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RESEARCH ARTICLE

Topological methods in zero-sum Ramsey theory

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Abstract

A landmark result of Erdős, Ginzburg, and Ziv (EGZ) states that any sequence of $2n-1$ elements in $\mathbb{Z}/n$ contains a zero-sum subsequence of length n. While algebraic techniques have predominated in deriving many deep generalizations of this theorem over the past sixty years, here we introduce topological approaches to zero-sum problems which have proven fruitful in other combinatorial contexts. Our main result is a topological criterion for determining when any $\mathbb{Z}/n$-coloring of an n-uniform hypergraph contains a zero-sum hyperedge. In addition to applications for Kneser hypergraphs, for complete hypergraphs our methods recover Olson's generalization of the EGZ theorem for arbitrary finite groups. Furthermore, we give a fractional generalization of the EGZ theorem with applications to balanced set families and provide a constrained EGZ theorem which imposes combinatorial restrictions on zero-sum sequences in the original result.

Contents

1. Introduction and statement of results

The classical Erdős–Ginzburg–Ziv (EGZ) theorem [19] states that any sequence $a_1, \dots, a_{2n-1}$ of $2n-1$ elements in $\mathbb{Z}/n$ contains a subsequence $a_{i_1}, \dots, a_{i_n}$ with $\sum_j a_{i_j} = 0$. Over the last sixty years, this result has inspired numerous generalizations and variants, collectively known as zero-sum Ramsey theory, a general viewpoint that seems to originate from a paper of Bialostocki and Dierker [7]; see [15, 21] for surveys. Algebraic techniques such as the Cauchy–Davenport and Chevalley–Warning theorems have proven to be particularly fruitful in deriving results of this type. Multiple different proofs of the original EGZ theorem are known (see e.g., [2]); these typically proceed by first establishing the result for n a prime, from which the general case follows by a simple induction on prime divisors.

Here we introduce equivariant topological methods to the study of zero-sum Ramsey problems. Such techniques have proven to be quite powerful in other combinatorial contexts, for example in Tverberg-type intersection theory (see, e.g., [12, 37, 11]) and in establishing the chromatic numbers of Kneser graphs [29] and hypergraphs [3]. We refer the reader to Matoušek [30], de Longueville [28], and Kozlov [25] for the basics of topological combinatorics. We remark that [36] presents an unrelated geometric approach to zero-sum Ramsey results and [24] develops topological methods for a problem in arithmetic combinatorics. Moreover, known results already show that the original EGZ theorem can be seen as a consequence of the colorful Carathéodory theorem [4], a central result in discrete geometry, since the latter implies a result of Drisko [18] on rainbow matchings in bipartite graphs from which [1] the EGZ theorem quickly follows.

In what follows, we give three topological proofs of the Erdős–Ginzburg–Ziv theorem that generalize in three distinct directions. In particular, when $n = p$ is prime we

(i) establish a topological criterion for when any $\mathbb{Z}/p$-coloring of a p-uniform hypergraph admits a hyperedge whose labels sum to zero, by which the Erdős–Ginzburg–Ziv theorem is recovered in the case of a complete hypergraph;
(ii) provide a fractional generalization of the Erdős–Ginzburg–Ziv theorem; and
(iii) prove a constrained version of the Erdős–Ginzburg–Ziv theorem which imposes combinatorial restrictions on zero-sum sequences in the original result.

We now state these generalizations and collect some consequences.

1.1. Hypergraph coloring generalizations of EGZ

Let H be an n-uniform hypergraph with vertex set V, that is, a collection of n-element subsets of V. A $\mathbb{Z}/n$**-coloring** of H is a map $c\colon V \longrightarrow \mathbb{Z}/n$, and a hyperedge $e \in H$ is said to be **zero-sum** with respect to c provided $\sum_{v\in e} c(v) = 0$. Thus an equivalent formulation of the EGZ theorem is that any $\mathbb{Z}/n$-coloring of the complete n-uniform hypergraph on $[2n-1] = \{1, 2, \dots, 2n-1\}$ has a zero-sum hyperedge.

When $n = p$ is prime, our topological criterion for when a given hypergraph H has a zero-sum hyperedge for any $\mathbb{Z}/p$-coloring is stated in terms of continuous maps $f\colon B(H) \longrightarrow S^{2p-3}$ from the **box complex** $B(H)$ of H to a $(2p-3)$-dimensional sphere. Box complexes are simplicial complexes which have been instrumental in establishing lower bounds for the chromatic number $\chi(H)$ of uniform hypergraphs; see Section 2.1 for the precise definition. As with any simplicial complex, one may think of $B(H)$ as a topological space glued from simplices, which in this case carries a free $\mathbb{Z}/p$-action.

Denote the d-dimensional sphere by S^d. Any p-th root of unity determines a $\mathbb{Z}/p$-action on $\mathbb{C} \cong \mathbb{R}^2$ given by multiplication. Considering all nontrivial roots of unity, one thereby has a free $\mathbb{Z}/p$-action on $S^{2p-3} \subset \mathbb{R}^{2p-2}$ given by the diagonal action on $\mathbb{R}^{2p-2} \cong (\mathbb{R}^2)^{p-1}$. We recall that a continuous map $f\colon X \longrightarrow Y$ between two spaces X and Y equipped with a $\mathbb{Z}/p$-action is *equivariant* if it commutes with the action.

We may now state our first main result:

Theorem 1.1. *Let $p \geq 2$ be a prime, let H be a p-uniform hypergraph, and suppose that there is no $\mathbb{Z}/p$-equivariant map $B(H) \longrightarrow S^{2p-3}$. Then for any $\mathbb{Z}/p$-coloring of H there is a zero-sum hyperedge in H.*

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An elementary Borsuk–Ulam type theorem of Dold [16] applies to show that the box complex of the complete p-uniform hypergraph on $2p-1$ vertices does not admit a $\mathbb{Z}/p$-equivariant map to S^{2p-3}. Thus Theorem 1.1 recovers the Erdős–Ginzburg–Ziv theorem when p is prime.

If one replaces continuous maps with *linear* ones, then the box complex construction allows for the consideration of colorings of n-uniform hypergraphs by any group of order n (see Theorem 2.5). By restricting to complete uniform hypergraphs and now applying the "linear Borsuk–Ulam theorem" of Sarkaria [33], one obtains an extension of the EGZ theorem to arbitrary finite groups due to Olson [32] as an immediate consequence:

Theorem 1.2. *Let $a_1, \dots, a_{2n-1}$ be a sequence of elements of a group G of order n. Then there are n distinct indices $i_1, \dots, i_n$ such that $a_{i_1} \cdots a_{i_n} = 1$, where 1 is the identity element of G.*

It follows from a result of Kříž that any hypergraph satisfying the criterion of Theorem 1.1 must have chromatic number at least three [27, Theorem 2.4]. Such a lower bound for the chromatic number does not in itself guarantee a zero-sum hyperedge; see Remark 2.3 for a simple counterexample. Nonetheless, in the special case of Kneser hypergraphs, one can use Theorem 1.1 to give a purely combinatorial criterion for the existence of zero-sum hyperedges. Recall that for a set family $\mathcal{F}$ on ground set $[m] = \{1, 2, \dots, m\}$, the **n-uniform Kneser hypergraph** $\mathrm{KG}^n(\mathcal{F})$ has $\mathcal{F}$ as its vertex set and $A_1, \dots, A_n \in \mathcal{F}$ form a hyperedge if the A_i are pairwise disjoint. The **n-colorability defect** $\mathrm{cd}^n(\mathcal{F})$ of $\mathcal{F}$, introduced by Dolnikov [17] for $n = 2$ and in general by Kříž [27], is defined by

$$\mathrm{cd}^n(\mathcal{F}) = \\ m - \max\left\{\sum_{i=1}^{n} |A_i| \mid A_1, \dots, A_n \subset [m] \text{ pairwise disjoint and } F \not\subset A_i \text{ for all } F \in \mathcal{F} \text{ and all } i \in [n]\right\}.$$

Kříž proved the fundamental inequality $(n-1)\chi(\mathrm{KG}^n(\mathcal{F})) \geq \mathrm{cd}^n(\mathcal{F})$ relating the n-colorability defect and the chromatic number of the Kneser hypergraph, while for $\mathbb{Z}/n$-colorings Theorem 1.1 will imply the following:

Theorem 1.3. *Let $n \geq 2$ be an integer, and let $\mathcal{F}$ be a set system with $\mathrm{cd}^n(\mathcal{F}) \geq 2n-1$. Then any $\mathbb{Z}/n$-coloring of $\mathrm{KG}^n(\mathcal{F})$ has a zero-sum hyperedge.*

As a special case of Theorem 1.3, let $k \geq 1$ be an integer and let m be an integer satisfying $m \geq n(k+1) - 1$. Considering the set system $\mathcal{F}$ consisting of all k-element subsets of $[m]$, it is easily seen that $\mathrm{cd}^n(\mathcal{F}) = m - n(k-1) \geq 2n - 1$. Thus Theorem 1.3 applied to $\mathrm{KG}^n(\mathcal{F})$ recovers the fact (see [6, 15]) that $\mathcal{F}$ contains a zero-sum *matching* of size n for any $\mathbb{Z}/n$-coloring of the hyperedges of $\mathcal{F}$. In particular, the Erdős–Ginzburg–Ziv theorem is recovered by letting $k = 1$.

