^{(s-1)}$, but at most $n^{s-1}$ copies of $K_s^{(s-1)}$ altogether.*

*Proof.* Take a set $B \subseteq [n/s]$, $|B| \geq n/e^{O\sqrt{\log n}}$, with no non-trivial solution to $y_1 + y_2 = 2y_3$, $y_1, y_2, y_3 \in B$. Take pairwise-disjoint sets $V_1, \ldots, V_s$ of size $n$ each, and identify each $V_i$ with $[n]$. For each $x_1, \ldots, x_{s-2} \in [n/s]$ and $y \in B$, add to $G$ a copy $K_{x_1,\ldots,x_{s-2},y}$ of $K_s^{(s-1)}$ on the vertices

$$x_1 \in V_1, \quad x_2 \in V_2, \quad \ldots \quad x_{s-2} \in V_{s-2}, \quad y + \sum_{i=1}^{s-2} x_i \in V_{s-1}, \quad 2y + \sum_{i=1}^{s-2} x_i \in V_s.$$

It is easy to see that these copies are $(s-1)$-disjoint, because fixing any $s-1$ of the $s$ coordinates allows to solve for $x_1, \ldots, x_{s-2}, y$. Also, the number of copies thus placed is $(n/s)^{s-2} \cdot |B| \geq n^{s-1}/e^{O\sqrt{\log n}}$. Let us show that there are no other copies of $K_s^{(s-1)}$ in $G$. This would imply that the total number of copies of $K_s^{(s-1)}$ in $G$ is $(n/s)^{s-2} \cdot |B| \leq n^{s-1}$. So suppose that $v_1 \in V_1, \ldots, v_s \in V_s$ form a copy of $K_s^{(s-1)}$. Let $x^{(i)} = (x_1^{(i)}, \ldots, x_{s-2}^{(i)}) \in$

$[n/s]^{s-2}$ and $y_i \in B$, $i = 1, 2, 3$, be such that $\{v_2, \ldots, v_s\} \in K_{x^{(1)}, y_1}$, $\{v_1, \ldots, v_{s-1}\} \in K_{x^{(2)}, y_2}$ and $\{v_1, \ldots, v_{s-2}, v_s\} \in K_{x^{(3)}, y_3}$. Then $x_1^{(2)} = x_1^{(3)} = v_1$ and

$$x_j^{(1)} = x_j^{(2)} = x_j^{(3)} = v_j \text{ for every } 2 \leq j \leq s-2. \tag{2.2}$$

Also, $v_s - v_{s-1} = y_1$, $v_{s-1} - v_1 = x_2^{(2)} + \cdots + x_{s-2}^{(2)} + y_2$ and $v_s - v_1 = x_2^{(3)} + \cdots + x_{s-2}^{(3)} + 2y_3$. Combining these three equations and using (2·2), we get $y_1 + y_2 = 2y_3$, and so $y_1 = y_2 = y_3 =: y$ by our choice of $B$. Also, $x_1^{(1)} = v_{s-1} - (v_2 + \cdots + v_{s-2} + y) = x_1^{(2)}$. So $x^{(1)} = x^{(2)} = x^{(3)}$.

We now prove two lemmas, Lemmas 2·8 and 2·9, which imply Lemmas 1·1 and 1·2, respectively. Recall that for a $k$-graph $F$ and $2 \leq \ell \leq k$, the *$\ell$-shadow* of $F$, denoted $\partial_\ell F$, is the $\ell$-graph consisting of all $f \in \binom{V(F)}{\ell}$ such that there is $e \in E(F)$ with $f \subseteq e$.

LEMMA 2·8. *Let $k \geq 2$, let $F$ be a core $k$-graph, and suppose that $\partial_2 F$ has an induced cycle of length at least 4. Then for every large enough $n$ there is a $k$-graph $H$ with $v(F) \cdot n$ vertices which is homomorphic to $F$, has a collection of $n^k / e^{O(\sqrt{\log n})} = n^{k-o(1)}$ edge-disjoint copies of $F$, but has at most $n^{v(F)-1}$ copies of $F$ altogether.*

*Proof.* It will be convenient to write $|V(F)| = t + r$ and assume that $V(F) = [t + r]$, where $(1, 2, \ldots, t, 1)$ is an induced cycle in $\partial_2 F$ and $t \geq 4$. It follows that $|e \cap \{1, \ldots, t\}| \leq 2$ for every $e \in E(F)$. Take disjoint sets $V_1, \ldots, V_{t+r}$ of size $n$ each. Let $G$ be the $t$-partite graph with sides $V_1, \ldots, V_t$ given by Lemma 2·6. Let $\mathcal{S}$ be a collection of $n^2 / e^{O(\sqrt{\log n})}$ 2-disjoint canonical copies of $C_t$ in $G$. Apply Lemma 2·3 to[4] $\mathcal{S}$ with $s = t$ and $\ell = 2$ to obtain a family $\mathcal{F} \subseteq V_1 \times \ldots \times V_{t+r}$ satisfying Items 1-3 in that lemma. Note that $r \geq k - 2 = k - \ell$, because each edge of $F$ contains at most two vertices from $\{1, \ldots, t\}$ and hence at least $k - 2$ vertices from $\{t + 1, \ldots, t + r\}$. Therefore, the conditions of Lemma 2·3 are satisfied. Define the hypergraph $H$ by placing a canonical copy of $F$ on each $F' \in \mathcal{F}$. We claim that these copies of $F$ are edge-disjoint. Indeed, suppose by contradiction that the copies on $F_1, F_2 \in \mathcal{F}$ share an edge $e$. Then $|F_1 \cap F_2| \geq k$. By Lemma 2·3(iii), we have $\#\{t + 1 \leq i \leq t + r : F_1(i) = F_2(i)\} \leq k - 3$. This implies that $\#\{1 \leq i \leq t : e \cap V_i \neq \emptyset\} \geq 3$. But this means that in $F$ there is an edge which intersects $\{1, \ldots, t\}$ in at least 3 vertices, a contradiction. So the $F$-copies in $\mathcal{F}$ are indeed edge-disjoint. Their number is $|\mathcal{F}| \geq \Omega(|\mathcal{S}| n^{k-2}) \geq n^k / e^{O(\sqrt{\log n})}$, by Lemma 2·3(ii).

To complete the proof, it remains to show that $H$ has at most $n^{t+r-1}$ copies of $F$. Observe that $H$ is homomorphic to $F$; indeed, the map $\varphi$ which sends $V_j \mapsto j$, $j = 1, \ldots, t + r$, is such a homomorphism. Let $F^*$ be a copy of $F$ in $H$. Since $F$ is a core and $\varphi$ is a homomorphism from $H$ to $F$, we can apply Claim 2·2 to conclude that $F^*$ must have the form $v_1, \ldots, v_{t+r}$, with $v_i \in V_i$ playing the role of $i$ for each $i = 1, \ldots, t + r$. We claim that $v_1, \ldots, v_t$ form a canonical copy of $C_t$ **in**[5] $G$. To see this, fix any $1 \leq i \leq t$ and let us show that $\{v_i, v_{i+1}\} \in E(G)$, with indices taken modulo $t$. Since $\{i, i + 1\}$ is an edge of $\partial_2 F$, there must be an edge

[4] Strictly speaking, we apply Lemma 2·3 to the vertex-sets of the copies of $C_t$ in $\mathcal{S}$.

[5] Note that the subgraph of $\partial_2(F^*)$ induced by $v_1, \ldots, v_t$ is a canonical copy of $C_t$ in the 2-shadow of $H$. The first key point is that this copy of $C_t$ must appear in $G$. Also, note that this fact is trivial if $F^*$ is one of the canonical copies of $F$ we placed in $H$ when defining it. The second key point is that this holds for every copy $F^*$ of $F$ in $H$.

$e \in E(F)$ containing $i, i+1$. Then $\{v_a : a \in e\} \in E(F^*) \subseteq E(H) = \bigcup_{F' \in \mathcal{F}} E(F')$. Let $F' \in \mathcal{F}$ such that $\{v_a : a \in e\} \in E(F')$. By Lemma 2·3(i), we have $S' := F'|_{V_1 \times \ldots \times V_t} \in \mathcal{S}$. Now, $S'$ is the vertex set of a canonical copy of $C_t$ in $G$, and hence $\{v_i, v_{i+1}\} \in E(G)$, as required. This proves our claim that $v_1, \ldots, v_t$ form a canonical copy of $C_t$ in $G$. Summarising, every copy of $F$ in $H$ contains the vertices of a canonical copy of $C_t$ in $G$. By the guarantees of Lemma 2·6, the number of canonical copies of $C_t$ in $G$ is at most $n^{t-1}$. Hence, the number of copies of $F$ in $H$ is at most $n^{t-1} \cdot n^r = n^{t+r-1}$, as required.