1.2. Fractional and constrained extensions of EGZ

Our remaining results have for their starting point the characterization of zero-sum sequences in $\mathbb{Z}/n$ originally due to Marshall Hall [22]. We will give a topological proof of this fact when n is prime; see Section 4.2.

Theorem 1.4. *A sequence $a_1, \dots, a_n \in \mathbb{Z}/n$ is zero-sum if and only if there are permutations $\{b_1, \dots, b_n\}$ and $\{c_1, \dots, c_n\}$ of $\mathbb{Z}/n$ with $a_i = b_i - c_i$ for all $i \in [n]$.*

Thus an equivalent reformulation of the Erdős–Ginzburg–Ziv theorem is that any sequence in $\mathbb{Z}/n$ of length $2n-1$ contains a subsequence of length n that is a difference of two permutations. When $n = p$ is prime, we will strengthen this reformulation of the EGZ theorem in two distinct ways. First, we replace sequences of elements in $\mathbb{Z}/p$ with sequences of arbitrary probability measures on $\mathbb{Z}/p$. Secondly, we give additional constraints for those a_i which are the difference of two permutations.

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Let $j \in \mathbb{Z}/p$. For any subset $A \subset \mathbb{Z}/p$ we denote by $A+j$ the shifted set $\{a+j \mid a \in A\}$, and likewise for any probability measure μ on $\mathbb{Z}/p$, we denote by $\mu + j$ the shifted probability measure defined by $(\mu+j)(A) = \mu(A-j)$ for $A \subset \mathbb{Z}/p$. Our second main result is as follows:

Theorem 1.5. *Let $\mu_1, \dots, \mu_{2p-1}$ be a sequence of probability measures on $\mathbb{Z}/p$. Then there is an injective map $\pi\colon \mathbb{Z}/p \to [2p-1]$ and convex coefficients $\lambda_i \geq 0$, $i \in \mathbb{Z}/p$, with $\sum_{i\in\mathbb{Z}_p} \lambda_i = 1$ such that $\sum_{i\in\mathbb{Z}/p} \lambda_i(\mu_{\pi(i)} + i)$ is the uniform probability measure on $\mathbb{Z}/p$.*

If the μ_i are Dirac measures concentrated at a single $a_i \in \mathbb{Z}/p$, then realizing the uniform probability measure as a convex combination of the $\mu_{\pi(i)} + i$ requires that the $a_{\pi(i)} - i$ are pairwise distinct elements of $\mathbb{Z}/p$ and so Theorem 1.5 recovers the Erdős–Ginzburg–Ziv theorem. In the special case where each μ_i is the uniform measure on a subset $A_i \subset \mathbb{Z}/p$, we derive a corollary for balanced set systems which specializes to the EGZ theorem in the case that all the A_i are singletons. Recall that a family of sets $\mathcal{F}$ is **balanced** if it admits a perfect fractional matching, that is, if there is a function $m\colon \mathcal{F} \to [0,1]$ such that $\sum_{A\in\mathcal{F}\mid v\in A} m(A) = 1$ for every v in the ground set of $\mathcal{F}$. One then has the following:

Corollary 1.6. *Let $A_1, \dots, A_{2p-1} \subset \mathbb{Z}/p$ be a sequence of nonempty subsets of $\mathbb{Z}/p$. Then there is an injective map $\pi\colon \mathbb{Z}/p \to [2p-1]$ such that $A_{\pi(i)} + i$, $i \in \mathbb{Z}/p$, is a balanced collection of subsets.*

By Theorem 1.4, the Erdős–Ginzburg–Ziv theorem is equivalent to the statement that for any sequence $a_1, \dots, a_{2p-1}$ of elements in $\mathbb{Z}/p$ there is a subsequence $a_{i_1}, \dots, a_{i_p}$ of length p and pairwise distinct elements $b_1, \dots, b_p \in \mathbb{Z}/p$ such that $\{a_{i_1} + b_1, \dots, a_{i_p} + b_p\} = \mathbb{Z}/p$. Lastly, we will use our topological approach to zero-sum problems to prove restrictions on the permutation $b_1, \dots, b_p$:

Theorem 1.7. *Let $a_1, \dots, a_{2p-1}$ be a sequence in $\mathbb{Z}/p$ and fix $d_1, \dots, d_{p-1} \in \mathbb{Z}/p \setminus \{0\}$. Then there is a subsequence $a_{i_1}, \dots, a_{i_p}$, $i_1 < i_2 < \cdots < i_p$, and pairwise distinct $b_1, \dots, b_p \in \mathbb{Z}/p$ such that $\{a_{i_1} + b_1, \dots, a_{i_p} + b_p\} = \mathbb{Z}/p$ with the following additional constraint: If for any $j \in [p-1]$ we have that i_j is odd and $i_{j+1} = i_j + 1$, then we may prescribe that $b_{j+1} = b_j + d_j$.*

2. A box complex criterion for zero-sums

2.1. Box complexes

We give the definition of the box complex of an n-uniform hypergraph in terms of the "deleted join" construction commonly used in Tverberg-type intersection theory.

Given a finite simplicial complex Σ and any integer n, its *n-fold join* Σ^{*n} is the simplicial complex

$$\Sigma^{*n} = \{A_1 \uplus \cdots \uplus A_n \mid A_1, \dots, A_n \in \Sigma\},$$

where $A_1 \uplus \cdots \uplus A_n = (A_1 \times \{1\}) \cup \cdots \cup (A_n \times \{n\})$ denotes the disjoint union of the faces $A_1, \dots, A_n$ of Σ. The subcomplex

$$\Sigma^{*n}_{\Delta} = \{A_1 \uplus \cdots \uplus A_n \in \Sigma^{*n} \mid A_i \cap A_j = \emptyset \text{ for all } i \neq j\}$$

consisting of all n pairwise disjoint faces of Σ (possibly including empty faces) is called the *deleted n-fold join* of Σ. The geometric realization of Σ^{*n}_{Δ} consists of all formal convex combinations $\sum_{i=1}^n \lambda_i x_i$ where each x_i lies in the simplex determined by A_i for all $i \in [n]$. If one identifies $[n]$ with a group G of order n, then left multiplication by elements of the group determines a free G-action on Σ^{*n}_{Δ} which permutes the join factors of the $A_1 \uplus \cdots \uplus A_n$.

Now let H be an n-uniform hypergraph with vertex set V of size m and let Δ_{m-1} be the simplex determined by V; that is, as an abstract simplicial complex Δ_{m-1} contains all subsets of V, whereas its geometric realization is the standard regular simplex

$$\Delta_{m-1} = \{x \in \mathbb{R}^V \mid x_v \geq 0 \text{ for all } v \in V,\ \sum_{v\in V} x_v = 1\}.$$

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We will use the notation Δ_{m-1} for both the abstract simplicial complex and its geometric realization. The box complex $B(H)$ is then the subcomplex of $(\Delta_{m-1})^{*n}_\Delta$ consisting of all $A_1 \uplus \cdots \uplus A_n \in (\Delta_{m-1})^{*n}_\Delta$ such that $\{a_1, \ldots, a_n\} \in H$ for all $a_1 \in A_1, \ldots, a_n \in A_n$. Thus for nonempty $A_i \subset V$ we have that $A_1 \uplus \cdots \uplus A_n \in B(H)$ if and only if the A_i are pairwise disjoint and H contains the complete n-partite hypergraph determined by the A_i. (If at least one of the A_j is empty then $A_1 \uplus \cdots \uplus A_n \in B(H)$ whenever the A_i are pairwise disjoint independent of the hypergraph H.) For any finite group G of order n, one has a free G-action on $B(H)$ by restricting the G-action on $(\Delta_{m-1})^{*n}_\Delta$. As an important special case, observe that $B(H) = (\Delta_{m-1})^{*n}_\Delta$ if H is the complete n-uniform hypergraph on $[m]$, since in that case H contains any n-element subset of $[m]$ as a hyperedge.

2.2. Proof of Theorem 1.1

Our proof of Theorem 1.1 relies on the following lemma, a simple consequence of Hall's matching criterion for bipartite graphs (see, e.g., [23], which is used in a standard proof of the Birkhoff–von Neumann theorem [8]).

Lemma 2.1. *Let $A = (a_{ij})_{i,j} \in \mathbb{R}^{n\times n}$ be a doubly stochastic matrix, that is, all entries $a_{ij} \geq 0$ are nonnegative and the entries of each column and of each row sum to* 1. *Then there is a permutation $\pi\colon [n] \longrightarrow [n]$ such that $a_{i\pi(i)} > 0$ for all $i \in [n]$.*

A reformulation of this yields a characterization of those subsets of vertices of the product of two regular $(n-1)$-simplices $\Delta_{n-1} \times \Delta_{n-1}$ whose convex hull captures the barycenter of $\Delta_{n-1} \times \Delta_{n-1}$.

Lemma 2.2. *Let $x_0 \in \Delta_{n-1} \times \Delta_{n-1}$ denote the barycenter of the product of two regular $(n-1)$-simplices. Identify the vertex set of $\Delta_{n-1} \times \Delta_{n-1}$ with $[n] \times [n]$. Then for $A \subset [n] \times [n]$ we have that $x_0 \in \operatorname{conv} A$ if and only if $\{(i, \pi(i)) \mid i \in [n]\} \subset A$ for some permutation $\pi\colon [n] \longrightarrow [n]$.*

Proof. The vertex set of $\Delta_{n-1} \times \Delta_{n-1}$ consists of all pairs (e_i, e_j) of standard basis vectors in $\mathbb{R}^n$. The barycenter x_0 of $\Delta_{n-1} \times \Delta_{n-1}$ is $x_0 = (\frac{1}{n}, \ldots, \frac{1}{n}) \in \mathbb{R}^n \times \mathbb{R}^n$. Let $\sum_{i,j} \lambda_{i,j}(e_i, e_j)$ be a convex combination of pairs of standard basis vectors (e_i, e_j). Associate the matrix $M = (\lambda_{i,j})_{i,j}$ to this convex combination. Then $x_0 = \sum_{i,j} \lambda_{i,j}(e_i, e_j)$ if and only if $n \cdot M$ is doubly stochastic. If A is a set of pairs of standard basis vectors (e_i, e_j) with $x_0 \in \operatorname{conv} A$ then by Lemma 2.1 there is permutation $\pi\colon [n] \longrightarrow [n]$ such that $\lambda_{i,\pi(i)} > 0$ for all $i \in [n]$. In particular, the set A contains $(e_i, e_{\pi(i)})$ for all $i \in [n]$. Conversely, if A contains $(e_i, e_{\pi(i)})$ for all $i \in [n]$ for some permutation $\pi\colon [n] \longrightarrow [n]$, then $\frac{1}{n}\sum_{i=1}^n (e_i, e_{\pi(i)}) = x_0$. □

Proof of Theorem 1.1. Denote the vertex set of H by V, and let $c\colon V \longrightarrow \mathbb{Z}/p$ be a $\mathbb{Z}/p$-coloring without zero-sum hyperedge. We will construct a $\mathbb{Z}/p$-equivariant map $B(H) \longrightarrow S^{2p-3}$. Recall that the vertex set of $B(H)$ consists of p disjoint copies of V, which we will represent as $V \times \mathbb{Z}/p$. Identify the vertex set of $\Delta_{p-1} \times \Delta_{p-1}$ with $\mathbb{Z}/p \times \mathbb{Z}/p$. The diagonal action of $\mathbb{Z}/p$ on $\mathbb{Z}/p \times \mathbb{Z}/p$ extends to $\Delta_{p-1} \times \Delta_{p-1}$. Define a map

$$f\colon B(H) \longrightarrow \Delta_{p-1} \times \Delta_{p-1} \subset \mathbb{R}^p \times \mathbb{R}^p$$

on vertices of $B(H)$ by $f(v, g) = (g, c(v) + g)$ and extend linearly onto the faces of $B(H)$. The map f is equivariant: $f(v, g+h) = (g+h, g+c(v)+h)$, which is g acting diagonally on $(h, c(v)+h)$.

We claim that the image of f misses the barycenter x_0 of $\Delta_{p-1} \times \Delta_{p-1}$. Otherwise by Lemma 2.2 there would be a permutation $\pi\colon \mathbb{Z}/p \longrightarrow \mathbb{Z}/p$ and a face $A_0 \uplus \cdots \uplus A_{p-1}$ of $B(H)$ such that $f(A_0 \uplus \cdots \uplus A_{p-1})$ contains the vertices $(g, \pi(g))$ for all $g \in \mathbb{Z}/p$. Then for every $g \in \mathbb{Z}/p$ there is a $v_g \in A_g$ with $f(v_g, g) = (g, \pi(g))$, and so $\pi(g) = c(v_g) + g$. Then $\sum_{g\in\mathbb{Z}/p} g = \sum_{g\in\mathbb{Z}/p} \pi(g) = \sum_{g\in\mathbb{Z}/p} c(v_g) + \sum_{g\in\mathbb{Z}/p} g$. This implies $\sum_{g\in\mathbb{Z}/p} c(v_g) = 0$. By definition of box complex $\{v_g \mid g \in \mathbb{Z}/p\} \in H$, in contradiction to c having no zero-sum hyperedges. Since f misses the barycenter of $\Delta_{p-1} \times \Delta_{p-1}$ it equivariantly retracts to the boundary of $\Delta_{p-1} \times \Delta_{p-1}$, which is S^{2p-3}. □

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Remark 2.3. For p-uniform hypergraphs H, it follows from work of Kříž [27, Theorems 2.2 and 2.6] that $\chi(H) \geq \frac{d+1}{p-1}$, where d is the minimum dimension of a sphere S^d equipped with a free $\mathbb{Z}/p$-action for which there exists a continuous $\mathbb{Z}/p$-equivariant map $B(H) \longrightarrow S^d$. Thus the condition $d \geq 2p-2$ given by Theorem 1.1 coincides with Kříž's criterion for $\chi(H) \geq 3$. However, the topological condition ensuring a zero-sum hyperedge given by Theorem 1.1 is stronger than the combinatorial condition that the hypergraph is not two-colorable. As a simple example, let $p = 3$ and let H be the 3-uniform hypergraph whose vertex set is $V = \{v_0, v_1, v_2, v_3, v_4, v_5, v_6\}$ and whose hyperedges consist of all subsets of three vertices from V where exactly two vertices come from either $A = \{v_1, v_2, v_3\}$ or $B = \{v_4, v_5, v_6\}$. It is easy to see that $\chi(H) = 3$, but there is no zero-sum hyperedge corresponding to the coloring $c\colon V \longrightarrow \mathbb{Z}/3$ defined by $c(v_i) = 0$ for $i \in \{1, 2, 3\}$, $c(v_i) = 1$ for $i \in \{4, 5, 6\}$, and $c(v_0) = 2$.