LEMMA 2·9. *Let $k \geq 2$, let $F$ be a core $k$-graph and suppose that there are $3 \leq s \leq k+1$ and a set $I \subseteq V(F)$ such that $(\partial_{s-1}F)[I] \cong K_s^{(s-1)}$ and $|e \cap I| \leq s-1$ for every $e \in E(F)$. Then for every large enough $n$ there is a $k$-graph $H$ with $v(F) \cdot n$ vertices which is homomorphic to $F$, has a collection of $n^k/e^{O(\sqrt{\log n})} = n^{k-o(1)}$ edge-disjoint copies of $F$, but has at most $n^{v(F)-1}$ copies of $F$ altogether.*

*Proof.* The proof is very similar to that of Lemma 2·8. Assume that $I = [s]$, $V(F) = [s+r]$. Take disjoint sets $V_1, \ldots, V_{s+r}$ of size $n$ each. Let $G$ be the $s$-partite $(s-1)$-graph with sides $V_1, \ldots, V_s$ given by Lemma 2·7. Let $\mathcal{S}$ be a collection of $n^{s-1}/e^{O(\sqrt{\log n})}$ $(s-1)$-disjoint copies of $K_s^{(s-1)}$ in $G$. Apply Lemma 2·3 to $\mathcal{S}$ with $\ell = s-1$ to obtain a family $\mathcal{F} \subseteq V_1 \times \ldots \times V_{s+r}$ satisfying (i)-(iii) in that lemma. Define the hypergraph $H$ by placing a canonical copy of $F$ on each $F' \in \mathcal{F}$. These copies of $F$ are edge-disjoint. Indeed, suppose by contradiction that the copies on $F_1, F_2 \in \mathcal{F}$ share an edge $e$. Then $|F_1 \cap F_2| \geq k$, and hence $\#\{s+1 \leq i \leq s+r : F_1(i) = F_2(i)\} \leq k - \ell - 1 = k - s$ by Lemma 2·3(iii). But then $\#\{1 \leq i \leq s : e \cap V_i \neq \emptyset\} = s$, meaning that there is an edge of $F$ which contains $I = [s]$, a contradiction to the assumption of the lemma. So the $F$-copies in $\mathcal{F}$ are indeed edge-disjoint. Also, $|\mathcal{F}| \geq \Omega(|\mathcal{S}| n^{k-s+1}) \geq n^k/e^{O(\sqrt{\log n})}$, using Lemma 2·3(ii).

The map $V_j \mapsto j$, $j = 1, \ldots, s+r$ is a homomorphism from $H$ to $F$. Let us bound the number of copies of $F$ in $H$. By Claim 2·2, every copy $F^*$ of $F$ must be of the form $v_1, \ldots, v_{s+r}$, with $v_i \in V_i$ playing the role of $i$ for each $i = 1, \ldots, s+r$. We claim that $v_1, \ldots, v_s$ span a copy of $K_s^{(s-1)}$ in $G$. So let $J \in \binom{[s]}{s-1}$. Since $(\partial_{s-1}F)[I] \cong K_s^{(s-1)}$, there is an edge $e \in E(F)$ with $J \subseteq e$. Since $F^*$ is a canonical copy of $F$, we have $\{v_i : i \in e\} \in E(F^*) \subseteq E(H) = \bigcup_{F' \in \mathcal{F}} E(F')$. Let $F' \in \mathcal{F}$ be such that $\{v_i : i \in e\} \in E(F')$. By Lemma 2·3(i), we have $S' := F'|_{V_1 \times \ldots \times V_s} \in \mathcal{S}$. Now, $S'$ is a canonical copy of $K_s^{(s-1)}$ in $G$, and hence $\{v_i : i \in J\} \in E(G)$, as required. So we see that every copy of $F$ in $H$ contains the vertices of a copy of $K_s^{(s-1)}$ in $G$. By the guarantees of Lemma 2·6, $G$ has at most $n^{s-1}$ copies of $K_s^{(s-1)}$. Hence, $H$ has at most $n^{s-1} \cdot n^r = n^{s+r-1}$ copies of $F$, as required.

Observe that Lemma 1·1 follows by combining Lemmas 2·5 and 2·8. Let us prove Lemma 1·2.

*Proof of Lemma 1·2.* Let $X$ be a clique of size $k+1$ in $\partial_2 F$. Let $I$ be a smallest subset of $X$ which is not contained in an edge of $F$. Note that $I$ is well-defined (because $X$ itself is not contained in any edge of $F$, as $|X| = k+1$). Also, $|I| \geq 3$ because every pair of vertices in $X$ is contained in some edge, as $X$ is a clique in $\partial_2 F$. Put $s = |I|$. Then $(\partial_{s-1}F)[I] \cong K_s^{(s-1)}$ and $|e \cap I| \leq s-1$ for every $e \in E(F)$, by the choice of $I$. Now the assertion of Lemma 1·2 follows by combining Lemmas 2·5 and 2·9.

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