Remark 2.4. Let G be a graph with k edges. Following Caro's survey on zero-sum Ramsey theory [15], we denote by $R(G, 2)$ the Ramsey number of G, that is the smallest n such that in any 2-coloring of the edges of the complete graph K_n, there is a monochromatic copy of G. We let H_n be the k-uniform hypergraph whose vertex set consists of the edges of K_n, with hyperedges for any k edges of K_n that form a copy of G. Thus $R(G, 2)$ is the smallest n such that H_n has chromatic number at least 3. Now denote by $R(G, \mathbb{Z}/k)$ the zero-sum Ramsey number of G, that is, the smallest n such that every $\mathbb{Z}/k$-coloring of H_n has a zero-sum hyperedge. It is known that $R(G, \mathbb{Z}/k) \geq R(G, 2)$, and that this inequality is often strict as in Remark 2.3 above. Clearly, $R(G, \mathbb{Z}/k) \leq R(G, k)$, where $R(G, k)$ is the smallest n with $\chi(H_n) > k$. While it is an open problem whether the asymptotics of $R(G, \mathbb{Z}/k)$ are aligned more closely with $R(G, 2)$ or with $R(G, k)$, Theorem 1.1 does show that the *topological* lower bounds on $R(G, \mathbb{Z}/k)$ and $R(G, 2)$ are identical.

2.3. Colorings by an arbitrary group

We will now extend the arguments of the previous subsection to arbitrary finite groups, starting with the box complex construction. Let H be an n-uniform hypergraph on vertex set V, and let G be any group of order n (written multiplicatively, and with identity element 1). As before, a G-coloring of the hypergraph is a map $c\colon V \longrightarrow G$, and we now say that a hyperedge e is zero-sum if there is some ordering $e = \{v_1, \dots, v_n\}$ of its vertices such that $\Pi_{i=1}^n c(v_i) = 1$.

Consider the group ring $\mathbb{C}[G] = \{\sum_{g\in G} z_g\, g \mid z_g \in \mathbb{C}\}$ of the group G, on which G acts by linearly extending the action on G afforded by left multiplication. The subrepresentation $U_{\mathbb{C}}[G] = \{\sum_{g\in G} z_g\, g \in \mathbb{C}[G] \mid \sum_{g\in G} z_g = 0\}$ is G-invariant and has the origin as the only point that is fixed by every element of the group.

Theorem 2.5. *Let $n \geq 2$ be an integer, let H be an n-uniform hypergraph, and let G be group of order n. If every equivariant linear map $B(H) \longrightarrow U_{\mathbb{C}}[G]$ has a zero, then any G-coloring of H admits a zero-sum hyperedge.*

We briefly defer the proof of Theorem 2.5, preferring instead to first derive Theorem 1.2 as an immediate corollary. As with the EGZ theorem, the latter has the equivalent reformulation that any G-coloring of the complete n-uniform hypergraph H on $[2n-1]$ has a zero-sum hyperedge. We shall need the following result of Sarkaria [33]:

Theorem 2.6. *Let $m, n \geq 2$ be integers and let U be a real $(m-1)$-dimensional representation of a group G of order n. If the origin is the unique element of U that is fixed by the action, then any linear G-equivariant map $f\colon (\Delta_{m-1})^{*n}_{\Delta} \longrightarrow U$ has a zero.*

Proof of Theorem 1.2. Let H be the complete n-uniform hypergraph on $[2n-1]$, which we have seen is precisely the deleted n-fold join of the $(2n-2)$-dimensional simplex. By Theorem 2.6, any linear G-equivariant map $f\colon B(H) \longrightarrow U_{\mathbb{C}}[G]$ must have a zero, and so by Theorem 2.5 any G-coloring of H results in a zero-sum hyperedge. □

To prove Theorem 2.5, we will need the following elementary fact.

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Proposition 2.7. *Let G be any group, let $S = \{g_1, \ldots, g_m\} = \{h_1, \ldots, h_m\}$ be two orderings of a subset of order m, and let $x_i := g_i^{-1} h_i$ for all $i \in [m]$. Then there exists a permutation π of $[m]$ such that $\prod_{i=1}^m x_{\pi(i)} = 1$.*

Proof. We proceed by induction, the case $m = 1$ being immediate. For $m \geq 2$, consider $x_1 = g_1^{-1} h_1$. If $h_1 = g_1$, then we let $S' = \{g_2, \ldots, g_m\} = \{h_2, \ldots, h_m\}$ and we are finished by induction. Supposing $h_1 \neq g_1$, consider the unique $j \neq 1$ such that $g_j = h_1$. Without loss of generality, we may suppose $g_2 = h_1$. Thus $x_1 x_2 = g_1^{-1} h_2$. If $g_1 = h_2$, then we let $S'' = \{g_3, \ldots, g_m\} = \{h_3, \ldots, h_m\}$ and again we are done by induction. If not, we consider the unique $j \neq 1, 2$ such that $g_j = h_2$. Again without loss of generality we may suppose $j = 3$, in which case we have $x_1 x_2 x_3 = g_1^{-1} h_3$. Continuing in this fashion, we will eventually be done by induction or else, without loss of generality, we have that $h_1 = g_2, h_2 = g_3, \ldots, h_{m-2} = g_{m-1}$ and $h_m = g_1$. Thus $g_m = h_{m-1}$, so $x_1 \cdots x_{m-1} x_m = g_1^{-1} h_{m-1} g_m^{-1} h_m = 1$. □

We now prove Theorem 2.5.

Proof of Theorem 2.5. Let V denote the vertex set of H and suppose that $c \colon V \to G$ is a G-coloring of H without a zero-sum hyperedge. We will show there must exist a nonvanishing linear equivariant map $B(H) \to U_{\mathbb{C}}[G]$, where now we view $B(H)$ as a simplicial complex on $V \times G$. To that end, let $\mathcal{Z}$ be the set of all permutations of G and let Y denote the simplicial complex on $G \times G$ defined by specifying that $\sigma \in Y$ for $\sigma \subseteq G \times G$ if and only if σ does not contain the graph $\Gamma(z) = \{(g, z(g)) \mid g \in G\}$ of any $z \in \mathcal{Z}$.

We now define a simplicial map $f \colon B(H) \to Y$ by setting $f(v, g) = (g, gc(v))$ for $(v, g) \in V \times G$ and extending simplicially to the faces of $B(H)$. First, we verify that our map is well-defined. Letting $\sigma \in B(H)$, we show that $f(\sigma) \subseteq G \times G$ does not contain the graph $\Gamma(z)$ of any $z \in \mathcal{Z}$. So let $\sigma = \cup_{g \in G} (A_g \times \{g\})$ be a face of $B(H)$ and let $z \colon G \to G$ be a permutation of G. By definition, any set $\{a_g \mid g \in G\}$ with $a_g \in A_g$ for all $g \in G$ is a hyperedge of H. If $\Gamma(z) \subset f(\sigma)$, then there would be some collection $\{a_g\}_{g \in G}$ with $a_g \in A_g$ for all $g \in G$ such that $\{gc(a_g) \colon g \in G\} = \{z(g) \colon i \in G\}$, and therefore $\{gc(a_g) \colon g \in G\} = G$. Fixing an ordering $G = \{g_1, \ldots, g_n\}$ and applying Proposition 2.7 to $S = G$ and $h_i = g_i c(a_{g_i})$, we see there is some permutation ψ of $[n]$ such that $\Pi_{i=1}^n c(a_{g_{\psi(i)}}) = 1$. This contradicts the assumption that H does not contain a zero-sum hyperedge, so σ does not contain the graph of any $z \in \mathcal{Z}$ and the map f is well-defined.

As with the box complex $B(H)$, the simplicial complex Y has a natural G-action arising from group multiplication, namely by letting G act diagonally on the vertex set $G \times G$ of Y and extending this action to each face of Y. It is then easily observed that the simplicial map $f \colon B(H) \to Y$ is equivariant with respect to the described actions.

We will now construct a never-vanishing linear G-equivariant map $Y \to U_{\mathbb{C}}[G]$. Composition then gives a never-vanishing linear equivariant map $B(H) \to Y \to U_{\mathbb{C}}[G]$, completing the proof. To that end, first consider the real group ring $\mathbb{R}[G] = \{\sum_{g \in G} r_g\, g \mid r_g \in \mathbb{R}\}$ and let $h \colon Y \to \mathbb{R}[G] \times \mathbb{R}[G]$ be the map that is defined on vertices $(g_i, g_\ell) \in G \times G$ by $h(g_i, g_\ell) = (g_i, g_\ell)$ and otherwise interpolates linearly, which is clearly G-equivariant. We observe that the image of h is contained in U, the real $(2n-2)$-dimensional affine subspace of $\mathbb{R}[G] \times \mathbb{R}[G]$ for which both the first and last n coordinates sum to 1. We observe that U is invariant under the diagonal G-action on $\mathbb{R}[G] \times \mathbb{R}[G]$. Geometrically, this action simultaneously permutes the vertices of two regular $(n-1)$-simplices lying in orthogonal $(n-1)$-dimensional subspaces. We now claim that $(\sum_{g \in G} \frac{1}{n} g, \sum_{g \in G} \frac{1}{n} g)$ does not lie in the image of h. This follows in the same way as in the proof of Theorem 1.1 using Lemma 2.2. □

3. Proof of Theorem 1.3

We now prove Theorem 1.3. While this follows from Theorem 1.1 together with the work of Kříž [27], whose construction uses a different (albeit related) notion of a box complex from our own. Instead of adapting Křiž's construction, we shall instead give an argument that allows us to obtain Theorem 1.3 directly from Theorems 1.1 and 3.1 below; see [20] for similar techniques. We refer the reader to

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Matoušek and Ziegler [31] for an exposition of the various notions of box complexes and how they relate in the case $p = 2$.

We give some details on the difference between Křiž's construction and our box complex. Let H be a p-uniform hypergraph on V and denote by $B_{\text{chain}}(H)$ the partially ordered set on $V \times \mathbb{Z}/p$, where $A_0 \times \{0\} \cup \cdots \cup A_{p-1} \times \{p-1\}$ is in $B_{\text{chain}}(H)$ if all the A_i are nonempty and if for all $a_0 \in A_0, \ldots, a_{p-1} \in A_{p-1}$ one has that $\{a_0, \ldots, a_{p-1}\} \in H$. Thus $B_{\text{chain}}(H)$ differs from $B(H)$ only in that it excludes those $A_0 \times \{0\} \cup \cdots \cup A_{p-1} \times \{p-1\}$ where some of the A_i are empty. While the set $B_{\text{chain}}(H)$ is not closed under taking subsets and is therefore not in itself a simplicial complex, as a poset it has an associated order complex $C(H)$; this is the simplicial complex which Kříž considers. We observe that this complex $C(H)$ is a subcomplex of the barycentric subdivision of our box complex $B(H)$; we will use this later geometric version of $B_{\text{chain}}(H)$ in our proof.

We shall need the following special case of an elementary Borsuk–Ulam type theorem due to Dold [16].

Theorem 3.1. *Let $m \geq 1$ be an integer and let G be a nontrivial finite group. Suppose that X is an m-connected simplicial complex on which G acts and that S^m is an m-sphere on which G acts freely. Then there is no continuous G-equivariant map $f\colon X \to S^m$.*

Lemma 3.2. *Let $p \geq 2$ be a prime, let $m \geq 2$ be an integer, and suppose that $\mathbb{Z}/p$ acts freely on the sphere S^{m-2}. Suppose that $\mathcal{F}$ is a set system that is upwards-closed, that is, $A' \in \mathcal{F}$ whenever $A \in \mathcal{F}$ and $A' \supset A$. If $\mathrm{cd}^p(\mathcal{F}) \geq m$, then there is no $\mathbb{Z}/p$-equivariant map $B(\mathrm{KG}^p(\mathcal{F})) \to S^{m-2}$.*

Proof. Assume that the ground set of $\mathcal{F}$ is $[n]$. We let $N = (p-1)(d+1) + m$, where $d \geq 0$ is chosen to be an integer such that $N \geq n$. We let Σ be the simplicial complex on vertex set $[N]$ defined by specifying that $\sigma \subseteq [N]$ is a face of Σ if and only if $\sigma \notin \mathcal{F}$. As $\mathcal{F}$ is upward-closed, Σ is downward-closed and therefore a simplicial complex. By definition of the colorability defect $\mathrm{cd}^p(\mathcal{F})$, a p-tuple of pairwise disjoint faces of Σ can involve at most $N - m = (p-1)(d+1)$ vertices of $[N]$.

We now show that there exists a $\mathbb{Z}/p$-equivariant map $\widetilde{h}\colon \Sigma^{*p}_{\Delta} \to S^{(p-1)(d+1)-1}$. This follows from elementary obstruction theory.[1] Indeed, the dimension of Σ^{*p}_{Δ} and $S^{(p-1)(d+1)-1}$ are the same and latter is connected up to top dimension. Since the action on the domain is free, one therefore has an equivariant map from Σ^{*p}_{Δ} to $S^{(p-1)(d+1)-1}$ by extending an arbitrary equivariant map defined on the vertex set of Σ^{*p}_{Δ}.

Now let $\mathrm{sd}((\Delta_{N-1})^{*p}_{\Delta})$ denote the barycentric subdivision of the simplicial complex $(\Delta_{N-1})^{*p}_{\Delta}$. We shall construct a $\mathbb{Z}/p$-equivariant simplicial map

$$\Phi\colon \mathrm{sd}((\Delta_{N-1})^{*p}_{\Delta}) \to \mathrm{sd}(\Sigma^{*p}_{\Delta}) * B(\mathrm{KG}^p(\mathcal{F})).$$

To that end, let v be a vertex of $\mathrm{sd}((\Delta_{N-1})^{*p}_{\Delta})$. This vertex uniquely corresponds to a face of the complex $(\Delta_{N-1})^{*p}_{\Delta}$, or in other words to a p-tuple $(A_1, \ldots, A_p)$ of pairwise disjoint subsets of $[N]$. If none of the A_i are in $\mathcal{F}$, then each A_i determines a face of Σ and we define $\Phi(v) \in \mathrm{sd}(\Sigma^{*p}_{\Delta})$ be the vertex of the barycentric subdivision of Σ^{*p}_{Δ} that corresponds to the face $A_1 \uplus \cdots \uplus A_p$. On the other hand, suppose that $A_i \in \mathcal{F}$ for at least one $i \in [p]$. Then we set $A_i = B_i$ for any such $i \in [p]$, and we set $B_i = \emptyset$ for any $i \in [p]$ with $A_i \notin \mathcal{F}$. We then define $\Phi(v)$ to be the vertex of the box complex $B(\mathrm{KG}^p(\mathcal{F}))$ that corresponds to $(B_1, \ldots, B_p)$. It is now easily verified that Φ is $\mathbb{Z}/p$-equivariant.

To complete the proof, suppose that $\mathbb{Z}/p$ acts freely on S^{m-2} and that there is some $\mathbb{Z}/p$-equivariant map $B(\mathrm{KG}^p(\mathcal{F})) \to S^{m-2}$. Joining this map with $\widetilde{h}$ above then gives a $\mathbb{Z}/p$-equivariant map $\Sigma^{*p}_{\Delta} * B(\mathrm{KG}^p(\mathcal{F})) \to S^{(p-1)(d+1)-1} * S^{m-2} \cong S^{N-2}$, and composition of this map with Φ thereby yields a $\mathbb{Z}/p$-equivariant map $(\Delta_{N-1})^{*p}_{\Delta} \to S^{N-2}$. Noting that $(\Delta_{N-1})^{*p}_{\Delta} \cong [p]^{*(N-1)}$ is $(N-2)$-connected and that the $\mathbb{Z}/p$-action on the join sphere S^{N-2} is free, we have reached a contradiction with Theorem 3.1. Thus there is no $\mathbb{Z}/p$-equivariant map $B(\mathrm{KG}^p(\mathcal{F})) \to S^{m-2}$. □

[1] We thank an anonymous referee for this observation, which simplified our previous argument.

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Theorem 1.3 now follows immediately from Lemma 3.2 and Theorem 1.1 when p is prime, and for arbitrary integers n by a standard induction on prime divisors as in [3, 26], and the original proof of the EGZ theorem [19].

Proof of Theorem 1.3. First, let p be prime. We note that we may assume that $\mathcal{F}$ is upward-closed. This is because if $\mathcal{G}$ is the set family obtained from $\mathcal{F}$ by including any supersets of the $A \in \mathcal{F}$, then it is immediate from the definition of colorability defect that $\mathrm{cd}^p(\mathcal{G}) \geq \mathrm{cd}^p(\mathcal{F}) \geq 2n-1$. Given a coloring $c\colon \mathcal{F} \longrightarrow \mathbb{Z}/p$, we greedily extend it to a $\mathbb{Z}/p$-coloring $\widetilde{c}\colon \mathcal{G} \longrightarrow \mathbb{Z}/p$. Namely, if $A_1, \ldots, A_k$ are the inclusion-maximal elements of $\mathcal{F}$, then we set $c(A_1') = A_1$ for all supersets $A_1' \supset A_1$, $c(A_2') = C(A_2)$ for all supersets $A_2' \supset A_2$ which are not supersets of A_1, and so on. Considering the $\mathbb{Z}/p$-action on $U_{\mathbb{C}}[\mathbb{Z}/p]$ and letting $m = 2p-1$ in Lemma 3.2, we have $\mathrm{cd}^p(\mathcal{G}) \geq 2p-1$ and so by Lemma 3.2 there is no $\mathbb{Z}/p$-equivariant map $B(\mathrm{KG}^p(\mathcal{G})) \longrightarrow S^{2p-3}$. Thus $\mathrm{KG}^p(\mathcal{G})$ contains a zero-sum hyperedge $\{A_1', \ldots, A_p'\}$ by Theorem 1.1, and therefore $\mathrm{KG}^p(\mathcal{F})$ contains a hyperedge $\{A_1, \ldots, A_p\}$.

For general n, let $\mathcal{F}$ be a set system on $[m]$ with $\mathrm{cd}^n(\mathcal{F}) \geq 2n-1$. Given any integer-valued function $c\colon \mathcal{F} \longrightarrow \mathbb{Z}$, we show that there are pairwise disjoint $A_1, \ldots, A_n \in \mathcal{F}$ such that n divides $\sum_{i=1}^n c(A_i)$. As in [19], we induct on the number of prime divisors of n and so we may assume that $n = pq$, where p is a prime and $q \geq 2$ is an integer. We now follow the same reasoning as given by Kříž [26]: Defining $\Gamma = \{E \subseteq [m] \mid \mathrm{cd}^q(\mathcal{F}|_E) \geq 2q-1\}$, Kříž shows that the assumption that $\mathrm{cd}^n(\mathcal{F}) \geq 2n-1$ implies that $\mathrm{cd}^p(\Gamma) \geq 2p-1$. Now let $E \in \Gamma$ be arbitrary. By induction, any $\mathbb{Z}/q$-coloring of $\mathrm{KG}^q(\mathcal{F}|_E)$ has a zero-sum hyperedge, that is, there are pairwise disjoint sets $A_{E,1}, \ldots, A_{E,q} \in \mathcal{F}|_E$ such that q divides $\sum_{i=1}^q c(A_{E,i})$. Defining $\widehat{c}\colon \Gamma \longrightarrow \mathbb{Z}$ by $\widehat{c}(E) = \sum_{i=1}^q c(A_{E,i})$ and using the fact that $\mathrm{cd}^p(\Gamma) \geq 2p-1$ shows that there are pairwise disjoint $E_1, \ldots, E_p \in \Gamma$ such that p divides $\sum_{i=1}^p \widehat{c}(E_i)$. We therefore have that n divides $\sum_{i=1}^p \sum_{j=1}^q c(A_{E_i,j})$, so that $\{A_{E_i,j} \mid i \in [p], j \in [q]\}$ is the desired zero-sum hyperedge. □

4. Zero-sums via the topology of chessboard complexes

4.1. Chessboard complexes

Our fractional generalization of the EGZ theorem and our proof of Hall's characterization of zero-sum sequences both rely crucially on the topological properties of chessboard complexes, a simplicial complex that encodes the nonattacking rook placements on an $m \times n$ chessboard. These complexes have seen extensive application to geometric combinatorics, notably in the context of colorful extensions of Tverberg's theorem (see, e.g., [5, 37, 35, 11]).

Definition 4.1 (Chessboard Complex). For positive integers n and m, the **chessboard complex** $\Delta_{m,n}$ is the simplicial complex where $\sigma \subseteq [m] \times [n]$ is a face of $\Delta_{m,n}$ if and only if $(i_1, j_1), (i_2, j_2) \in \sigma$ whenever $i_1 \neq i_2$ and $j_1 \neq j_2$.

As an abstract simplicial complex, $\Delta_{m,n}$ consists of all matchings of the complete bipartite graph $K_{m,n}$. For any chessboard complex $\Delta_{m,n}$ one has a $\mathbb{Z}/m$-action given by permuting the rows of the chessboard, as well as a free $\mathbb{Z}/n$-action obtained by permuting the columns.

4.2. A topological proof of Hall's zero-sum criterion

Our proof of Theorem 1.4 relies on the fact (see [11, 35]) that any chessboard complex of the form $\Delta_{n-1,n}$ is an $(n-2)$-dimensional orientable pseudomanifold when $n \geq 3$. Considering the $\mathbb{Z}/n$ action on $\Delta_{n-1,n}$ induced by permuting the columns of an $(n-1) \times n$ chessboard, arguments of [10, 11] (see also [35, Proposition 2]) show that when $p \geq 3$ is prime then any $\mathbb{Z}/p$-equivariant map $\Delta_{p-1,p} \longrightarrow \partial\Delta_{p-1}$ has nonzero degree. Since any $\mathbb{Z}/p$-equivariant map $\Delta_{p-1,p} \longrightarrow \partial\Delta_{p-1}$ that fails to be surjective has degree zero one has the following lemma:

Lemma 4.2. *If $p \geq 3$ is prime then any continuous $\mathbb{Z}/p$-equivariant mapping $f\colon \Delta_{p-1,p} \longrightarrow \partial\Delta_{p-1}$ is surjective.*

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Lemma 4.2 implies the following result on partial transversals from which Hall's criterion for zero-sums in $\mathbb{Z}/p$ follows quickly:

Theorem 4.3. *Let $p \geq 2$ be a prime, and let $a_1, a_2, \dots, a_{p-1} \in \mathbb{Z}/p$ be a sequence of length $p-1$. Then there are pairwise distinct $b_1, b_2, \dots, b_{p-1} \in \mathbb{Z}/p$ such that*

$$\{a_1 + b_1, a_2 + b_2, \dots, a_{p-1} + b_{p-1}\} = \{0, 1, \dots, p-2\}.$$

Proof. We identify the vertex set of the chessboard complex $\Delta_{p-1,p}$ with $[p-1] \times \mathbb{Z}/p$ and the vertex set of Δ_{p-1} with $\mathbb{Z}/p$ as well (i.e., Δ_{p-1} is the standard simplex in the regular representation $\mathbb{R}[\mathbb{Z}/p]$). The map $f\colon \mathbb{Z}/p \times [p-1] \to \mathbb{Z}/p$ given by $f(i,j) = a_j + i$ induces a simplicial and $\mathbb{Z}/p$-equivariant map $f\colon \Delta_{p-1,p} \to \Delta_{p-1}$. Any such map is surjective onto $\partial\Delta_{p-1}$ by Lemma 4.2, so in particular there must be a maximal face of $\Delta_{p-1,p}$ which is mapped onto the face $\{0, 1, \dots, p-2\}$ of $\partial\Delta_{p-1}$. Thus there is an injective map $\pi\colon [p-1] \to \mathbb{Z}/p$ such that $\{0, 1, \dots, p-2\} = \{f(\pi(i), i) \mid i \in [p-1]\} = \{a_i + \pi(i) \mid i \in [p-1]\}$, completing the proof. □

Proof of Theorem 1.4 for primes. First, let $b = \sum_{i \in Z/p} i$ (thus $b = 1$ if $p = 2$ and is zero otherwise). We define the simplicial complex Y on $\mathbb{Z}/p \times \mathbb{Z}/p$ to consist of all subsets of $\mathbb{Z}/p \times \mathbb{Z}/p$ that do not contain the graph of any function $z\colon \mathbb{Z}/p \to \mathbb{Z}/p$ with $\sum_i z(i) = b$. We now let Y' be the simplicial complex $Y \cup \Delta_{p,p}$ and we equip Y' with the $\mathbb{Z}/p$-action that is defined on vertices $(i, \ell) \in \mathbb{Z}/p \times \mathbb{Z}/p$ by $j \cdot (i, \ell) = (i, \ell + j)$ for all $j \in \mathbb{Z}/p$.

We now show that any $\mathbb{Z}/p$-equivariant simplicial map from $Y' \to \Delta_{p-1}$ is surjective. To see this, first observe that the restriction of Y' to $\{0, 1, \dots, p-2\} \times \mathbb{Z}/p$ equivariantly contains $\Delta_{p-1,p}$. By Lemma 4.2, any $\mathbb{Z}/p$-equivariant map $\Delta_{p-1,p} \to \partial\Delta_{p-1}$ is surjective, so there is some injective map $\pi\colon \{0, 1, \dots, p-2\} \to \mathbb{Z}/p$ so that the face $\{(i, \pi(i)) \mid i \in \{0, 1, \dots, p-2\}\}$ of $\Delta_{p-1,p}$ is mapped simplicially onto the face $\{0, 1, \dots, p-2\}$ of $\partial\Delta_{p-1}$. We may now extend this face to a larger face in Y' that will surject onto Δ_{p-1}, as follows: there must be a unique vertex of the form $(p-1, j)$ of Y' that maps to the vertex $p-1$ of Δ_{p-1}. We claim that

$$\sigma\colon = \{(i, \pi(i)) \mid i \in \{0, 1, \dots, p-2\}\} \cup \{(p-1, j)\}$$

is a face of Y' and thus that $Y' \to \Delta_{p-1}$ is surjective. To see this, note that if $j \neq \pi(i)$ for all $i \in \{0, 1, \dots, p-2\}$, then σ lies in $\Delta_{p,p} \subseteq Y'$. On the other hand, if $j = \pi(i)$ for some $i \in \{0, 1, \dots, p-2\}$, then we must have $j + \sum_i \pi(i) \neq b$, so that now σ lies in $Y \subset Y'$. Thus any $\mathbb{Z}/p$-equivariant simplicial map from Y' to Δ_{p-1} is surjective.

Finally, let $a_1, \dots, a_p \in \mathbb{Z}/p$ be a given zero-sum sequence. We must show there is some permutation π of $\mathbb{Z}/p$ such that $\{a_1 + \pi(1), \dots, a_p + \pi(p)\} = \mathbb{Z}/p$. For this, we define a simplicial map $f\colon Y' \to \Delta_{p-1}$ by specifying that $f(i,j) = a_j + i$ for each vertex (i,j). Since f is $\mathbb{Z}/p$-equivariant and is therefore surjective, there exists a function $\pi\colon \mathbb{Z}/p \to \mathbb{Z}/p$ such that f carries some maximal face $\tau := \{(i, \pi(i)) \mid i \in \mathbb{Z}/p\}$ of Y' onto Δ_{p-1}. To complete the proof, we show that τ is not a face of Y, hence that τ is a face of $\Delta_{p,p}$, and therefore that π is indeed a permutation of $\mathbb{Z}/p$. To see this, observe that by definition $\sum_i \pi'(i) \neq b$ were τ in Y. However, since $\{\pi'(i) + a_i\} = \mathbb{Z}/p$ and $\sum_i a_i = 0$ we must have $\sum_i \pi'(i) = \sum_i \pi'(i) + a_i = b$. □

4.3. *Proof of Theorem 1.5*

We now give a second proof of the original EGZ theorem, now based on the topological connectivity of chessboard complexes (see, e.g., [9]). The proof method will then easily yield the probabilistic extension Theorem 1.5. First we need the following Lemma, which can also be obtained as a direct consequence of cohomological index computations for more general chessboard complexes [12, Theorem 6.8]. For this, we consider the free $\mathbb{Z}/p$-action on $\Delta_{p,2p-1}$ arising from permuting the rows of the $p \times (2p-1)$ chessboard.

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As in the complex setting, for a finite group G we let $U_{\mathbb{R}}[G]$ denote the sub-representation of the group ring $\mathbb{R}[G] = \sum_{g\in G} r_g\, g \mid r_g \in \mathbb{R}\}$ consisting of all formal sums $\sum_{g\in G} r_g\, g$ for which $\sum_{g\in G} r_g = 0$.

Lemma 4.4. *If $p \geq 2$ is prime, then for any continuous $\mathbb{Z}/p$-equivariant map $\Delta_{p,2p-1} \to \Delta_{p-1}$ there is some point in $\Delta_{p,2p-1}$ whose image is the barycenter of Δ_{p-1}.*

Proof. As shown in [9], $\Delta_{p,2p-1}$ is $(p-2)$-connected while $S(U_{\mathbb{R}}[\mathbb{Z}_p])$ has dimension $p-2$, so by Theorem 3.1 any continuous $\mathbb{Z}/p$-equivariant map $\Delta_{p,2p-1} \to U_{\mathbb{R}}[\mathbb{Z}/p]$ must have a zero. Viewing Δ_{p-1} as the standard simplex inside $\mathbb{R}[\mathbb{Z}/p]$, translation by the barycenter of Δ_{p-1} shows that any $\mathbb{Z}/p$-equivariant map $\Delta_{p,2p-1} \to \Delta_{p-1}$ results in equivariant map $\Delta_{p,2p-1} \to U_{\mathbb{R}}[\mathbb{Z}/p]$. As such a map must have a zero, the map $\Delta_{p,2p-1} \to \Delta_{p-1}$ must hit the barycenter. $\square$

Second Proof of the EGZ theorem for primes p. Let $a_1, \ldots, a_{2p-1} \in \mathbb{Z}/p$ be an arbitrary sequence. Identifying the vertex set of the chessboard complex $\Delta_{p,2p-1}$ with $\mathbb{Z}/p \times [2p-1]$, we let $f\colon \mathbb{Z}/p \times [2p-1] \to \mathbb{Z}/p$ be the simplicial map defined on the vertex set by $f(i,j) = a_j + i$. This map is $\mathbb{Z}/p$-equivariant after identifying the vertex set of Δ_{p-1} with $\mathbb{Z}/p$. Any such map must hit the barycenter of Δ_{p-1}, and since the map is simplicial there must be a face of the chessboard complex $\Delta_{p,2p-1}$ which is mapped onto Δ_{p-1}. Thus there is an injective map $\pi\colon \mathbb{Z}/p \to [2p-1]$ such that $\mathbb{Z}/p = \{f(i,\pi(i)) \mid i \in \mathbb{Z}/p\} = \{a_{\pi(i)} + i \mid i \in \mathbb{Z}/p\}$. This implies $\sum_{i\in\mathbb{Z}/p} i = \sum_{i\in\mathbb{Z}/p} a_{\pi(i)} + i$ and so that $\sum a_{\pi(i)} = 0$. $\square$

Replacing simplicial maps with linear ones proves our fractional generalization of the EGZ theorem.

Proof of Thm 1.5. First, observe that any probability measure μ on $\mathbb{Z}/p$ may be identified with a point x_μ in Δ_{p-1} since both uniquely describe convex coefficients for the vertices of Δ_{p-1}. Here we again identify $\mathbb{Z}/p$ with the vertices of Δ_{p-1}. Explicitly, this bijective correspondence is given by $x_\mu = \sum_{i\in\mathbb{Z}/p} \mu(\{i\})\cdot i$. Repeating the proof of the EGZ theorem above, given a sequence $\mu_1, \ldots, \mu_{2p-1}$ of measures on $\mathbb{Z}/p$ we define the continuous $\mathbb{Z}/p$-equivariant map $f\colon \Delta_{p,2p-1} \to \Delta_{p-1}$ by setting $f(i,j) = \mu_j + i$ on the vertices and extending to the faces of $\Delta_{p,2p-1}$ by linear interpolation. As each μ_j is a probability measure on $\mathbb{Z}/p$, the image $f(i,j)$ of each vertex lies in Δ_{p-1} and so f does indeed map to Δ_{p-1}. While this map is not simplicial (unless each of the μ_j are Dirac measures), Lemma 4.4 applies nonetheless and so there is a face σ of $\Delta_{p,2p-1}$ such that $f(\sigma)$ contains the barycenter of Δ_{p-1}. Thus there exists a permutation π of $\mathbb{Z}/p$ and convex coefficients λ_i such that $\sum_i \mu_{\pi(i)} + i$ equals the barycenter of Δ_{p-1}. As the barycenter of Δ_{p-1} corresponds to the uniform probability measure on $\mathbb{Z}/p$, the proof is complete. $\square$

Corollary 1.6 immediately follows from Theorem 1.5:

Proof of Cor. 1.6. Let $A_1, \ldots, A_{2p-1} \subseteq \mathbb{Z}/p$ be nonempty subsets of $\mathbb{Z}/p$ and associate to each A_i the uniform probability distribution μ_i supported on A_i. Applying Theorem 1.5 to the sequence $\mu_1, \ldots, \mu_{2p-1}$ thereby completes the proof. $\square$

We conclude this section with a remark concerning Lemma 4.4. As we have seen, our chessboard proof of the EGZ theorem for cyclic groups of prime order relied only on the fact that any equivariant *simplicial* map $\Delta_{p,2p-1} \to \Delta_{p-1}$ hits the barycenter of the simplex, or equivalently that the simplicial map is surjective. Thus if it could be shown that any $\mathbb{Z}/n$-equivariant simplicial map $\Delta_{n,2n-1} \to \Delta_{n-1}$ is surjective for any integer $n \geq 2$, our proof technique would imply the EGZ theorem for arbitrary cyclic groups. As we now show, it actually follows immediately from the EGZ theorem that any $\mathbb{Z}/n$-equivariant simplicial map $\Delta_{n,2n-1} \to \Delta_{n-1}$ is indeed surjective.

Theorem 4.5. *Let $n \geq 2$ be an integer. Then any $\mathbb{Z}/n$-equivariant simplicial map $f\colon \Delta_{n,2n-1} \to \Delta_{n-1}$ is surjective.*

Proof. As before, we identify the vertex set of Δ_{n-1} by $\mathbb{Z}/n$. By the remarks above, we only need to show that the EGZ theorem implies that a given $\mathbb{Z}/n$-equivariant simplicial map $f\colon \Delta_{n,2n-1} \to \Delta_{n-1}$ hits the barycenter of Δ_{n-1}. To that end, consider the sequence $a_1 = f(0,1), \ldots, a_{2n-1} = f(0, 2n-1)$ in $\mathbb{Z}/n$. This has a zero-sum subsequence of length n by the EGZ theorem, and thus there is a permutation π of

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$\mathbb{Z}/n$ such that $\sum_i f(0,\pi(i)) = 0$. The set $\{(i,\pi(i)) \mid i \in \mathbb{Z}/p\}$ is a face of $\Delta_{n,2n-1}$, and by equivariance we have that $f(i,\pi(i)) = a_{\pi(i)} + i$. Letting $x = \sum_i \frac{1}{n}(i,\pi(i))$, we now have that $f(x) = \sum_i \frac{1}{n} f(i,\pi(i)) = \sum_i \frac{1}{n} a_{\pi(i)} + \sum_i \frac{1}{n} i = \sum_i \frac{1}{n} i$ is the barycenter of Δ_{n-1}. □

5. Erdős–Ginzburg–Ziv plus constraints

We conclude with the proof of our constrained version of the EGZ theorem.

Proof of Thm. 1.7. Let $a_1,\dots,a_{2p-1} \in \mathbb{Z}/p$. We identify the vertex set of $(\Delta_{2p-2})^{*p}_{\Delta}$ with $\mathbb{Z}/p\times[2p-1]$. Suppose now that $X \subseteq (\Delta_{2p-2})^{*p}_{\Delta}$ is a $\mathbb{Z}/p$-equivariant $(2p-3)$-connected subcomplex of $(\Delta_{2p-2})^{*p}_{\Delta}$. For any $i \in \mathbb{Z}/p$, we let Y_i be the subcomplex of $(\Delta_{2p-2})^{*p}_{\Delta}$ defined by letting $\sigma = \{0\}\times A_0 \cup\dots\cup \{p-1\}\times A_{p-1}$ be a face of Y_i if $|A_i| \le 1$. Thus $\cap_{i\in\mathbb{Z}/p} Y_i = \Delta_{p,2p-1}$ is precisely the $p\times(2p-1)$ chessboard complex. Letting $d\colon X\times X \to [0,\infty)$ be any metric on X compatible with the topology of X as a simplicial complex (e.g., the ℓ_1-metric), for each $x \in X$ we denote by $d(x,Y_i) = \min_{y\in Y_i} d(x,y_i)$ the distance of x to the subcomplex Y_i. We thus have $x \in \Delta_{p,2p-1}$ if and only if $d(x,Y_i) = 0$ for all $i \in \mathbb{Z}/p$.

As in our second proof of the EGZ theorem, we define $f\colon \mathbb{Z}/p\times[2p-1] \to \mathbb{Z}/p$ by $f(i,j) = a_j + i$, which induces a $\mathbb{Z}/p$-equivariant simplicial map $f\colon X \to \Delta_{p-1}$. Thinking of Δ_{p-1} as the standard simplex in $\mathbb{R}[\mathbb{Z}/p]$ and letting b denote its barycenter, we now define

$$F\colon X \to (U_{\mathbb{R}}[\mathbb{Z}/p])^{\oplus 2},\ x \mapsto (f(x)-b, d(x,Y_0)-a(x),\dots,d(x,Y_{p-1})-a(x)),$$

where $a(x) = \frac{1}{p}\sum_i d(x,Y_i)$ is the average distance of $x \in X$ to the Y_i.

The map F is $\mathbb{Z}/p$-equivariant, and since by assumption X is $(2p-3)$-connected, it follows from Theorem 3.1 that F must have a zero x. Thus $f(x)$ is at the barycenter of Δ_{p-1}, and moreover we have that $d(x,Y_0) = \dots = d(x,Y_{p-1}) = a(x)$. Let $\sigma = \{0\}\times A_0\cup\dots\cup\{p-1\}\times A_{p-1}$ be the inclusion-minimal face of X that contains x. We now claim that $a(x) = 0$, so that $d(x,Y_i) = 0$ for all $i \in Z/p$ and $x \in \Delta_{p,2p-1}$. Indeed, if $a(x) > 0$, then $d(x,Y_j) > 0$ for some $j \in \mathbb{Z}/p$ and therefore that $d(x,Y_i) > 0$ for all $i \in \mathbb{Z}/p$. We would therefore have that $|A_i| > 1$ for all $i \in \mathbb{Z}/p$. Since by definition of $(\Delta_{2p-1})^{*p}_{\Delta}$ the A_j are pairwise disjoint, this implies that $|\bigcup_j A_j| \ge 2p$, a contradiction since $\bigcup_j A_j \subseteq [2p-1]$. Thus $|A_j| \le 1$ for all j. On the other hand, since $f(\sigma) = \Delta_{p-1}$, we therefore must have $|A_j| = 1$ for all j. Thus $\sigma = \{(i,\pi(i)) \mid i \in \mathbb{Z}/p\}$ for some injective map $\pi\colon \mathbb{Z}/p \to [2p-1]$, and since $\{f(i,\pi(i)) \mid i \in \mathbb{Z}/p\} = \mathbb{Z}/p$ we have that $\sum a_{\pi(i)} = 0$ as before.

To finish the proof, we therefore only need to verify that the constraints of Theorem 1.7 give rise to a $(2p-3)$-connected subcomplex $X \subseteq (\Delta_{2p-2})^{*p}_{\Delta}$. The subcomplex X is defined by the property that for every face σ of $(\Delta_{2p-2})^{*p}_{\Delta}$, we have that whenever $(i,2j-1) \in \sigma$ then $(i+x,2j) \notin \sigma$ unless $x = d_j$. The complex X is indeed highly connected, since it is the join of $(p-1)$ circles (corresponding to $\{a_1,a_2\},\dots,\{a_{2p-3},a_{2p-2}\}$) and p discrete points (corresponding to a_{2p-1}). Restricting the vertex set of X to $\mathbb{Z}/p\times\{2j-1,2j\}$ yields a $\mathbb{Z}/p$-equivariant cycle of length $2p$ that traverses $(0,2j-1),(d_j,2j),(d_j,2j-1),(2d_j,2j),\dots,(pd_j,2j)$. This closes up only after $2p$ steps because p is prime. As there are no further constraints, X is the join of these cycles and the p vertices corresponding to the restriction of X to $\mathbb{Z}/p\times\{2p-1\}$. As the $(p-1)$-fold join of the circle S^1 is a $(2p-3)$-dimensional sphere S^{2p-3}, we see that $X = \cup_{i\in\mathbb{Z}/p} D_i^{2p-2}$ is the union of the p cones $D_i^{2p-2} = S^{2p-3} * \{i\}$, each of which is a $(2p-2)$-dimensional disk. Thus X is $(2p-3)$-connected. □

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Competing interest. The authors have no competing interest to declare.

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